# Two–scale symbol and autocorrelation symbol for B-splines with multiple knots

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**Abstract**

The Fourier transforms of B-splines with multiple integer knots are shown to satisfy a simple recursion relation. This recursion formula is applied to derive a generalized two–scale relation for B-splines with multiple knots. Furthermore, the structure of the corresponding autocorrelation symbol is investigated.
In particular, it can be observed that the solvability of the cardinal Hermite spline interpolation problem for spline functions of degree $2m+1$ and defect $r$, first considered by P.R. Lipow and I.J. Schoenberg [9], is equivalent to the Riesz basis property of our B-splines with degree $m$ and defect $r$. In this way we obtain a new, simple proof for the assertion that the cardinal Hermite spline interpolation problem in [9] has a unique solution.

**1991 Mathematics Subject Classification**. Primary 41A15, 42A38, 65D07.
**Key words and Phrases**. B-splines with multiple knots, two–scale symbol, autocorrelation symbol, Riesz basis, cardinal Hermite spline interpolation

# 1 Introduction

Let $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ be given integers. We consider equidistant knots with multiplicity $r$

$$x_l^r := \lfloor l/r \rfloor \qquad (l \in \mathbb{Z}), \tag{1.1}$$

where $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$.
Let $N_k^{m,r} \in C^{m-r}(\mathbb{R})$ $(m \geq r,\ k \in \mathbb{Z})$ denote the normalized B-splines of degree $m$ and defect $r$ with the knots $x_k, ..., x_{k+m+1}$. The class $S_{m,r}(\mathbb{Z})$ $(m \geq r)$ of cardinal B-splines of degree $m$ with integer knots of multiplicity $r$ consists of functions $s$, which are polynomials of degree $m$ in each interval $[\nu, \nu+1]$ $(\nu \in \mathbb{Z})$ and belong to $C^{m-r}(\mathbb{R})$.
Note that the B-splines

$$N_{k+rl}^{m,r} = N_k^{m,r}(\cdot - l) \quad (k = 0, ..., r-1;\ l \in \mathbb{Z}) \tag{1.2}$$

form a basis of the spline space $S_{m,r}(\mathbb{Z})$ (cf. [1]). For the well-known normalized B-splines $N_m$ of defect 1 it follows the notation $N_m(\cdot - l) := N_l^{m,1}$.
We introduce the B-spline vector $\boldsymbol{N}_m^r := (N_k^{m,r})_{k=0}^{r-1}$ of length $r$. The Fourier transform of $\boldsymbol{N}_m^r$ is denoted by $\hat{\boldsymbol{N}}_m^r := (\hat{N}_k^{m,r})_{k=0}^{r-1}$ with

$$\hat{N}_k^{m,r} := \int_{-\infty}^{\infty} N_k^{m,r}(x)\, e^{-i \cdot x}\ \mathrm{d}x \quad (k = 0, ..., r-1).$$

In Section 2 we shall derive a recursion formula for the Fourier transformed B-spline vector $\hat{\boldsymbol{N}}_m^r$. This recursion formula can now be applied to various problems in Sections 3 – 5.

In Section 3, we are going to find a recursive scheme for the computation of the $(r,r)$-symbol

$$\boldsymbol{P}_m^r(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \boldsymbol{P}_n z^n \tag{1.3}$$

which is defined by the generalized two–scale relation (refinement equation)

$$\boldsymbol{N}_m^r := \sum_{n=-\infty}^{\infty} \boldsymbol{P}_n\, \boldsymbol{N}_m^r(2 \cdot -n). \tag{1.4}$$

Here and in the following let $z \in \mathbb{T}$ with $\mathbb{T} := \{z \in \mathbb{C},\ |z| = 1\}$. The two–scale relation (1.4) is needed for a generalized multiresolution analysis with $r$ scaling functions $N_k^{m,r}$ $(k = 0, ..., r-1)$ and for the construction of Hermite spline wavelets. In this way new spline wavelets with very small support can be found (cf. [4, 5]).
The well-known two–scale relation for the normalized B-splines $N_m := \boldsymbol{N}_m^1$ of defect 1

$$\hat{N}_m = \left( \frac{1 + e^{-i\cdot}}{2} \right)^{m+1} \hat{N}_m(\cdot/2)$$

will be obtained in the special case $r = 1$ (cf. [2, 3]).

In Section 4 we consider the autocorrelation functions

$$F_{k,l} := \int_{-\infty}^{\infty} N_k^{m,r}(\cdot + y)\, N_l^{m,r}(y)\ \mathrm{d}y \quad (k, l = 0, ..., r-1) \tag{1.5}$$

and the corresponding $(r,r)$-matrices $\boldsymbol{F}(n) := (F_{k,l}(n))_{k,l=0}^{r-1}$ $(n \in \mathbb{Z})$. The autocorrelation symbol is defined by

$$\boldsymbol{\Phi}_m^r(z) := \sum_{n=-\infty}^{\infty} \boldsymbol{F}(n) z^k. \tag{1.6}$$

We shall investigate the structure of $\boldsymbol{\Phi}_m^r$. Some interesting properties of $\boldsymbol{\Phi}_m^r$ can be observed which are reminiscent of the well-known case with one scaling function. In particular, the equivalence between invertibility of $\boldsymbol{\Phi}_m^r(z)$ for $z \in \mathbb{T}$ and the Riesz basis property of $N_k^{m,r}(\cdot - l)$ $(l \in \mathbb{Z};\ k = 0, \ldots, r-1)$ in $S_{m,r}(\mathbb{Z}) \bigcap \mathbb{L}_2(\mathbb{R})$ can be shown. Note that $\boldsymbol{\Phi}_m^1$ coincides with the well-known Euler–Frobenius polynomial

$$\boldsymbol{\Phi}_m^1(z) := \sum_{n=-\infty}^{\infty} N_{2m+1}(n)\, z^n$$

(cf. [2, 3, 13]).

Finally, in Section 5 the cardinal Hermite spline interpolation problem, first considered by P.R. Lipow and I.J. Schoenberg [9], will be handled. With the help of results of the previous sections a new, simple proof will be given for the assertion that the cardinal Hermite spline interpolation problem in [9] is uniquely solvable. Furthermore, the new conclusions will be compared with known results in [9] and [8].

