Periodic spline interpolation with shifted nodes

Gerlind Plonka

Abstract

Interpolation problems with periodic splines of defect 2 on an equidistant lattice with two shifted interpolation nodes in each knot interval are considered. Then the periodic Hermite-spline interpolation problem is obtained as a special case. Using generalized Euler-Frobenius polynomials and exponential Euler splines, a simple criterion for the existence and uniqueness of solutions of the considered interpolation problem can be given. This solves an old open problem and generalizes the well-known result on periodic Lagrange-spline interpolation obtained by G. Meinardus, G. Merz and H. ter Morsche. An extension to cardinal spline interpolation is also described.

1. Introduction

It has been known for a long time that the investigations concerning the construction of spline interpolants as well as the problem of existence and uniqueness of solutions of spline interpolation problems on an equidistant lattice unavoidably lead to the Euler-Frobenius polynomials and their generalizations (cf. [11]). Recently we have described an efficient algorithm for the computation of periodic Hermite-spline interpolants on the equidistant lattice $\mathbf{Z}$ (cf. [8]). This method uses a generalization of Euler-Frobenius polynomials which is based on B-splines with multiple knots, and can be extended to shifted nodes too.

Now we are mainly interested in the investigation of the existence and uniqueness of solutions. This problem has been completely solved only in the case of Lagrange-spline interpolation ($r = 1$) [4], [7]. For $r \geq 2$ results on the correctness of cardinal and periodic Hermite-spline interpolation on an equidistant lattice without shifted nodes may be found in [3], [2] and [5].

In this paper we consider a periodic spline interpolation problem based on spline functions of defect 2 with two shifted interpolation nodes $\tau_0 + j$ and $\tau_1 + j$ ($\tau_0, \tau_1 \in (0, 1]$) in each knot interval $[j, j+1]$. Then we obtain the periodic Hermite-spline interpolation problem in the special case $\tau_0 = \tau_1$.

The purpose of this paper is to present a simple criterion for the existence and uniqueness of solutions of our extended spline interpolation problem. Contrary to [6], [9] and [10] we prefer a new generalization of Euler-Frobenius polynomials which is based on B-splines with double knots. A representation of the symbol of the considered interpolation problem will be given and its behaviour on the unit circle will be studied. Our result can easily be extended to cardinal spline interpolation.

2. Main results

Let $N, m \in \boldsymbol{N}$ and $r \in \{1, ..., m\}$ be fixed. By $S_{m,r}^N$ we denote the linear space of all N-periodic real functions $s \in C^{m-r}(\boldsymbol{R})$ with

$$s(j+t) = p_j(t), \qquad p_j = p_{j+N} \in \boldsymbol{P}_m$$

for all $t \in [0,1]$ and for all $j \in \boldsymbol{Z}$, where $\boldsymbol{P}_m$ signifies the set of all real polynomials of degree $\leq m$ defined on $[0,1]$. The elements of $S_{m,r}^N$ are called N-periodic spline functions of degree m and defect r on the equidistant lattice $\boldsymbol{Z}$. It is well-known that $\dim S_{m,r}^N = rN$. Furthermore, let $y_j^{(k)} \in \boldsymbol{R}$ $(j \in \boldsymbol{Z},\ k = 0, ..., r-1)$ with $y_j^{(k)} = y_{j+N}^{(k)}$ be given N-periodic data, which can be completely described by the vectors

$$\boldsymbol{y}^{(k)} = (y_0^{(k)}, y_1^{(k)}, ..., y_{N-1}^{(k)})^T \in \boldsymbol{R}^N \qquad (k = 0, ..., r-1).$$

In the case of Lagrange-spline interpolation $(r = 1)$ with shift parameter $\tau \in (0, 1]$, we wish to find a N-periodic spline function $s \in S_{m,1}^N$ satisfying the interpolation conditions

$$s(j+\tau) = y_j^{(0)} \qquad (j = 0, ..., N-1). \tag{1}$$

Then the well-known existence- and uniqueness theorem of G. Meinardus, G. Merz and H. ter Morsche holds :

Theorem 1 (cf. [4], [7]) . *Let $N, m \in \boldsymbol{N}$ and $\tau \in (0,1]$ be fixed. Then the interpolation problem* (1) *is uniquely solvable for any data vector $\boldsymbol{y}^{(0)} \in \boldsymbol{R}^N$ if and only if one of the following conditions is satisfied:*

(i) N odd,

(ii) N even and m even and $\tau \in (0,1)$,

(iii) N even and m odd and $\tau \neq 1/2$.

Remark: The m-th Euler polynomial E_m on $[0,1]$ may be defined by

$$E_0(t) \equiv 1 \qquad (t \in [0,1]),$$

$$E_m'(t) = E_{m-1}(t) \qquad (m \in \boldsymbol{N}), \qquad E_m(0) + E_m(1) = 0, \qquad (m \in \boldsymbol{N}).$$

Then the conditions (ii) and (iii) of Theorem 1 are equivalent to the condition

(iv) N even and $E_m(\tau) \neq 0$. ♠

In the case $r = 2$, we consider the following spline interpolation problem: For given shift parameters $\tau_0, \tau_1 \in \boldsymbol{R}$ with $0 < \tau_0 \leq \tau_1 \leq 1$, we try to find a N-periodic spline function $s \in S_{m,2}^N$ such that

$$\begin{aligned} s(j+\tau_0) &= y_j^{(0)} \qquad (j \in \boldsymbol{Z}), \\ s[j+\tau_0, j+\tau_1] &= y_j^{(1)} \qquad (j \in \boldsymbol{Z}), \end{aligned} \tag{2}$$

where $s[j+\tau_0, j+\tau_1]$ denotes the second order divided difference.

The j-th Bernoulli polynomial B_j on $[0,1]$ may be defined by

$$B_0(t) \equiv 1 \qquad (t \in [0,1]),$$

and

$$B'_{j+1}(t) = B_j(t) \qquad (j \in \boldsymbol{N}_0), \qquad \int_0^1 B_j(t)dt = 0 \qquad (j \in \boldsymbol{N}).$$

Our main result, proved in Section 6, is the following

Theorem 2 . *Let $N, m \in \boldsymbol{N}$ $(N, m \geq 2)$ and $\tau_0, \tau_1 \in \boldsymbol{R}$ with $0 < \tau_0 \leq \tau_1 \leq 1$ be fixed. Then the spline interpolation problem* (2) *possesses a unique solution for any given data vectors $\boldsymbol{y}^{(0)}$, $\boldsymbol{y}^{(1)} \in \boldsymbol{R}^N$ if and only if*

$$B_m[\tau_0, \tau_1] \neq 0. \tag{3}$$

Here $B_m[\tau_0, \tau_1]$ denotes the divided difference of the m-th Bernoulli polynomial.