## 2 Fourier transform of the B-spline vector $\boldsymbol{N}_m^r$

The B-splines $N_k^{m,r}$ $(k \in \mathbb{Z})$ of degree $m \in \mathbb{N}_0$ and defect $r \in \mathbb{N}$ possess the following properties:

**Theorem 2.1** *For $m \in \mathbb{N}_0$, $r \in \mathbb{N}$ and $k = 0, \ldots, r-1$ we have:*

(i)
$$N_k^{m,r} \in C^{m-r}(\mathbb{R}) \quad (r \leq m),$$

(ii)
$$\operatorname{supp} N_k^{m,r} = [0, \lfloor (m+1+k)/r \rfloor],$$
$$N_k^{m,r}(x) > 0 \quad (x \in (0, \lfloor (m+1+k)/r \rfloor)),$$

(iii)
$$N_k^{m,r} | [j, j+1] \in \boldsymbol{P}_m \qquad (j \in \mathbb{Z}),$$

(iv)
$$\sum_{j=-\infty}^{\infty} \sum_{k=0}^{r-1} N_k^{m,r}(\cdot - j) = 1$$

(v)
$$\hat{N}_k^{m,r}(0) = \int_{-\infty}^{\infty} N_k^{m,r}(x)\ \mathrm{d}x = \frac{\lfloor (m+k+1)/r \rfloor}{m+1}.$$

*In particular,*

$$\sum_{k=0}^{r-1} \hat{N}_k^{m,r}(0) = \int_{-\infty}^{\infty} \sum_{k=0}^{r-1} N_k^{m,r}(x)\, \mathrm{d}x = 1.$$

*Furthermore,*

$$\sum_{k=0}^{r-1} \hat{N}_k^{m,r}(2\pi l) = 0 \quad (l \in \mathbb{Z} \setminus \{\nvdash\}).$$

(vi) *For* $m > r-1$ *we have*

$$N_k^{m,r}(x) = \frac{x - x_k^r}{x_{k+m}^r - x_k^r} N_k^{m-1,r}(x) + \frac{x_{k+m+1}^r - x}{x_{k+m+1}^r - x_{k+1}^r} N_{k+1}^{m-1,r}(x). \quad (x \in \mathbb{R}).$$

(vii) *For* $m > r-1$ *we obtain*

$$\mathrm{D} N_k^{m,r} = m \left( \frac{1}{x_{k+m}^r - x_k^r} N_k^{m-1,r} - \frac{1}{x_{k+m+1}^r - x_{k+1}^r} N_{k+1}^{m-1,r} \right),$$

*where* $\mathrm{D}$ *denotes the differential operator* $\mathrm{D} := \mathrm{d}/\,\mathrm{d}\cdot$.

For a proof of Theorem 2.1 we refer to [1].

For $r = m+1$, the B-splines $N_k^{m,m+1}$ $(k = 0, \ldots, m)$ coincide with the Bernstein polynomials

$$N_k^{m,m+1}(x) = B_k^m(x) := \begin{cases} \binom{m}{k} x^k (1-x)^{m-k} & x \in [0,1], \\ 0 & x \notin [0,1]. \end{cases} \tag{2.1}$$

According to the distribution theory let $N_k^{m,r}$ be defined for $r > m+1$ and $k = 0, \ldots, r-m-2$ as follows

$$N_k^{m,r} := \frac{D^{r-m-2-k}\,\delta}{r-1-k},$$

where $\delta$ denotes the Dirac distribution. For the Fourier transformed B-spline vector $\hat{\boldsymbol{N}}_m^r$ of length $r > m+1$ we find

$$\hat{\boldsymbol{N}}_m^r(u) = \Big( \frac{(iu)^{r-m-2}}{r-1}, \ldots, \frac{(iu)^0}{m+1}, \hat{\boldsymbol{N}}_m^{m+1}(u)^T \Big)^T, \tag{2.2}$$

where $\boldsymbol{N}_m^{m+1}$ denotes the vector of the $m+1$ Bernstein polynomials of degree $m$. In particular, we have

$$\hat{\boldsymbol{N}}_0^r(u) = \Big( \frac{(iu)^{r-2}}{r-1}, \ldots, \frac{(iu)^0}{1}, \frac{(1-e^{-iu})}{iu} \Big)^T. \tag{2.3}$$

Further, we put

$$\hat{\boldsymbol{N}}_{-1}^r(u) := \Big( \frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^1}{1}, 1 \Big)^T \qquad (u \in \mathbb{R}). \tag{2.4}$$

The following recursion relation for $\hat{\boldsymbol{N}}_m^r$ can be found:

**Theorem 2.2** *For $m \in \mathbb{N}_{\not\varkappa}$, $r \in \mathbb{N}$ we have*

$$(iu)\, \hat{\boldsymbol{N}}^r_m(u) = \boldsymbol{A}^r_m(e^{-iu})\, \hat{\boldsymbol{N}}^r_{m-1}(u) \quad (u \in \mathbb{R}). \tag{2.5}$$

*The $(r,r)$-matrices $\boldsymbol{A}^r_m(z)$ $(z \in \mathbb{T})$ are defined for $m > r-1$ by*

$$\boldsymbol{A}^r_m(z) := m \begin{pmatrix} \frac{1}{x^r_m} & -\frac{1}{x^r_{m+1}} & \dots & 0 & 0 \\ 0 & \frac{1}{x^r_{m+1}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{x^r_{m+r-2}} & -\frac{1}{x^r_{m+r-1}} \\ -\frac{z}{x^r_m} & 0 & \dots & 0 & \frac{1}{x^r_{m+r-1}} \end{pmatrix}, \tag{2.6}$$

*where $x^r_{m+k}$ $(k = 0, \dots, r-1)$ are given in (1.1). For $m = r-1 > 0$ let*

$$\boldsymbol{A}^{m+1}_m(z) := m \begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ -z & 0 & \dots & 0 & 1 \end{pmatrix} \tag{2.7}$$

*and for $0 \le m < r-1$*

$$\boldsymbol{A}^r_m(z) := \begin{pmatrix} \boldsymbol{I}_{r-m-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^{m+1}_m(z) \end{pmatrix}, \tag{2.8}$$

*where $\boldsymbol{A}^1_0(z) := 1 - z$. Further, $\boldsymbol{I}_{r-m-1}$ denotes the $(r-m-1)$-th unit matrix and $\boldsymbol{0}$ a zero matrix.*

**Proof:**
1. Applying the Fourier transform to the relation (vii) of Theorem 2.1 we obtain for $k = 0, \dots, r-1$ and $m > r-1$:

$$(iu)\hat{N}^{m,r}_k(u) = m\Big(\frac{1}{x^r_{k+m}}\hat{N}^{m-1,r}_k(u) - \frac{1}{x^r_{k+m+1} - x^r_{k+1}}\hat{N}^{m-1,r}_{k+1}(u)\Big).$$

Using (1.2) it follows that

$$\hat{N}^{m-1,r}_r(u) = [N^{m-1,r}_0(\cdot - 1)]^\wedge(u) = e^{-iu}\hat{N}^{m-1,r}_0(u)$$

and thus

$$(iu)\hat{\boldsymbol{N}}^r_m(u) = \boldsymbol{A}^r_m(e^{-iu})\hat{\boldsymbol{N}}^r_{m-1}(u) \quad (u \in \mathbb{R}) \tag{2.9}$$

with $\boldsymbol{A}^r_m(e^{-iu})$ in (2.6).
2. For $m = r-1$ we find

$$\mathrm{D}N^{m,m+1}_k = m\Big(N^{m-1,m+1}_k - N^{m-1,m+1}_{k+1}\Big) \quad (k = 0, \dots, m)$$

with $N^{m-1,m+1}_0 := \delta/m$ and $N^{m-1,m+1}_{m+1} := N^{m-1,m+1}_0(\cdot - 1) = \delta(\cdot - 1)/m$, where $\delta$ denotes the Dirac distribution. For the vector $\boldsymbol{N}^{m+1}_m$ it follows by $(N^{m-1,m+1}_k)^m_{k=1} = \boldsymbol{N}^m_{m-1}$

$$(\,\mathrm{D}\boldsymbol{N}^{m+1}_m)^\wedge(u) = (iu)\hat{\boldsymbol{N}}^{m+1}_m(u) = \boldsymbol{A}^{m+1}_m(e^{-iu})\hat{\boldsymbol{N}}^{m+1}_{m-1}(u) \tag{2.10}$$

with the $(m+1,m+1)$-matrix $\boldsymbol{A}_m^{m+1}(e^{-iu})$ in (2.7) and $\hat{\boldsymbol{N}}_{m-1}^{m+1}(u) = (1/m, \hat{\boldsymbol{N}}_{m-1}^{m}(u)^T)^T$.
3. The formula (2.10) and the definitions (2.2) – (2.4) lead for $0 \le m < r-1$ to

$$(iu)\hat{\boldsymbol{N}}_m^r(u) = \boldsymbol{A}_m^r(e^{-iu})\hat{\boldsymbol{N}}_{m-1}^r(u)$$

with $\boldsymbol{A}_m^r(e^{-iu})$ defined in (2.8) and $\boldsymbol{A}_0^1(z) = (1-z)$. ∎

**Example 2.3**
For $r = 1$ and $m \ge 0$ we have $\boldsymbol{A}_m^1(z) = (1-z)$.
For $r = 2$ and even $m > 1$ we find

$$\boldsymbol{A}_m^2(z) = 2\begin{pmatrix} 1 & -1 \\ -z & 1 \end{pmatrix}.$$

For $r = 2$ and odd $m = 2n+1 > 1$ it follows

$$\boldsymbol{A}_m^2(z) = \begin{pmatrix} 2+1/n & -2+1/(n+1) \\ -z(2+1/n) & 2-1/(n+1) \end{pmatrix}.$$

From the recursion relation (2.5) it follows in the special case $r = 1$ the well-known formula

$$\hat{\boldsymbol{N}}_m^1(u) = \hat{N}_m(u) = \frac{1}{iu}(1-e^{-iu})\hat{N}_{m-1}(u),$$

i.e.

$$\hat{N}_m(u) = \Big(\frac{1-e^{-iu}}{iu}\Big)^{m+1} \qquad (u \in \mathbb{R}).$$

In the case $r = 2$ we have

$$\begin{aligned}
\hat{\boldsymbol{N}}_0^2(u) &= \frac{1}{iu}\begin{pmatrix} 1 & 0 \\ 0 & 1-e^{-iu} \end{pmatrix}\begin{pmatrix} iu \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ (1-e^{-iu})/iu \end{pmatrix}, \\
\hat{\boldsymbol{N}}_1^2(u) &= \frac{1}{iu}\begin{pmatrix} 1 & -1 \\ -e^{-iu} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ (1-e^{-iu})/iu \end{pmatrix} = \frac{1}{(iu)^2}\begin{pmatrix} iu-1+e^{-iu} \\ 1-(1+iu)e^{-iu} \end{pmatrix}, \\
\hat{\boldsymbol{N}}_2^2(u) &= \frac{2}{iu}\begin{pmatrix} 1 & -1 \\ -e^{-iu} & 1 \end{pmatrix}\frac{1}{(iu)^2}\begin{pmatrix} iu-1+e^{-iu} \\ 1-(1+iu)e^{-iu} \end{pmatrix} \\
&= \frac{2}{(iu)^3}\begin{pmatrix} iu-2+(2+iu)e^{-iu} \\ 1-2iue^{-iu}-e^{-2iu} \end{pmatrix} \qquad (u \in \mathbb{R}). \quad \spadesuit
\end{aligned}$$

**Remark 2.4**
1. Note that for $m > r-1$

$$\det\, \boldsymbol{A}_m^r(z) = (1-z)\prod_{l=0}^{r-1}\frac{m}{x_{m+l}^r}$$

and for $0 < m \le r-1$

$$\det\, \boldsymbol{A}_m^r(z) = \det\, \boldsymbol{A}_m^{m+1}(z) = m^{m+1}\,(1-z).$$

2. By Theorem 2.2 we have the recursion formula for the computation of all Fourier transforms of B-spline vectors with multiple knots. Since $\hat{\boldsymbol{N}}_m^r$ is continuous at 0 we obtain

$$\hat{\boldsymbol{N}}_m^r(0) = \lim_{u\to 0} \frac{1}{iu} \boldsymbol{A}_m^r(e^{-iu}) \hat{\boldsymbol{N}}_{m-1}^r(u) = \Big( \frac{\lfloor (m+1+k)/r \rfloor}{m+1} \Big)_{k=0}^{r-1} \quad (u \in \mathbb{R}).$$

3. Fourier transforms of special B-splines and fundamental splines for cardinal Hermite spline interpolation, introduced in [9] and [14], were already treated in [7]. In contrast to our approach, in [7] an integral representation for the Fourier transforms, based on exponential Hermite Euler splines, was given. ♠

# 3 Two–scale symbol matrix

A central role in the construction of multiresolution analysis and wavelets is played by the following two–scale relation or refinement equation of a given function $\phi \in L_2(\mathbb{R})$:

$$\phi = \sum_{n=-\infty}^{\infty} p_n \phi(2 \cdot -n) \quad ((p_n) \in l_2).$$

The Fourier transformed two–scale relation reads

$$\hat{\phi} = P(e^{-i\cdot/2})\, \hat{\phi}(\cdot/2)$$

with the two–scale symbol

$$P(z) := \frac{1}{2} \sum_{n=-\infty}^{\infty} p_n\, z^n.$$

We want to generalize this two–scale relation for more than one scaling function and wish to find matrices $\boldsymbol{P}_n$ $(n \in \mathbb{Z})$ with

$$\boldsymbol{N}_m^r = \sum_{n=-\infty}^{\infty} \boldsymbol{P}_n \boldsymbol{N}_m^r(2 \cdot -n) \tag{3.1}$$

or

$$\hat{\boldsymbol{N}}_m^r = \boldsymbol{P}_m^r(e^{-i\cdot/2})\, \hat{\boldsymbol{N}}_m^r(\cdot/2) \quad (m \in \mathbb{N}_{\nvdash},\ \diagdown \in \mathbb{N}). \tag{3.2}$$

The $(r,r)$-matrix

$$\boldsymbol{P}_m^r(z) := \frac{1}{2} \sum_{n=-\infty}^{\infty} \boldsymbol{P}_n z^n$$

is called *two–scale symbol* or *refinement mask of* $\boldsymbol{N}_m^r$.