Remark: In the case $\tau = \tau_0 = \tau_1$ of Hermite-spline interpolation, the condition (3) is equivalent to

$$B_{m-1}(\tau) \neq 0.$$

The behaviour of the zeros of the Bernoulli polynomials on $[0,1]$ is known. In particular, if m is even, then the Hermite-spline interpolation problem (2) is uniquely solvable if and only if $\tau \notin \{1/2,\ 1\}$. ♠

Examples: In the case of quadratic spline interpolation $(m, r) = (2, 2)$, the condition (3) is equivalent to

$$\tau_0 + \tau_1 - 1 \neq 0.$$

In the cubic case $(m, r) = (3, 2)$, we obtain from (3):

$$2(\tau_0^2 + \tau_0\tau_1 + \tau_1^2) - 3(\tau_0 + \tau_1) + 1 \neq 0.$$

In particular, if $\tau = \tau_0 = \tau_1$, then the corresponding Hermite-spline interpolation problem (2) has a unique solution if and only if

$$\tau \notin \left\{ \frac{1}{2}(1 + \sqrt{\frac{1}{3}})\ ,\ \frac{1}{2}(1 - \sqrt{\frac{1}{3}}) \right\}. \qquad ♠$$

3. Generalized Euler-Frobenius polynomials

Now we will introduce the generalized Euler-Frobenius polynomials with the help of B-splines. Consider equidistant knots with multiplicity r:

$$x_{j+rk} := k \qquad (k \in \boldsymbol{Z},\ j = 0, ..., r-1).$$

Let $B^r_{k,m} \in C^{m-r}(\boldsymbol{R})$ denote the normalized B-spline of degree m and defect r with the knots $x_k, x_{k+1}, ..., x_{k+m+1}$. Then the N-periodic B-spline $P^r_{k,m}$ is given by

$$P^r_{k,m}(x) := \sum_{n=-\infty}^{\infty} B^r_{k,m}(x+nN), \qquad (x \in \boldsymbol{R}).$$

Observe that the N-periodic B-splines

$$P^r_{j+rk,m}(x) = P^r_{j,m}(x-k) \qquad (j=0,...,r-1;\ k=0,...,N-1)$$

form a basis of the spline space $S^N_{m,r}$.
The *m-th Euler-Frobenius polynomial H^1_m of multiplicity* 1 *and with shift parameter $t \in$* $[0,1]$ is defined by the equation (cf. [12])

$$H^1_m(t,z) := \sum_{\nu=0}^{m} B^1_{0,m}(\nu+t)z^\nu,$$

where $z \in \boldsymbol{C}$, $m \in \boldsymbol{N}$. Note that the classical Euler-Frobenius polynomial reads $m!H^1_m(1,z)$ (cf. [11]).
If $s \in S^N_{m,1}$ of the form

$$s(x) = \sum_{k=0}^{N-1} c_k P^1_{0,m}(x-k)$$

satisfies the Lagrange-interpolation condition (1), then it holds

$$H^1_m(\tau, \boldsymbol{V})\boldsymbol{c} = \boldsymbol{y}^{(0)}$$

with $\boldsymbol{c} := (c_0, ..., c_{N-1})^T \in \boldsymbol{R}^N$, where

$$\boldsymbol{V} := \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

denotes the fundamental circulant matrix (cf. [8]). Therefore the interpolation problem (1) is uniquely solvable if and only if the circulant matrix $H^1_m(\tau, \boldsymbol{V})$ is nonsingular. This is satisfied, if all eigenvalues of $H^1_m(\tau, \boldsymbol{V})$ are different from zero, i.e., if $H^1_m(\tau, w^j) \neq 0$ for $j = 0, 1, ..., N-1$, where $w := e^{2\pi i/N}$. We call the polynomial H^1_m the *symbol of periodic Lagrange-spline interpolation with shift parameter τ.*

The following hold:

$$zH^1_m(1,z) = H^1_m(0,z) \qquad (m \geq 1), \tag{4}$$

$$H^1_m(t,1) = 1 \qquad (m \geq 1), \tag{5}$$

$$H^1_m(t,0) = \frac{1}{m!}t^m \qquad (m \geq 1), \tag{6}$$

$$\frac{\partial}{\partial t}H^1_m(t,z) = (1-z)H^1_{m-1}(t,z) \qquad (m \geq 2). \tag{7}$$

A list of properties of $m!H^1_m$ can be found in [12].

Now it is our goal to define a symbol for the generalized interpolation problem (2) like H_m^1 in the Lagrange case.
With the help of the *generalized Euler-Frobenius polynomials of multiplicity* 2 *with shift parameter* $t \in [0,1]$, given by

$$
\begin{aligned}
H^2_{0,m}(t,z) &:= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} B^2_{0,m}(j+t)z^j, \\
H^2_{1,m}(t,z) &:= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} B^2_{1,m}(j+t)z^j \ ,
\end{aligned}
$$

we define for $m \geq 2$:

$$
H^2_m(t_0,t_1,z) := \begin{vmatrix} H^2_{0,m}(t_0,z) & H^2_{1,m}(t_0,z) \\ H^2_{0,m}[t_0,t_1](z) & H^2_{1,m}[t_0,t_1](z) \end{vmatrix}
$$

with $t_0, t_1 \in [0,1], \quad z \in \boldsymbol{C}$.
Again $H^2_{0,m}[t_0,t_1](z)$ and $H^2_{1,m}[t_0,t_1](z)$ denote divided differences with respect to t.

Theorem 3 . *Let* $N, m \in \boldsymbol{N}$ $(N, m \geq 2)$ *and* $0 < \tau_0 \leq \tau_1 \leq 1$ *be given. Then the interpolation problem* (2) *is uniquely solvable if and only if*

$$
H^2_m(\tau_0,\tau_1,w^j) \neq 0 \qquad (j = 0, ..., N-1),
$$

where $w := e^{2\pi i/N}$.

Proof: We follow the ideas in [8] for the computation of Hermite-spline interpolants. Let $\boldsymbol{y}^{(0)}, \boldsymbol{y}^{(1)} \in \boldsymbol{R}$ be the given data vectors. If $s \in S^N_{m,2}$ of the form

$$
s(x) = \sum_{k=0}^{N-1} (c_k P^2_{0,m}(x-k) + d_k P^2_{1,m}(x-k))
$$

satisfies the interpolation conditions (2), then we get

$$
\begin{pmatrix} H^2_{0,m}(\tau_0,\boldsymbol{V}) & H^2_{1,m}(\tau_0,\boldsymbol{V}) \\ H^2_{0,m}[\tau_0,\tau_1](\boldsymbol{V}) & H^2_{1,m}[\tau_0,\tau_1](\boldsymbol{V}) \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} \boldsymbol{y}^{(0)} \\ \boldsymbol{y}^{(1)} \end{pmatrix}
$$

with $\boldsymbol{c} := (c_0, ..., c_{N-1})^T$, $\boldsymbol{d} := (d_0, ..., d_{N-1})^T \in \boldsymbol{R}^N$. By definition of H^2_m it follows immediately that

$$
\begin{aligned}
H^2_m(\tau_0,\tau_1,\boldsymbol{V})\boldsymbol{c} &= H^2_{1,m}[\tau_0,\tau_1](\boldsymbol{V})\boldsymbol{y}^{(0)} - H^2_{1,m}(\tau_0,\boldsymbol{V})\boldsymbol{y}^{(1)}, \\
H^2_m(\tau_0,\tau_1,\boldsymbol{V})\boldsymbol{d} &= H^2_{0,m}(\tau_0,\boldsymbol{V})\boldsymbol{y}^{(1)} - H^2_{0,m}[\tau_0,\tau_1](\boldsymbol{V})\boldsymbol{y}^{(0)}.
\end{aligned}
$$

Hence, our periodic interpolation problem (2) is uniquely solvable for any data vectors $\boldsymbol{y}^{(0)}$ and $\boldsymbol{y}^{(1)}$ if and only if the circulant matrix $H^2_m(\tau_0,\tau_1,\boldsymbol{V})$ is nonsingular, i.e., if $H^2_m(\tau_0,\tau_1,w^j) \neq 0$ for $j = 0, ..., N-1$. ■
We call the polynomial H^2_m the *symbol of the spline interpolation problem* (2).