We are going to find a recursive scheme for the computation of $\boldsymbol{P}_m^r$ for all vectors $\boldsymbol{N}_m^r$ $(m \in \mathbb{N}_{\nvdash},\ \diagdown \in \mathbb{N})$. The well-known two–scale relation for the normalized B-splines of defect 1

$$\hat{N}_m = \left( \frac{1 + e^{-i\cdot}}{2} \right)^{m+1} \hat{N}_m(\cdot/2)$$

will be obtained in the special case $r = 1$. Furthermore, for $r = m+1$ we will find the two–scale relation for the vector of Bernstein polynomials $\boldsymbol{N}_m^{m+1} = \boldsymbol{B}^m := (B_0^m, \ldots, B_m^m)^T$ (see (2.1)).

**Theorem 3.1** *Let $m \in \mathbb{N}_{\nvdash}$ and $r \in \mathbb{N}$ be given. Then the two–scale symbol $\boldsymbol{P}_m^r$ is finite and satisfies for $m \geq 0$ the following recursion formula*

$$\boldsymbol{P}_m^r(z) = \frac{1}{2}\boldsymbol{A}_m^r(z^2)\boldsymbol{P}_{m-1}^r(z)\boldsymbol{A}_m^r(z)^{-1} \quad (z \in \mathbb{T} \setminus \{\nVdash\}) \tag{3.3}$$

*with $\boldsymbol{A}_m^r(z)$ defined in* (2.6) – (2.8) *and the $(r,r)$-diagonal matrix*

$$\boldsymbol{P}_{-1}^r(z) := \operatorname{diag}\,(2^{r-1}, \ldots, 2^0)^T.$$

*For $z = 1$ we have*

$$\boldsymbol{P}_m^r(1) = \frac{1}{2}\lim_{u \to 0} \boldsymbol{A}_m^r(e^{-2iu})\boldsymbol{P}_{m-1}^r(e^{-iu})\boldsymbol{A}_m^r(e^{-iu})^{-1} \quad (u \in \mathbb{R}).$$

**Proof:** By Remark 2.4 the matrix $\boldsymbol{A}_m^r$ is invertible for $z \neq 1$. Applying the formula (3.3) for $m = 0$ we obtain

$$\begin{aligned}
\boldsymbol{P}_0^r(z) &= \frac{1}{2}\boldsymbol{A}_0^r(z^2)\boldsymbol{P}_{-1}^r(z)\boldsymbol{A}_0^r(z)^{-1} \\
&= \frac{1}{2}\begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & 1-z^2 \end{pmatrix}\begin{pmatrix} 2^{r-1} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 2 & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & (1-z)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{P}_{-1}^{r-1}(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_0^1(z) \end{pmatrix}.
\end{aligned} \tag{3.4}$$

With (2.3) it can be easily verified that the assertion is true for $m = 0$, i.e., the matrix $\boldsymbol{P}_0^r$ computed in (3.4) satisfies

$$\hat{\boldsymbol{N}}_0^r = \boldsymbol{P}_0^r(e^{-i\cdot/2})\hat{\boldsymbol{N}}_0^r(\cdot/2).$$

Now let $m > 0$ and let (3.3) be satisfied for $m-1$. On the one hand, by (3.2) and using the recursion (2.5) it follows for $u \in \mathbb{R} \setminus \{\nvdash\}$ that

$$\begin{aligned}
\hat{\boldsymbol{N}}_m^r(u) = \frac{1}{iu}\boldsymbol{A}_m^r(e^{-iu})\hat{\boldsymbol{N}}_{m-1}^r(u) &= \boldsymbol{P}_m^r(e^{-iu/2})\hat{\boldsymbol{N}}_m^r(u/2) \\
&= \boldsymbol{P}_m^r(e^{-iu/2})\frac{2}{iu}\boldsymbol{A}_m^r(e^{-iu/2})\hat{\boldsymbol{N}}_{m-1}^r(u/2),
\end{aligned}$$

i.e.

$$\hat{\boldsymbol{N}}_{m-1}^r(u) = 2\boldsymbol{A}_m^r(e^{-iu})^{-1}\boldsymbol{P}_m^r(e^{-iu/2})\boldsymbol{A}_m^r(e^{-iu/2})\,\hat{\boldsymbol{N}}_{m-1}^r(u/2).$$

On the other hand, we have

$$\hat{\boldsymbol{N}}_{m-1}^r(u) = \boldsymbol{P}_{m-1}^r(e^{-iu/2})\hat{\boldsymbol{N}}_{m-1}^r(u/2).$$

Thus,

$$\boldsymbol{P}_m^r(z) = \frac{1}{2}\boldsymbol{A}_m^r(z^2)\boldsymbol{P}_{m-1}^r(z)\boldsymbol{A}_m^r(z)^{-1} \quad (z \in \mathbb{T} \setminus \{\nVdash\}).$$

Since the B-splines $N_k^{m,r}$ $(k \in \mathbb{Z})$ are spline functions of $S_{m,r}(\mathbb{Z})$ with minimal support and in particular

$$\text{clos}\,(\text{supp} \sum_{k=-\infty}^{\infty} \alpha_k N_k^{m,r}) = \text{clos}\,(\bigcup_{\alpha_k \neq 0} \text{supp } N_k^{m,r})$$

(cf. [1]), the two–scale relation is finite. It follows that $\boldsymbol{P}_m^r(z)$ is a matrix polynomial in $z$. By continuity of $\hat{\boldsymbol{N}}_m^r(u)$ at $u = 0$ we obtain $\boldsymbol{P}_m^r(1)$ by the limiting process

$$\boldsymbol{P}_m^r(1) = \lim_{u \to 0} \frac{1}{2} \boldsymbol{A}_m^r(e^{-2iu}) \boldsymbol{P}_{m-1}^r(e^{-iu}) \boldsymbol{A}_m^r(e^{-iu})^{-1} \quad (u \in \mathbb{R}). \qquad \blacksquare$$

By Theorem 3.1 all two–scale symbol matrices $\boldsymbol{P}_m^r$ $(m \in \mathbb{N}_{\nvDash};\ \searrow \in \mathbb{N})$ can be recursively computed. The inverse $(\boldsymbol{A}_m^r)^{-1}$ can be explicitly given.