Examples: For $m = 2$ we have:

$$H^2_{0,2}(t,z) = 2t(1-t), \qquad H^2_{1,2}(t,z) = t^2 + (1-t)^2 z,$$

$$H^2_2(t_0,t_1,z) = 2(t_0t_1 - (1-t_0)(1-t_1)z).$$

For $m = 3$:

$$\begin{aligned} H^2_{0,3}(t,z) &= t^2(3-\frac{5}{2}t) + \frac{1}{2}(1-t)^3 z, \\ H^2_{1,3}(t,z) &= \frac{t^3}{2} + (1-t)^2(\frac{5}{2}t + \frac{1}{2})z, \end{aligned}$$

$$H^2_3(t_0,t_1,z) = \frac{3}{2}\{t_0^2t_1^2 - [t_0(1-t_0) + t_1(1-t_1) + 2t_0t_1(1-t_0)(1-t_1)]z \\ +(1-t_0)^2(1-t_1)^2z^2\}. \quad \spadesuit$$

4. Euler-Frobenius polynomials of multiplicity 1 and 2

First we establish some properties of the Euler-Frobenius polynomials $H^2_{0,m}$ and $H^2_{1,m}$.

Lemma 1 . *Let $m \in \boldsymbol{N}$ $(m \geq 2)$, $t \in [0,1]$ be fixed. Then we have*

$$zH^2_{j,m}(1,z) = H^2_{j,m}(0,z) \qquad (j = 0,1), \tag{8}$$

$$H^2_{0,m}(t,1) + H^2_{1,m}(t,1) \equiv 1, \tag{9}$$

$$\begin{aligned} \frac{\partial}{\partial t}H^2_{0,m}(t,z) &= m\left\{\frac{1}{\lfloor\frac{m}{2}\rfloor}H^2_{0,m-1}(t,z) - \frac{1}{\lfloor\frac{m+1}{2}\rfloor}H^2_{1,m-1}(t,z)\right\}, \\ \frac{\partial}{\partial t}H^2_{1,m}(t,z) &= m\left\{\frac{1}{\lfloor\frac{m+1}{2}\rfloor}H^2_{1,m-1}(t,z) - \frac{z}{\lfloor\frac{m}{2}\rfloor}H^2_{0,m-1}(t,z)\right\}. \end{aligned} \tag{10}$$

Proof: 1°. Since $B^2_{0,m}(0) = B^2_{1,m}(0) = 0$ and $B^2_{0,m}(\lfloor\frac{m}{2}\rfloor + 1) = B^2_{1,m}(\lfloor\frac{m}{2}\rfloor + 1) = 0$, we see for $m \geq 2$ and $j = 1,2$ that

$$\begin{aligned} H^2_{j,m}(0,z) &= \sum_{k=1}^{\lfloor\frac{m}{2}\rfloor} B^2_{j,m}(k)z^k = \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor-1} B^2_{j,m}(k+1)z^{k+1} \\ &= z\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} B^2_{j,m}(k+1)z^k = zH^2_{j,m}(1,z). \end{aligned}$$

2°. Let $t \in [0,1]$. By the well-known partition of unity property of the B-splines we get

$$H^2_{0,m}(t,1) + H^2_{1,m}(t,1) = \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(B^2_{0,m}(k+t) + B^2_{1,m}(k+t)) \equiv 1.$$

3°. The relation (10) follows immediately from the recursion formulas for B-splines with double knots

$$\frac{\mathrm{d}}{\mathrm{d}x} B^2_{0,m}(x) = m \left\{ \frac{1}{\lfloor \frac{m}{2} \rfloor} B^2_{0,m-1}(x) - \frac{1}{\lfloor \frac{m+1}{2} \rfloor} B^2_{1,m-1}(x) \right\},$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B^2_{1,m}(x) = m \left\{ \frac{1}{\lfloor \frac{m+1}{2} \rfloor} B^2_{1,m-1}(x) - \frac{1}{\lfloor \frac{m}{2} \rfloor} B^2_{0,m-1}(x-1) \right\} \quad (x \in \boldsymbol{R}). \quad \blacksquare$$

In order to analyze H^2_m we introduce the determinant Δ^2_m by

$$\Delta^2_m(t_0, t_1, z) := \begin{vmatrix} H^1_m(t_0, z) & H^1_{m-1}(t_0, z) \\ H^1_m[t_0, t_1](z) & H^1_{m-1}[t_0, t_1](z) \end{vmatrix}$$

with $t_0, t_1 \in [0, 1]$, $z \in \boldsymbol{C}$, where $H^1_n[t_0, t_1](z)$ $(n = m, m-1)$ denotes the divided difference of $H^1_n(t, z)$ with respect to the variable t.

Theorem 4 . *Let $t_0, t_1 \in (0, 1]$ and $z \in \boldsymbol{C}$ be given. Then we have*

$$\Delta^2_m(t_0, t_1, z) = c_m (1-z)^m H^2_m(t_0, t_1, z)$$

with

$$c_{2n} := -\frac{(n!)^4}{(2n)!n}, \qquad c_{2n+1} := -\frac{(n!)^4 (n+1)}{(2n+1)!}.$$

Proof: The proof will follow from several statements.
1°. *The Euler-Frobenius polynomial H^1_m $(m \geq 2)$ is uniquely determined by H^1_{m-1}, if the relations* (4),(5) *and* (7) *are satisfied, where $H^1_1(t, z) := t(1-z) + z$.*
Assume that the functions $P_1(t, z)$ and $P_2(t, z)$ satisfy the relations

$$\frac{\partial}{\partial t} P_1(t, z) = \frac{\partial}{\partial t} P_2(t, z) = (1-z) H^1_{m-1}(t, z), \tag{11}$$

$$z P_1(1, z) = P_1(0, z), \qquad z P_2(1, z) = P_2(0, z) \tag{12}$$

and

$$P_1(t, 1) = P_2(t, 1) = 1. \tag{13}$$

We consider $Q(t, z) := P_1(t, z) - P_2(t, z)$. Then (11) implies that $\frac{\partial}{\partial t} Q(t, z) = 0$, i.e., $Q(t, z) = q(z)$. Using (12), we find that $(1-z)q(z) = 0$. Hence $q(z) \equiv 0$ for $z \in \boldsymbol{C}$, $z \neq 1$. Finally from (13) we have $q(1) = 0$ and therefore $Q(t, z) \equiv 0$.