**Lemma 3.2** *For $z \in \mathbb{T} \setminus \{\nVdash\}$ the $(r,r)$-matrix $\boldsymbol{A}_m^r(z)$ $(r > 1)$ is invertible and we have for $m > r-1$*

$$\boldsymbol{A}_m^r(z)^{-1} = \frac{1}{m(1-z)} \begin{pmatrix} x_m^r & x_m^r & \dots & x_m^r & x_m^r \\ z x_{m+1}^r & x_{m+1}^r & \dots & x_{m+1}^r & x_{m+1}^r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z x_{m+r-2}^r & z x_{m+r-2}^r & \dots & x_{m+r-2}^r & x_{m+r-2}^r \\ z x_{m+r-1}^r & z x_{m+r-1}^r & \dots & z x_{m+r-1}^r & x_{m+r-1}^r \end{pmatrix}. \tag{3.5}$$

*For $m = r-1$ we obtain*

$$\boldsymbol{A}_m^{m+1}(z)^{-1} = \frac{1}{m(1-z)} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ z & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & \dots & 1 & 1 \\ z & z & \dots & z & 1 \end{pmatrix} \tag{3.6}$$

*and for $m < r-1$*

$$\boldsymbol{A}_m^r(z)^{-1} = \begin{pmatrix} \boldsymbol{I}_{r-m-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_m^{m+1}(z)^{-1} \end{pmatrix}, \tag{3.7}$$

*where $\boldsymbol{I}_{r-m-1}$ denotes the $(r-m-1)$-th unit matrix and $\boldsymbol{0}$ a zero matrix.*

**Proof:** The assertion immediately follows by the observation that the matrices $\boldsymbol{A}_m^r(z)^{-1}$ defined in (3.5) – (3.7) satisfy the identity

$$\boldsymbol{A}_m^r(z) \boldsymbol{A}_m^r(z)^{-1} = \boldsymbol{I}_r. \quad \blacksquare$$

**Example 3.3** For $r = 1$ we only have $(1,1)$-matrices. Since

$$\boldsymbol{A}_m^1(z^2) \boldsymbol{A}_m^1(z)^{-1} = \frac{1-z^2}{1-z} = 1 + z,$$

it follows that

$$\boldsymbol{P}_m^1(z) = \frac{1+z}{2}\boldsymbol{P}_{m-1}^1(z),$$

that is

$$\boldsymbol{P}_m^1(z) = P_m(z) = \left(\frac{1+z}{2}\right)^{m+1}.$$

For $r = 2$ we obtain for $m = 0, 1, 2$:

$$\begin{aligned}
\boldsymbol{P}_0^2(z) &= \begin{pmatrix} 1 & 0 \\ 0 & (1+z)/2 \end{pmatrix}, \\
\boldsymbol{P}_1^2(z) &= \frac{1}{2(1-z)} \begin{pmatrix} 1 & -1 \\ -z^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1+z)/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2+z & 1 \\ z & 2z+1 \end{pmatrix}, \\
\boldsymbol{P}_2^2(z) &= \frac{1}{2(1-z)} \begin{pmatrix} 1 & -1 \\ -z^2 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 2+z & 1 \\ z & 2z+1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix} \\
&= \frac{1}{8} \begin{pmatrix} 2+2z & 2 \\ 2z+2z^2 & 1+4z+z^2 \end{pmatrix}.
\end{aligned}$$

For $r = 3$ and $m = 2$ we find

$$\begin{aligned}
\boldsymbol{P}_2^3(z) &= \frac{1}{2(1-z)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -z^2 & 0 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2+z & 1 \\ 0 & z & 2z+1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ z & 1 & 1 \\ z & z & 1 \end{pmatrix} \\
&= \frac{1}{8} \begin{pmatrix} 4+z & 2 & 1 \\ 2z & 2z+2 & 2 \\ z & 2z & 4z+1 \end{pmatrix}. \quad \spadesuit
\end{aligned}$$

Furthermore, using the recursion relation (3.3) we find

**Corollary 3.4** *For $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ we have*

$$\det \boldsymbol{P}_m^r(z) = 2^{-rm+r(r-3)/2}\,(1+z)^{m+1} \quad (z \in \mathbb{T}). \tag{3.8}$$

**Proof:** For $m = 0$ we have by (3.4)

$$\begin{aligned}
\det \boldsymbol{P}_0^r(z) &= \frac{1}{2^r}(1-z^2)2^{r(r-1)/2}(1-z)^{-1} \\
&= 2^{r(r-3)/2}(1+z).
\end{aligned}$$

Now let (3.8) be satisfied for $\boldsymbol{P}_{m-1}^r$ $(m > 0)$. Then we obtain by (3.3) and Remark 2.4

$$\begin{aligned}
\det \boldsymbol{P}_m^r(z) &= \frac{1}{2^r}(1+z)\det \boldsymbol{P}_{m-1}^r(z) \\
&= 2^{-mr+r(r-3)/2}\,(1+z)^{m+1}. \qquad \blacksquare
\end{aligned}$$

# 4 Autocorrelation symbol of $\boldsymbol{N}_m^r$

As in the simple case with one scaling function let us consider the autocorrelation functions

$$F_{k,l} := \int_{-\infty}^{\infty} N_k^{m,r}(\cdot + y)\, N_l^{m,r}(y)\ \mathrm{d}y \quad (k,l = 0,\ldots,r-1). \tag{4.1}$$

The *autocorrelation symbol of* $\boldsymbol{N}_m^r$ is defined by the sequences $\boldsymbol{F}(n) := (F_{k,l}(n))_{k,l=0}^{r-1}$ $(n \in \mathbb{Z})$, namely

$$\boldsymbol{\Phi}_m^r(z) := \sum_{n=-\infty}^{\infty} \boldsymbol{F}(n) z^n. \tag{4.2}$$

The following properties hold:

**Theorem 4.1** *Let* $m \in \mathbb{N}_{\not 0}$ *and* $r \in \mathbb{N}$. *Then we have*
(i) *For all* $u \in \mathbb{R}$

$$\boldsymbol{\Phi}_m^r(e^{-iu}) = \sum_{k=-\infty}^{\infty} \hat{\boldsymbol{N}}_m^r(u + 2\pi k)\, \hat{\boldsymbol{N}}_m^r(u + 2\pi k)^\star \tag{4.3}$$

*with* $\hat{\boldsymbol{N}}_m^r(u)^\star := \overline{\hat{\boldsymbol{N}}_m^r(u)^T}$. *In particular,* $\boldsymbol{\Phi}_m^r(z)$ *is Hermitian and positive semidefinite.*
(ii) *For* $z \in \mathbb{T}$,

$$\boldsymbol{\Phi}_m^r(z^2) = \boldsymbol{P}_m^r(z)\boldsymbol{\Phi}_m^r(z)\boldsymbol{P}_m^r(z)^\star + \boldsymbol{P}_m^r(-z)\boldsymbol{\Phi}_m^r(-z)\boldsymbol{P}_m^r(-z)^\star$$

*with* $\boldsymbol{P}_m^r(z)^\star := \overline{\boldsymbol{P}_m^r(z)^T}$.

**Proof:** The assertion (4.3) follows from the Poisson summation formula. Applying (4.3) and the two–scale relation (3.2) we find for $u \in \mathbb{R}$

$$\begin{aligned}
\boldsymbol{\Phi}_m^r(e^{-iu}) &= \sum_{k=-\infty}^{\infty} \boldsymbol{P}_m^r(e^{-i(u/2+k\pi)})\hat{\boldsymbol{N}}_m^r(u/2+k\pi)\hat{\boldsymbol{N}}_m^r(u/2+k\pi)^\star \boldsymbol{P}_m^r(e^{-i(u/2+k\pi)})^\star \\
&= \sum_{l=-\infty}^{\infty} \boldsymbol{P}_m^r(e^{-iu/2})\hat{\boldsymbol{N}}_m^r(u/2+2\pi l)\hat{\boldsymbol{N}}_m^r(u/2+2\pi l)^\star \boldsymbol{P}_m^r(e^{-iu/2})^\star \\
&\quad + \sum_{l=-\infty}^{\infty} \boldsymbol{P}_m^r(-e^{-iu/2})\hat{\boldsymbol{N}}_m^r(u/2+(2l+1)\pi)\hat{\boldsymbol{N}}_m^r(u/2+(2l+1)\pi)^\star \boldsymbol{P}_m^r(-e^{-iu/2})^\star \\
&= \boldsymbol{P}_m^r(e^{-iu/2})\boldsymbol{\Phi}_m^r(e^{-iu/2})\boldsymbol{P}_m^r(e^{-iu/2})^\star + \boldsymbol{P}_m^r(-e^{-iu/2})\boldsymbol{\Phi}_m^r(-e^{-iu/2})\boldsymbol{P}_m^r(-e^{-iu/2})^\star. \quad \blacksquare
\end{aligned}$$

**Example 4.2** For $r = 1$ we have

$$\begin{aligned}
\boldsymbol{\Phi}_0^1(z) &= 1, \\
6\boldsymbol{\Phi}_1^1(z) &= z^{-1} + 4 + z, \\
120\boldsymbol{\Phi}_2^1(z) &= z^{-2} + 26z^{-1} + 66 + 26z + z^2, \\
5040\boldsymbol{\Phi}_3^1(z) &= z^{-3} + 120z^{-2} + 1191z^{-1} + 2416 + 1191z + 120z^2 + z^3.
\end{aligned}$$

For $r = 2$ we obtain

$$\begin{aligned} 6\mathbf{\Phi}_1^2(z) &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad 30\mathbf{\Phi}_2^2(z) = \begin{pmatrix} 4 & 3z^{-1}+3 \\ 3+3z & z^{-1}+12+z \end{pmatrix}, \\ 560\mathbf{\Phi}_3^2(z) &= \begin{pmatrix} 9z^{-1}+128+9z & 53z^{-1}+80+z \\ z^{-1}+80+53z & 9z^{-1}+128+9z \end{pmatrix}. \quad \spadesuit \end{aligned}$$

Now we want to analyze the structure of $\mathbf{\Phi}_m^r$ using the recursion formula (2.5) for $\hat{\boldsymbol{N}}_m^r$. Therefore we introduce the following $(r,r)$-matrix

$$\boldsymbol{H}_{2m+1}^r := \left(H_\mu^\nu\right)_{\nu,\mu=0}^{r-1} \tag{4.4}$$

with

$$H_\mu^\nu(z) := \sum_{l=-\infty}^{\infty} \mathrm{D}^\nu N_\mu^{2m+1,r}(l) z^l \quad (\mu,\ \nu = 0,1,\ldots,r-1,\ z \in \mathbb{T}), \tag{4.5}$$

where $\mathrm{D}^\nu f$ denotes the $\nu$-th derivative of $f$. For $2m+1-\nu \le r-1$ the functions $\mathrm{D}^\nu N_\mu^{2m+1,r}$ are understood according to the distribution theory. The functions $H_\mu^0$ in (4.5) are called *Euler–Frobenius functions* of $N_\mu^{2m+1,r}$. Therefore we call $\boldsymbol{H}_{2m+1}^r$ the *Euler–Frobenius matrix of* $\boldsymbol{N}_{2m+1}^r$.
For $r = 1$ we obtain the well-known Euler–Frobenius polynomial

$$\boldsymbol{H}_{2m+1}^1(z) = H_{2m+1}^0(z) = \sum_{l=-\infty}^{\infty} N_{2m+1}(l) z^l.$$

For $r = 2$ and $m \ge 1$ the matrix

$$\boldsymbol{H}_{2m+1}^2(z) = \begin{pmatrix} \sum_{k=-\infty}^{\infty} N_0^{2m+1,2}(k) z^k & \sum_{k=-\infty}^{\infty} N_1^{2m+1,2}(k) z^k \\ \sum_{k=-\infty}^{\infty} \mathrm{D} N_0^{2m+1,2}(k) z^k & \sum_{k=-\infty}^{\infty} \mathrm{D} N_1^{2m+1,2}(k) z^k \end{pmatrix}$$

is found. By the Poisson summation formula the matrix $\boldsymbol{H}_{2m+1}^r$ reads for $z = e^{-iu}$ as follows

$$\boldsymbol{H}_{2m+1}^r(e^{-iu}) = \sum_{l=-\infty}^{\infty} \left((i(u+2\pi l))^k\right)_{k=0}^{r-1} \hat{\boldsymbol{N}}_{2m+1}^r(u+2\pi l)^T \quad (u \in \mathbb{R}). \tag{4.6}$$

**Example 4.3** In particular, we have

$$\begin{aligned} H_1^1(z) &= z, \\ 6H_3^1(z) &= z(1+4z+z^2), \\ 120H_5^1(z) &= z(1+26z+66z^2+26z^3+z^4), \end{aligned}$$

$$2H_3^2(z) = z\begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}, \quad 12H_5^2(z) = z\begin{pmatrix} 5+z & 1+5z \\ 10-5z & 5-10z \end{pmatrix}. \quad \spadesuit$$

Later, in Section 5 we will see that the Euler–Frobenius matrix $\boldsymbol{H}_{2m+1}^r$ plays a crucial role in solving the cardinal Hermite spline interpolation problem.
The following connection between $\boldsymbol{H}_{2m+1}^r$ and $\mathbf{\Phi}_m^r$ can be shown:

**Theorem 4.4** *For $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ we have*

$$\boldsymbol{\Phi}_m^r(z) = \boldsymbol{D}_{m,0}^r(z)\, \boldsymbol{D}_r\, \overline{\boldsymbol{H}_{2m+1}^r(z)}\, (\boldsymbol{D}_{m,1}^r(z)^\star)^{-1} \quad (z \in \mathbb{T} \setminus \{1\}) \tag{4.7}$$