2°. *Let $0 < t \leq 1$ and $z \in \boldsymbol{C}$ be fixed. Furthermore let $\boldsymbol{A}_m$ $(m \geq 2)$ denote the square matrix which is recursively determined by*

$$\boldsymbol{A}_m(z) = \frac{1}{m} \boldsymbol{A}_{m-1}(z) \begin{pmatrix} \lfloor \frac{m}{2} \rfloor & \lfloor \frac{m}{2} \rfloor \\ \lfloor \frac{m+1}{2} \rfloor z & \lfloor \frac{m+1}{2} \rfloor \end{pmatrix} \quad (m \geq 3),$$

where

$$\boldsymbol{A}_2(z) := \frac{1}{2} \begin{pmatrix} 2z & 1+z \\ 1+z & 2 \end{pmatrix}.$$

Then for $m \geq 2$,

$$\begin{pmatrix} H^1_m(t,z) \\ H^1_{m-1}(t,z) \end{pmatrix} = \boldsymbol{A}_m(z) \begin{pmatrix} H^2_{0,m}(t,z) \\ H^2_{1,m}(t,z) \end{pmatrix}.$$

For $m = 2$ we have

$$\boldsymbol{A}_2(z) \begin{pmatrix} H^2_{0,2}(t,z) \\ H^2_{1,2}(t,z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2z & 1+z \\ 1+z & 2 \end{pmatrix} \begin{pmatrix} 2t(1-t) \\ t^2 + (1-t)^2 z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} t^2(1-z)^2 + 2t(1-z)z + z^2 + z \\ 2[t(1-z) + z] \end{pmatrix} = \begin{pmatrix} H^1_2(t,z) \\ H^1_1(t,z) \end{pmatrix}.$$

Using 1° we shall prove inductively, that for $m \geq 3$, the components of $\boldsymbol{A}_m(H^2_{0,m}, H^2_{1,m})^T$ satisfy the relations (4), (5) and (7). From (10) we get

$$\begin{aligned} & \boldsymbol{A}_m(z) \begin{pmatrix} \frac{\partial}{\partial t} H^2_{0,m}(t,z) \\ \frac{\partial}{\partial t} H^2_{1,m}(t,z) \end{pmatrix} \\ = \;& \frac{1}{m} \boldsymbol{A}_{m-1}(z) \begin{pmatrix} \lfloor \frac{m}{2} \rfloor & \lfloor \frac{m}{2} \rfloor \\ \lfloor \frac{m+1}{2} \rfloor z & \lfloor \frac{m+1}{2} \rfloor \end{pmatrix} m \begin{pmatrix} \frac{1}{\lfloor \frac{m}{2} \rfloor} H^2_{0,m-1}(t,z) - \frac{1}{\lfloor \frac{m+1}{2} \rfloor} H^2_{1,m-1}(t,z) \\ \frac{1}{\lfloor \frac{m+1}{2} \rfloor} H^2_{1,m-1}(t,z) - \frac{z}{\lfloor \frac{m}{2} \rfloor} H^2_{0,m-1}(t,z) \end{pmatrix} \\ = \;& (1-z) \boldsymbol{A}_{m-1}(z) \begin{pmatrix} H^2_{0,m-1}(t,z) \\ H^2_{1,m-1}(t,z) \end{pmatrix} = (1-z) \begin{pmatrix} H^1_{m-1}(t,z) \\ H^1_{m-2}(t,z) \end{pmatrix}. \end{aligned}$$

Hence (7) holds.
The relation (4) follows immediately from (8).
By (9) the relation (5) is established if $\boldsymbol{A}_m(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ holds. But this is a simple consequence of the definition of $\boldsymbol{A}_m$ and the fact that $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m+1}{2} \rfloor = m$. Now 2° follows from 1°.

3°.*The assertion of Theorem 4 holds.*
From 2° it follows that

$$\begin{pmatrix} H^1_m(t_0,z) & H^1_{m-1}(t_0,z) \\ H^1_m[t_0,t_1](z) & H^1_{m-1}[t_0,t_1](z) \end{pmatrix} = \begin{pmatrix} H^2_{0,m}(t_0,z) & H^2_{1,m}(t_0,z) \\ H^2_{0,m}[t_0,t_1](z) & H^2_{1,m}[t_0,t_1](z) \end{pmatrix} \boldsymbol{A}_m(z)^T.$$

Applying the multiplication rule of determinants we obtain

$$\Delta^2_m(t_0,t_1,z) = \det\,(\boldsymbol{A}_m(z))\; H^2_m(t_0,t_1,z).$$

We will prove inductively that for $m \geq 2$

$$\det\,(\boldsymbol{A}_m(z)) = c_m(1-z)^m \tag{14}$$

with

$$c_{2n} = -\frac{(n)!^4}{(2n)!n}, \qquad c_{2n+1} = -\frac{(n)!^4(n+1)}{(2n+1)!}.$$

For $m = 2$ we have

$$\frac{1}{2}\det\begin{pmatrix} 2z & 1+z \\ 1+z & 2 \end{pmatrix} = -\frac{1}{2}(1-z)^2.$$

Assume that the assertion holds for $m-1$. Then

$$\det(\boldsymbol{A}_m(z)) = \frac{1}{m}\det(\boldsymbol{A}_{m-1}(z))\det\begin{pmatrix} \lfloor\frac{m}{2}\rfloor & \lfloor\frac{m}{2}\rfloor \\ \lfloor\frac{m+1}{2}\rfloor z & \lfloor\frac{m+1}{2}\rfloor \end{pmatrix}$$

$$= \frac{1}{m}\lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor(1-z)\det(\boldsymbol{A}_{m-1}(z)).$$

Using the induction hypothesis we arrive at (14). ■

5. A property of the symbol H^2_m

In order to establish Theorem 2, we have to investigate the eigenvalues of $H^2_m(t_0, t_1, \boldsymbol{V})$. In Section 6 we will find that $|H^2_m(t_0, t_1, w^j)| > 0$ for $j = 1, ..., N-1$, $t_0, t_1 \in (0, 1]$. The remaining case $j = 0$ is treated by

Theorem 5 . *Let $t_0, t_1 \in (0, 1]$ be fixed. Then for $m \geq 2$ we have*

$$H^2_m(t_0, t_1, 1) = d_m B_m[t_0, t_1],$$

where $B_m[t_0, t_1]$ denotes the divided difference of the Bernoulli polynomial B_m. The constant d_m is independent of t_0 and t_1.