*with*

$$\begin{aligned} \boldsymbol{D}_{m,0}^r(z) &:= \boldsymbol{A}_m^r(z)\boldsymbol{A}_{m-1}^r(z)\ldots\boldsymbol{A}_0^r(z), \\ \boldsymbol{D}_{m,1}^r(z) &:= \boldsymbol{A}_{2m+1}^r(z)\boldsymbol{A}_{2m}^r(z)\ldots\boldsymbol{A}_{m+1}^r(z), \end{aligned}$$

*where the $(r,r)$-matrices $\boldsymbol{A}_k^r$ $(k = 0,\ldots,2m+1)$ are defined in* (2.6) – (2.8),

$$\boldsymbol{D}_r := (-1)^{m+1} \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{(-1)^{r-1}}{r-1} \\ 0 & 0 & \ldots & \frac{(-1)^{r-2}}{r-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & (-1)^1 & \ldots & 0 & 0 \\ (-1)^0 & 0 & \ldots & 0 & 0 \end{pmatrix} \quad (r > 1)$$

*and $\boldsymbol{D}_1 := (-1)^{m+1}$. For $z = 1$ we have*

$$\boldsymbol{\Phi}_m^r(1) = \lim_{u\to 0} \boldsymbol{D}_{m,0}^r(e^{-iu})\boldsymbol{D}_r\boldsymbol{H}_{2m+1}^r(e^{iu})(\boldsymbol{D}_{m,1}^r(e^{-iu})^\star)^{-1} \quad (u \in \mathbb{R}).$$

**Proof:** Let $z \neq 1$. The relations (2.5) and (4.3) lead to

$$\begin{aligned} \boldsymbol{\Phi}_m^r(e^{-iu}) &= \sum_{l=-\infty}^{\infty} \hat{\boldsymbol{N}}_m^r(u+2\pi l)\hat{\boldsymbol{N}}_m^r(u+2\pi l)^\star \\ &= \sum_{l=-\infty}^{\infty} \frac{(-i)(u+2\pi l)}{i(u+2\pi l)}\boldsymbol{A}_m^r(e^{-iu})\hat{\boldsymbol{N}}_{m-1}^r(u+2\pi l)\hat{\boldsymbol{N}}_{m+1}^r(u+2\pi l)^\star(\boldsymbol{A}_{m+1}^r(e^{-iu})^\star)^{-1} \\ &= (-1)\boldsymbol{A}_m^r(e^{-iu})\Big(\sum_{l=-\infty}^{\infty} \hat{\boldsymbol{N}}_{m-1}^r(u+2\pi l)\hat{\boldsymbol{N}}_{m+1}^r(u+2\pi l)^\star\Big)(\boldsymbol{A}_{m+1}^r(e^{-iu})^\star)^{-1}. \end{aligned}$$

Repeating this procedure we finally obtain

$$\boldsymbol{\Phi}_m^r(e^{-iu}) = (-1)^{m+1}\boldsymbol{D}_{m,0}^r(e^{-iu})\Big(\sum_{l=-\infty}^{\infty} \hat{\boldsymbol{N}}_{-1}^r(u+2\pi l)\hat{\boldsymbol{N}}_{2m+1}^r(u+2\pi l)^\star\Big)(\boldsymbol{D}_{m,1}^r(e^{-iu})^\star)^{-1}$$

with $\hat{\boldsymbol{N}}_{-1}^r$ defined in (2.4). By

$$\boldsymbol{D}_r\Big((-i(u+2\pi l))^k\Big)_{k=0}^{r-1} = (-1)^{m+1}\,\hat{\boldsymbol{N}}_{-1}^r(u+2\pi l)$$

and by the definition (4.6) of $\boldsymbol{H}_{2m+1}^r$ it follows the assertion for $z \neq 1$.
Since $\boldsymbol{\Phi}_m^r(1)$ and $\boldsymbol{H}_{2m+1}^r(1)$ are well-defined we also have

$$\boldsymbol{\Phi}_m^r(1) = \lim_{u\to 0} \boldsymbol{D}_{m,0}^r(e^{-iu})\boldsymbol{D}_r\boldsymbol{H}_{2m+1}^r(e^{iu})(\boldsymbol{D}_{m,1}^r(e^{-iu})^\star)^{-1} \quad (u \in \mathbb{R}). \quad \blacksquare$$

In particular, Theorem 4.4 yields by $\boldsymbol{\Phi}_m^r(z) = \boldsymbol{\Phi}_m^r(z)^\star$:

**Corollary 4.5** *Let $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ be given. Then for $z \in \mathbb{T}$ we have*

$$\det\, \boldsymbol{\Phi}_m^r(z) = c_{m,r}\, z^{-(m+1)} \det\, \boldsymbol{H}_{2m+1}^r(z)$$

*with a constant $c_{m,r}$ not depending on $z$.*

The proof directly follows from Theorem 4.4 and Remark 2.4.

# 5 Riesz basis and Hermite interpolation

In this section we will show that there is a strong connection between the Riesz basis property for the B-splines $N_k^{m,r}$ $(k \in \mathbb{Z})$ and the unique solvability of the following cardinal Hermite spline interpolation problem:
Let $m \in \mathbb{N}$ and $r \leq m+1$ be fixed. For given data sequences $(y_n^k)_{n=-\infty}^{\infty} \in l_1$ $(k = 0, \ldots, r-1)$ we wish to find a spline function $s \in S_{2m+1,r}(\mathbb{Z}) \bigcap \mathbb{L}_2(\mathbb{R})$ such that the Hermite interpolation conditions

$$\mathrm{D}^k s(n) = y_n^k \qquad (n \in \mathbb{Z};\ k = 0, \ldots, r-1) \tag{5.1}$$

are satisfied.

It is well-known that the problem of existence and uniqueness of solutions of spline interpolation problems on an equidistant lattice unavoidably leads to the Euler–Frobenius functions. However, we want to give a short proof for the following

**Theorem 5.1** *Let $m \in \mathbb{N}$ and $1 \leq r \leq m+1$ be fixed. Then the cardinal Hermite spline interpolation problem* (5.1) *possesses a unique solution $s \in S_{2m+1,r}(\mathbb{Z}) \bigcap \mathbb{L}_2(\mathbb{R})$ for any given data sequences $(y_n^k)_{n=-\infty}^{\infty} \in l_1$ $(k = 0, \ldots, r-1)$ if and only if the Euler–Frobenius matrix $\boldsymbol{H}_{2m+1}^r(z)$ is invertible for $z \in \mathbb{T}$, i.e.*

$$\det \boldsymbol{H}_{2m+1}^r(z) \neq 0 \quad (z \in \mathbb{T}).$$

**Proof:** The spline function $s \in S_{2m+1,r}(\mathbb{Z}) \bigcap \mathbb{L}_2(\mathbb{R})$ can uniquely be represented in the form

$$s = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} a_l^k \, N_k^{2m+1,r}(\cdot - l)$$

with $(a_l^k)_{l=-\infty}^{\infty} \in l_1$ $(k = 0, \ldots, r-1)$. From the interpolation conditions (5.1) it follows for $n \in \mathbb{Z}$

$$\sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} a_l^k \; \mathrm{D}^\nu N_k^{2m+1,r}(n-l) = y_n^\nu \quad (\nu = 0, \ldots, r-1). \tag{5.2}$$