Proof: $1°$.*For fixed $t_0, t_1 \in (0, 1]$ and $m \geq 2$, the following hold:*

$$H^2_m(t_0, t_1, 1) = H^2_{1,m}[t_0, t_1](1).$$

First we show the relation

$$H^2_{0,m}[t_0, t_1](1) = -H^2_{1,m}[t_0, t_1](1). \tag{15}$$

For $t_0 = t_1$ we get (15) differentiating of (9). For $t_0 \neq t_1$ we have

$$\begin{aligned} H^2_{0,m}[t_0, t_1](1) + H^2_{1,m}[t_0, t_1](1) &= \frac{1}{t_0 - t_1}(H^2_{0,m}(t_0, 1) + H^2_{1,m}(t_0, 1) - H^2_{0,m}(t_1, 1) \\ &\quad - H^2_{1,m}(t_1, 1)) = 0. \end{aligned}$$

Thus, from (9) it follows that

$$\begin{aligned} H^2_m(t_0, t_1, 1) &= H^2_{1,m}[t_0, t_1](1)\,(H^2_{0,m}(t_0, 1) + H^2_{1,m}(t_0, 1)) \\ &= H^2_{1,m}[t_0, t_1](1). \end{aligned}$$

$2°$.*The assertion of Theorem 5 holds for $t_0 = t_1 = t \in (0, 1]$, i.e.,*

$$H^2_m(t, t, 1) = d_m B'_m(t) = d_m B_{m-1}(t).$$

From 1° and (10) we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H^2_m(t,t,1) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}H^2_{1,m}(t,1)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{m\left(\frac{1}{\lfloor\frac{m+1}{2}\rfloor}H^2_{1,m-1}(t,1)-\frac{1}{\lfloor\frac{m}{2}\rfloor}H^2_{0,m-1}(t,1)\right)\right\}\end{aligned}$$

Now, by (9) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H^2_m(t,t,1) &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{m\left(\frac{1}{\lfloor\frac{m+1}{2}\rfloor}+\frac{1}{\lfloor\frac{m}{2}\rfloor}\right)H^2_{1,m-1}(t,1)\right\}\\ &= m\left(\frac{1}{\lfloor\frac{m+1}{2}\rfloor}+\frac{1}{\lfloor\frac{m}{2}\rfloor}\right)H^2_{m-1}(t,t,1). \qquad (16)\end{aligned}$$

Furthermore (8) implies that

$$\int_0^1 H^2_m(t,t,1)\ \mathrm{d}t = H^2_{1,m}(1,1)-H^2_{1,m}(0,1)=0. \qquad (17)$$

Note that $H^2_{1,m}[t,t](1)$ is a polynomial of degree $m-1$ with respect to t. Therefore, by the definition of B_m the assertion 2° follows from (16),(17) and the fact

$$H^2_2(t,t,1)=4t-2=4B_1(t).$$

3°. *The assertion of Theorem 5 holds for $t_0\neq t_1$.*
From 1° and 2° it follows that

$$H^2_{1,m}[t,t](1)=\frac{\mathrm{d}}{\mathrm{d}t}H^2_{1,m}(t,1)=d_mB_{m-1}(t).$$

By integration we find

$$H^2_{1,m}(t,1)=d_mB_m(t)+r_m,$$

with constants d_m and r_m. Thus,

$$H^2_m(t_0,t_1,1)=d_mB_m[t_0,t_1]. \qquad \blacksquare$$

Remark: The constant d_m can easily be computed. We find

$$d_{2n}=\frac{4^n((2n-1)!!)^2}{((n-1)!)^2n}, \qquad d_{2n+1}=\frac{4^n((2n+1)!!)^2}{(n!)^2(n+1)}. \qquad \spadesuit$$

6. Exponential Euler splines

Now we will show that for any fixed $t_0,t_1\in(0,1]$ and $B_m[t_0,t_1]\neq 0$ the matrix $H^2_m(t_0,t_1,\boldsymbol{V})$ is nonsingular, i.e., that

$$H^2_m(t_0,t_1,w^j)\neq 0 \qquad (j=0,1,...,N-1)$$

with $w := e^{2\pi i/N}$.
Unfortunately, similar ideas as in [4] and [7] can not be applied, because we are not able to describe the zeros of H_m^2 for any $t_0, t_1 \in [0,1]$ and for any $m \geq 2$.
Indeed, we are only interested in the case when $|z| = 1$. Our main tool will be the *symbol of cardinal interpolation with centered B-splines of degree* $m \in \boldsymbol{N}$, which is defined for a fixed shift parameter $x \in \boldsymbol{R}$ by

$$\varphi_m^1(x,u) := \sum_{j \in \boldsymbol{Z}} B_{0,m}^1(x + \frac{m+1}{2} - j)\, e^{iju}, \qquad u \in [-\pi, \pi].$$

(cf. [1]). The following identities and properties of φ_m^1 will be used in our further considerations.

Theorem 6 (cf. [1]) . *For* $m \in \boldsymbol{N}$, $x \in \boldsymbol{R}$ *and* $-\pi \leq u \leq \pi$, *we have*
(i).

$$\begin{aligned} \varphi_m^1(x \pm 1, u) &= e^{\pm iu} \varphi_m^1(x,u), \\ \varphi_m^1(x, -u) = \varphi_m^1(-x, u) &= \overline{\varphi_m^1(x,u)}, \\ e^{-iu}\varphi_m^1(x + \frac{1}{2}, u) &= \overline{\varphi_m^1(\frac{1}{2} - x, u)}, \end{aligned}$$

(ii).

$$\frac{\partial}{\partial x}\varphi_m^1(x,u) = (1 - e^{-iu})\varphi_{m-1}^1(x + \frac{1}{2}, u) \qquad (m \geq 2).$$

(iii). For $0 \leq x \leq 1/2$ *and* $0 < u < \pi$,

$$\varphi_m^1(x,u) = \alpha_{m-1}(x,u)e^{iu/2} + \beta_{m-1}(x,u),$$

where

$$\begin{aligned} \alpha_{m-1}(x,u) &= 2\int_{1/2-x}^{1/2} \mathrm{Re}[e^{-iu/2}\varphi_{m-1}^1(\tau,u)]\ d\tau, \\ \beta_{m-1}(x,u) &= 2\int_0^{1/2-x} \mathrm{Re}[\varphi_{m-1}^1(\tau,u)]\ d\tau. \end{aligned}$$

Now let $m \geq 1$ *and* $0 < u_0 < \pi$ *be fixed. Then we have*
(iv). The function $\arg \varphi_m^1(x,u_0)$ *is strictly increasing for* $x \in [0,1]$. *In particular,*
$\arg \varphi_m^1(0,u_0) = 0, \qquad \arg \varphi_m^1(1/2,u_0) = u_0/2, \qquad \arg \varphi_m^1(1,u_0) = u_0.$

(v). The function $|\varphi_m^1(x,u_0)|$ *is strictly decreasing for* $x \in [0,1/2]$ *and strictly increasing for* $x \in [1/2,1]$. *Particularly,* $|\varphi_m^1(x,u_0)| > 0$ *for* $x \in [0,1]$.

(vi). We have

$$\begin{aligned} 0 < \arg \varphi_m^1(x,u_0) < xu_0 &\qquad \text{for } 0 < x < 1/2, \\ xu_0 < \arg \varphi_m^1(x,u_0) < u_0 &\qquad \text{for } 1/2 < x < 1. \end{aligned}$$

(vii). Further, $\varphi_m^1(x,\pi)$ *is a real and for* $x \in [0,1]$ *strictly decreasing function with* $\varphi_m^1(1/2,\pi) = 0$.

We consider the determinant

$$\varphi_m^2(x_0,x_1,u) := \begin{vmatrix} \varphi_m^1(x_0,u) & \varphi_{m-1}^1(x_0+1/2,u) \\ \varphi_m^1[x_0,x_1](u) & \varphi_{m-1}^1[x_0+1/2,x_1+1/2](u) \end{vmatrix} \qquad (m \geq 2)$$

with $x_0, x_1 \in \boldsymbol{R}$, $u \in [-\pi,\pi]$, where $\varphi_m^1[x_0,x_1](u)$ and $\varphi_{m-1}^1[x_0,x_1](u)$ denote the divided differences with respect to the variable x. The connection between φ_m^2 and Δ_m^2 is described in the following

Lemma 2 . *For $t_0, t_1 \in (0,1]$, $-\pi < u \leq \pi$ and $m \in \boldsymbol{N}$ $(m \geq 2)$, we have*

$$\Delta_m^2(t_0,t_1,e^{iu}) = \begin{cases} e^{iu(m-2)}\,\varphi_m^2(1-t_0,1-t_1,u) & \textit{if } m \textit{ is odd} , \\ e^{iu(m-1)}\,\varphi_m^2(1/2-t_0,1/2-t_1,u) & \textit{if } m \textit{ is even} . \end{cases}$$

Proof: Using (i) and the symmetry relation $B_{0,m}^1(x) = B_{0,m}^1(m+1-x)$ it follows from the definition of φ_m^1 that for $m \geq 1$

$$H_m^1(x,e^{iu}) = \begin{cases} e^{iu(m-1)/2}\,\varphi_m^1(1-x,u) & \text{if } m \text{ is odd} , \\ e^{ium/2}\,\varphi_m^1(1/2-x,u) & \text{if } m \text{ is even} . \end{cases}$$

For $m \geq 2$ odd we see from (i) that:

$$\begin{aligned} \Delta_m^2(t_0,t_1,e^{iu}) &= e^{iu(m-1)} \begin{vmatrix} \varphi_m^1(1-t_0,u) & \varphi_{m-1}^1(1/2-t_0,u) \\ \varphi_m^1[1-t_0,1-t_1](u) & \varphi_{m-1}^1[1/2-t_0,1/2-t_1](u) \end{vmatrix} \\ &= e^{iu(m-2)}\,\varphi_m^2(1-t_0,1-t_1,u). \end{aligned}$$

The relation follows analogously for $m \geq 2$ even. ■

By Theorems 4 and 5 and Lemma 2 it suffices to show that

$$\varphi_m^2(x_0,x_1,\frac{2\pi j}{N}) \neq 0 \qquad (j = 1,...,N-1,\ x_0,x_1 \in \boldsymbol{R},\ |x_0-x_1| < 1).$$

The following identities are immediate consequences of the definition of φ_m^2 and (i).

Lemma 3 . *For $m \in \boldsymbol{N}$ $(m \geq 2)$, $x_0, x_1 \in \boldsymbol{R}$, $-\pi < u \leq \pi$, we have*

$$\varphi_m^2(x_0,x_1,u) = \varphi_m^2(x_1,x_0,u), \tag{18}$$

$$\varphi_m^2(x_0,x_1,-u) = -e^{-iu}\varphi_m^2(-x_0,-x_1,u) = \overline{\varphi_m^2(x_0,x_1,u)}, \tag{19}$$

$$\varphi_m^2(1/2+x_0,1/2+x_1,u) = -e^{3iu}\,\overline{\varphi_m^2(1/2-x_0,1/2-x_1,u)}. \tag{20}$$

For $x_1 - x_0 \notin \boldsymbol{Z}$,

$$\begin{aligned} (x_1-(x_0\pm 1))\varphi_m^2(x_0\pm 1,x_1,u) &= e^{\pm iu}(x_1-x_0)\varphi_m^2(x_0,x_1,u), \\ (x_1\pm 1-x_0)\varphi_m^2(x_0,x_1\pm 1,u) &= e^{\pm iu}(x_1-x_0)\varphi_m^2(x_0,x_1,u). \end{aligned} \tag{21}$$

For $x_1 - x_0 = k \in \boldsymbol{Z}\setminus\{0\}$,

$$\varphi_m^2(x_0+k,x_0,u) = 0.$$

For $x_0 = x_1$,

$$\begin{aligned} \varphi_m^2(x_0+1,x_0+1,u) &= e^{2iu}\,\varphi_m^2(x_0,x_0,u), \\ \varphi_m^2(x_0-1,x_0-1,u) &= e^{-2iu}\,\varphi_m^2(x_0,x_0,u). \end{aligned} \tag{22}$$

By Lemma 3 we may restrict our investigations to the intervals $0 < u \leq \pi$ and $x_0, x_1 \in (0, 1]$.

Theorem 7 . *Let $x_0, x_1 \in (0,1]$ be fixed. Then for $m \geq 3$ and $0 < u \leq \pi$, we have*

$$|\varphi_m^2(x_0, x_1, u)| > 0.$$

Proof: Since (18) we only have to consider the case $0 < x_0 \leq x_1 \leq 1$.
1°.*First let $x = x_0 = x_1$.*
By (20) it can be supposed that $0 \leq x \leq 1/2$. From the definition we have

$$\varphi_m^2(x, x, u) = \varphi_m^1(x, u) \frac{\partial}{\partial x} \varphi_{m-1}^1(x + 1/2, u) - \varphi_{m-1}^1(x + 1/2, u) \frac{\partial}{\partial x} \varphi_m^1(x, u).$$

Using the properties (i) and (ii) it follows for $m \geq 3$ that

$$\varphi_m^2(x, x, u) = (1 - e^{-iu})(e^{iu}\, \varphi_m^1(x, u)\varphi_{m-2}^1(x, u) - (\varphi_{m-1}^1(x + 1/2, u))^2).$$

Thus it remains to show that for $m \geq 3$ and $0 < u \leq \pi$,

$$e^{iu}\, \varphi_m^1(x, u)\varphi_{m-2}^1(x, u) \neq (\varphi_{m-1}^1(x + 1/2, u))^2. \tag{23}$$

The case $u = \pi$ follows readily from property (vii). Now let $u_0 \in (0, \pi)$ be fixed.
$1^\circ.1$. *The assertion holds for $x = 0$ and $x = 1/2$.*
On the one hand for $0 < t < 1/2$ and $m \geq 1$ we have from (vi)

$$0 < \arg \varphi_m^1(t, u_0) < u_0 t < \frac{\pi}{2}.$$

Therefore, we obtain $\operatorname{Im} \varphi_m^1(t, u_0) > 0$ and $\operatorname{Re} \varphi_m^1(t, u_0) > 0$. Hence we may write

$$0 < \frac{\operatorname{Im}\, [\varphi_m^1(t, u_0)]}{\operatorname{Re}\, [\varphi_m^1(t, u_0)]} < \tan u_0 t.$$

This yields

$$|\varphi_m^1(t, u_0)| < \operatorname{Re}\, [\varphi_m^1(t, u_0)](1 + \tan^2 u_0 t)^{1/2} = \frac{\operatorname{Re}\, [\varphi_m^1(t, u_0)]}{\cos u_0 t},$$

i.e.,

$$\operatorname{Re}\, [\varphi_m^1(t, u_0)] > |\varphi_m^1(t, u_0)| \cos u_0 t. \tag{24}$$

On the other hand it follows from (vi) that

$$-\frac{\pi}{2} < -\frac{u_0}{2} < \arg[e^{-iu_0/2}\varphi_m^1(t, u_0)] < u_0 t - \frac{u_0}{2} < 0.$$

Thus, we get $\operatorname{Im}\, [e^{-iu_0/2}\varphi_m^1(t, u_0)] < 0$ and $\operatorname{Re}\, [e^{-iu_0/2}\varphi_m^1(t, u_0)] > 0$. Hence,

$$\tan \frac{u_0}{2} > -\frac{\operatorname{Im}\, [e^{-iu_0/2}\varphi_m^1(t, u_0)]}{\operatorname{Re}\, [e^{-iu_0/2}\varphi_m^1(t, u_0)]} > \tan(1/2 - t)u_0,$$

which implies

$$|\varphi_m^1(t, u_0)| > \operatorname{Re}\, [e^{-iu_0/2}\varphi_m^1(t, u_0)](1 + \tan^2(1/2 - t)u_0)^{1/2} = \frac{\operatorname{Re}\, [e^{-iu_0/2}\varphi_m^1(t, u_0)]}{\cos(1/2 - t)u_0}$$

and therefore

$$\text{Re}\,[e^{-iu_0/2}\varphi_m^1(t,u_0)] < |\varphi_m^1(t,u_0)|\cos(1/2-t)u_0. \tag{25}$$

Now, using the recursion relation (iii) and monotonicity property (v) we find from (24) and (25):

$$\begin{aligned}|\varphi_{m+1}^1(0,u_0)| &= |\beta_m(0,u_0)| > 2\int_0^{1/2} |\varphi_m^1(t,u_0)|\cos u_0 t\ \mathrm{d}t \\ &> |\varphi_m^1(1/2,u_0)|\ \text{ sinc } u_0/2\end{aligned}$$

and

$$\begin{aligned}|\varphi_{m+1}^1(1/2,u_0)| &= |\alpha_m(1/2,u_0)| < 2\int_0^{1/2} |\varphi_m^1(t,u_0)|\cos(1/2-t)u_0\ \mathrm{d}t \\ &< |\varphi_m^1(0,u_0)|\text{ sinc } u_0/2.\end{aligned}$$

The required inequality (23) follows for $x=0$ and $x=1/2$, since

$$\frac{|\varphi_m^1(0,u_0)\varphi_{m-2}^1(0,u_0)|}{|\varphi_{m-1}^1(1/2,u_0)|^2} > \frac{|\varphi_{m-1}^1(1/2,u_0)\varphi_{m-2}^1(0,u_0)|\text{ sinc } u_0/2}{|\varphi_{m-1}^1(1/2,u_0)\varphi_{m-2}^1(0,u_0)|\text{ sinc } u_0/2} = 1,$$

$$\frac{|\varphi_m^1(1/2,u_0)\varphi_{m-2}^1(1/2,u_0)|}{|\varphi_{m-1}^1(0,u_0)|^2} < \frac{|\varphi_{m-1}^1(0,u_0)\varphi_{m-2}^1(1/2,u_0)|\text{ sinc } u_0/2}{|\varphi_{m-1}^1(0,u_0)\varphi_{m-2}^1(1/2,u_0)|\text{ sinc } u_0/2} = 1.$$

1°.2. *The assertion holds for* $0<x<1/2$.
Let $m \ge 3$ and $u_0 \in (0,\pi)$ be fixed. For $0<x<1/2$ we have from (vi),

$$u_0 < \arg[e^{iu_0}\varphi_m^1(x,u_0)\varphi_{m-2}^1(x,u_0)] < 2xu_0 + u_0.$$

Furthermore,

$$(x+1/2)u_0 < \arg[\varphi_{m-1}^1(x+1/2,u_0)] < u_0,$$

i.e.,

$$u_0 + 2xu_0 < \arg[(\varphi_{m-1}^1(x+1/2,u_0))^2] < 2u_0.$$

Hence,

$$\arg[e^{iu_0}\varphi_m^1(x,u_0)\varphi_{m-2}^1(x,u_0)] < \arg[(\varphi_{m-1}^1(x+1/2,u_0))^2].$$

Together with (v) this completes the proof of 1°.2.

2°. *Now let* $x_0 < x_1$.
Then φ_m^2 can be simplified to

$$\varphi_m^2(x_0,x_1,u) = \frac{1}{(x_1-x_0)}\begin{vmatrix} \varphi_m^1(x_0,u) & \varphi_{m-1}^1(x_0+1/2,u) \\ \varphi_m^1(x_1,u) & \varphi_{m-1}^1(x_1+1/2,u)\end{vmatrix}$$

with $m \ge 2$ and $u \in (0,\pi]$.
Thus it remains to show that for any $m \ge 3$, $0 < u \le \pi$,

$$\varphi_m^1(x_0,u)\varphi_{m-1}^1(x_1+1/2,u) \ne \varphi_m^1(x_1,u)\varphi_{m-1}^1(x_0+1/2,u). \tag{26}$$

First let $0 < x_0 < x_1 \le 1/2$. Then it follows from (v) and (vii) that for $0 < u_0 \le \pi$ and $m \ge 2$,

$$|\varphi_m^1(x_0, u_0)| > |\varphi_m^1(x_1, u_0)|, \qquad |\varphi_{m-1}^1(x_1 + 1/2, u_0)| > |\varphi_{m-1}^1(x_0 + 1/2, u_0)|.$$

Hence the inequality (26) holds.
Analogously the assertion can be shown for $1/2 \le x_0 < x_1 \le 1$. Therefore we only have to consider the case $0 < x_0 < 1/2 < x_1 \le 1$.
For $0 < x_0 < 1/2$ and $x_1 = 1$ we find from (i), (v) and (vii)

$$|\varphi_m^1(1, u_0)| = |\varphi_m^1(0, u_0)| > |\varphi_m^1(x_0, u_0)|,$$

$$|\varphi_{m-1}^1(x_0 + 1/2, u_0)| > |\varphi_{m-1}^1(1/2, u_0)| = |\varphi_{m-1}^1(3/2, u_0)| \qquad (m \ge 2),$$

which implies (26).
Finally, for $0 < x_0 < 1/2 < x_1 < 1$ and $0 < u_0 < \pi$ it follows from (i) and (vi) that

$$u_0 < \arg[\varphi_m^1(x_0, u_0)\varphi_{m-1}^1(x_1 + 1/2, u_0)] < \frac{u_0}{2} + (x_0 + x_1)u_0$$

and

$$\frac{u_0}{2} + (x_0 + x_1)u_0 < \arg[\varphi_m^1(x_1, u_0)\varphi_{m-1}^1(x_0 + 1/2, u_0)] < 2u_0 \qquad (m \ge 2).$$

Thus, by (v), we have $|\varphi_m^2(x_0, x_1, u_0)| > 0$.
For $u_0 = \pi$ and $0 < x_0 < 1/2 < x_1 < 1$ we find from (vii) that

$$\varphi_m^1(x_0, \pi)\varphi_{m-1}^1(x_1 + 1/2, \pi) < 0 < \varphi_{m-1}^1(x_0 + 1/2, \pi)\varphi_m^1(x_1, \pi). \qquad \blacksquare$$

Remark: Let E_j denote the j-th Euler polynomial restricted to the interval $[0, 1]$. Then we have

$$\varphi_m^1(1/2 + t, \pi) = (-1)^{(m+2)/2}\, 2^m E_m(t) \qquad \text{if } m \text{ is even},$$
$$\varphi_m^1(t, \pi) = (-1)^{(m+1)/2}\, 2^m E_m(t) \qquad \text{if } m \text{ is odd}.$$

Hence the statement $|\varphi_m^2(x_0, x_1, \pi)| > 0$ is equivalent to the Haar condition for the polynomials $E_m(.)$ and $E_{m-1}(.)$. ♠

Now we can show the following

Theorem 8 . *Let $\tau_0, \tau_1 \in (0, 1]$ be fixed. Then for $m \ge 2$, we have*

$$H_m^2(\tau_0, \tau_1, e^{iu}) = 0 \qquad (-\pi < u \le \pi)$$

if and only if

$$B_m[\tau_0, \tau_1] = 0.$$

Proof: The case $m = 2$ follows from the example. Consider $m \geq 3$. First assume that $u \neq 0$. Using (21), (19) and (22) it follows from Theorem 7 that for $m \geq 3$ and $t_0, t_1 \in \boldsymbol{R}$, $t_0 - t_1 \notin \boldsymbol{Z} \setminus \{0\}$:

$$|\varphi_m^2(t_0, t_1, u)| > 0$$

with $-\pi < u \leq \pi$, $u \neq 0$. Thus by Lemma 2,

$$|\Delta_m^2(\tau_0, \tau_1, e^{iu})| > 0 \qquad (-\pi < u \leq \pi,\ u \neq 0)$$

with $0 < \tau_0 \leq \tau_1 \leq 1$. Hence, by Theorem 4 we find

$$|H_m^2(\tau_0, \tau_1, e^{iu})| > 0 \qquad (-\pi < u \leq \pi,\ u \neq 0).$$

Together with Theorem 5 this completes the proof. ■

Now the assertion of Theorem 2 follows readily from Theorem 3 and Theorem 8.

7. Cardinal spline interpolation

The result of Theorem 2 on existence and uniqueness of solutions in the periodic case can be extended to cardinal spline interpolation. Let $m \in \boldsymbol{N}$ $(m \geq 2)$ and $r \in \{1, ..., m\}$ be fixed. By $S_{m,r}$ we denote the linear space of all real functions $s \in \boldsymbol{C}^{m-r}(\boldsymbol{R})$ with

$$s(j - 1 + t) = p_j(t) \qquad (p_j \in \boldsymbol{P}_m)$$

for all $t \in [0, 1]$.
We consider the following cardinal spline interpolation problem: For fixed real data sequences $(y_j^{(0)})_{j \in Z}$, $(y_j^{(1)})_{j \in Z}$ and given shift parameters $\tau_0, \tau_1 \in \boldsymbol{R}$ with $0 < \tau_0 \leq \tau_1 \leq 1$, we wish to find a spline function $s \in S_{m,2}$, such that

$$s(j - \tau_0) = y_j^{(0)} \qquad (j \in \boldsymbol{Z}),$$

$$s[j + \tau_0, j + \tau_1] = y_j^{(1)} \qquad (j \in \boldsymbol{Z}). \tag{27}$$

Introducing the linear operator $\boldsymbol{U}$ by

$$\boldsymbol{U}(y_j^{(k)})_{j \in Z} := (y_{j-1}^{(k)})_{j \in Z}, \qquad (k = 0, 1),$$

it follows that the spline interpolation problem (27) is uniquely solvable if and only if the infinite Toeplitz matrix $H_m^2(\tau_0, \tau_1, \boldsymbol{U})$ is nonsingular, i.e., if

$$H_m^2(\tau_0, \tau_1, z) \neq 0, \qquad (|z| = 1).$$

Hence we have as an immediate consequence of Theorem 8:

Theorem 9 . *Let $m \in \boldsymbol{N}$ $(m \geq 2)$ and $\tau_0, \tau_1 \in \boldsymbol{R}$ with $0 < \tau_0 \leq \tau_1 \leq 1$ be fixed. Then the cardinal spline interpolation problem* (27) *possesses a unique solution $s \in S_{m,2}$ for any data sequences $(y_j^{(0)})_{j \in Z}$, $(y_j^{(1)})_{j \in Z} \in l_2$ if and only if $B_m[\tau_0, \tau_1] \neq 0$.*

Acknowledgment

The author thanks Professor M. Tasche for helpful suggestions.

References

[1] Jetter, K., Riemenschneider, S.D., Sivakumar, N.: Schoenberg's exponential Euler spline curves. Proc. Roy. Soc. Edinburgh **118A**, 21 – 33 (1991).

[2] Lee, S.L., Sharma, A.: Cardinal lacunary interpolation by g-splines I. The characteristic polynomials. J. Approx. Theory **16**, 85 – 96 (1975).

[3] Lipow, P., Schoenberg, I.J.: Cardinal interpolation and spline functions. III. Cardinal Hermite interpolation. Linear Algebra Appl. **6**, 273 – 304 (1973).

[4] Meinardus, G., Merz, G.: Zur periodischen Spline-Interpolation. In: Böhmer, K., Meinardus, G., Schempp, W. (eds): Spline-Funktionen, pp. 177 – 195. Mannheim: Bibliographisches Institut 1974.

[5] Meinardus, G., Merz, G.: Hermite-Interpolation mit periodischen Spline-Funktionen. In: Collatz, L., Meinardus, G., Werner, H. (eds): Numerical methods of approximation theory, pp. 200 – 210. Birkhäuser, Basel 1980.

[6] Merz, G., Sippel, W.: Zur Konstruktion periodischer Hermite-Interpolationssplines bei äquidistanter Knotenverteilung. J. Approx. Theory **54**, 92 – 106 (1988).

[7] ter Morsche, H.: On the existence and convergence of interpolating periodic spline functions of arbitrary degree. In: Böhmer, K., Meinardus, G., Schempp, W. (eds): Spline-Funktionen, pp. 197 – 214. Mannheim: Bibliographisches Institut 1974.

[8] Plonka, G., Tasche, M.: Efficient algorithms for the periodic Hermite-spline interpolation. Math. Comp. **58**, 693 – 703 (1992).

[9] Reimer, M.: Cardinal Hermite-spline-interpolation on the equidistant lattice. Numer. Math. **56**, 345 – 357 (1989).

[10] Reimer, M.: Zur reellen Darstellung periodischer Hermite-Spline-Interpolierender bei äquidistantem Gitter mit Knotenshift. In: Schmidt, J.W., Späth, H. (eds): Splines in Numerical Analysis, pp. 125 – 134. Berlin: Akademie-Verlag 1989.

[11] Schoenberg, I.J.: Cardinal spline interpolation, SIAM, Philadelphia, 1973.

[12] Siepmann, D.: Kardinale Spline-Interpolation bezüglich äquidistant verteilter Knoten. Dissertation, Dortmund 1984.

Universität Rostock, Fachbereich Mathematik, Universitätsplatz 1, O-2500 Rostock, Germany