We put

$$\tilde{a}_k := \sum_{l=-\infty}^{\infty} a_l^k e^{-i \cdot l}, \quad \tilde{y}_k := \sum_{l=-\infty}^{\infty} y_l^k e^{-i \cdot l} \quad (k = 0, \ldots, r-1).$$

The functions $\tilde{a}_k$, $\tilde{y}_k$ $(k = 0, \ldots, r-1)$ are continuous. By (5.2) we find

$$\boldsymbol{H}_{2m+1}^r(e^{-i \cdot})(\tilde{a}_k)_{k=0}^{r-1} = (\tilde{y}_k)_{k=0}^{r-1},$$

i.e., the functions $\tilde{a}_k$ $(k = 0, \ldots, r-1)$ are uniquely determined by (5.2) for any data sequences $(y_l^k)_{l=-\infty}^{\infty} \in l_1$ $(k = 0, \ldots, r-1)$ if and only if $\boldsymbol{H}_{2m+1}^r(z)$ is invertible for $z \in \mathbb{T}$. ■

It is well-known that the B-splines satisfy the following Riesz basis property:

**Theorem 5.2** *Let $m \in \mathbb{N}_{\nvDash}$ and $1 \le r \le m+1$ be fixed. The functions $N_k^{m,r}(\cdot - l)$ ($l \in \mathbb{Z}$; $\urcorner = \nvdash, \ldots, \leftthreetimes - \nVdash$) form a Riesz basis (or $L_2(\mathbb{R})$-stable basis) of $S_{m,r}(\mathbb{Z}) \bigcap \mathbb{L}_{\nvDash}(\mathbb{R})$, i.e. there exist positive constants $0 < A \le B < \infty$ such that*

$$A \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} |c_l^k|^2 \le \| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c_l^k N_k^{m,r}(\cdot - l)\|_{L_2(\mathbb{R})}^2 \le B \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} |c_l^k|^2 \tag{5.3}$$

*for all sequences $(c_l^k)_{l=-\infty}^{\infty} \in l_2$ ($k = 0, \ldots, r-1$).*

For a proof we refer to [1]. As for one scaling function we can prove the following equivalence:

**Theorem 5.3** *Let $m \in \mathbb{N}_{\nvDash}$ and $1 \le r \le m+1$. The Riesz basis property* (5.3) *with Riesz bounds $A$ and $B$ is equivalent to the following condition:*
*The eigenvalues $\lambda_k(z)$ ($k = 0, \ldots, r-1$) of the autocorrelation symbol $\mathbf{\Phi}_m^r(z)$ satisfy for $z \in \mathbb{T}$*

$$A \le \lambda_k(z) \le B \quad (k = 0, \ldots, r-1). \tag{5.4}$$

*In particular, the integer translates of $N_k^{m,r}$ ($k = 0, \ldots, r-1$) form a Riesz basis of $S_{m,r}(\mathbb{Z}) \bigcap \mathbb{L}_{\nvDash}(\mathbb{R})$ if and only if the autocorrelation symbol $\mathbf{\Phi}_m^r(z)$ is positive definite on the unit circle $z \in \mathbb{T}$.*

**Proof:** For $(c_l^k)_{l=-\infty}^{\infty} \in l_2$ ($k = 0, \ldots, r-1$) let $C_k$ denote their $2\pi$-periodic symbols, that is

$$C_k := \sum_{l=-\infty}^{\infty} c_l^k \, e^{-i \cdot l} \quad (k = 0, \ldots, r-1).$$

Put

$$\boldsymbol{C} := (C_0, \ldots, C_{r-1})^T.$$

Then by the Parseval identity we find

$$\begin{aligned}
&\| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c_l^k N_k^{m,r}(\cdot - l)\|_{L_2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\boldsymbol{C}(u)^T \hat{\boldsymbol{N}}_m^r(u)|^2 \, \mathrm{d}u \\
= \; & \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} |\boldsymbol{C}(u)^T \hat{\boldsymbol{N}}_m^r(u + 2\pi l)|^2 \, \mathrm{d}u \\
= \; & \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} \boldsymbol{C}(u)^T \hat{\boldsymbol{N}}_m^r(u + 2\pi l) \hat{\boldsymbol{N}}_m^r(u + 2\pi l)^\star \, \overline{\boldsymbol{C}(u)} \, \mathrm{d}u \\
= \; & \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{C}(u)^T \mathbf{\Phi}_m^r(u) \overline{\boldsymbol{C}(u)} \, \mathrm{d}u.
\end{aligned}$$

Considering

$$g_k := \frac{C_k}{(\sum_{k=0}^{r-1} \|C_k\|_{L_2(\mathbb{R})}^2)^{1/2}}, \quad \boldsymbol{g} := (g_0, \ldots, g_{r-1})^T$$

and appealing to the Parseval identity

$$\sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} |c_l^k|^2 = \sum_{k=0}^{r-1} \|c_l^k\|_{l_2}^2 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{r-1} |C_k(u)|^2 \, \mathrm{d}u = \sum_{k=0}^{r-1} \|C_k\|_{L_2(\mathbb{R})}^2$$

it follows that (5.3) is equivalent to the assertion that

$$A \le \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{g}(u)^T \boldsymbol{\Phi}_m^r(e^{-iu}) \overline{\boldsymbol{g}(u)} \ \mathrm{d}u \le B \tag{5.5}$$

is satisfied for any $2\pi$-periodic function $\boldsymbol{g}(u)$ with

$$\|\boldsymbol{g}\|^2_{L_2(\mathbb{R})} := \sum_{k=0}^{r-1} \|g_k\|^2_{L_2(\mathbb{R})} = 1.$$

For the matrix of eigenvalues of the Hermitian matrix $\boldsymbol{\Phi}_m^r(z)$

$$\boldsymbol{\Lambda}(z) := \text{diag } (\lambda_0(z), \ldots, \lambda_{r-1}(z))^T$$

we obtain

$$A \le \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{h}(u)^T \boldsymbol{\Lambda}(e^{-iu}) \overline{\boldsymbol{h}(u)} \ \mathrm{d}u \le B$$

for any $2\pi$-periodic function $\boldsymbol{h}(u) := (h_0(u), \ldots, h_{r-1}(u))^T$ with

$$\|\boldsymbol{h}\|^2 = \sum_{k=0}^{r-1} \|h_k\|^2_{L_2(\mathbb{R})} = 1.$$

Thus, it follows by appropriate choice of $\boldsymbol{h}$ that

$$A \le \lambda_k(z) \le B \quad (z \in \mathbb{T};\ k = 0, \ldots, r-1).$$

To see that (5.5) follows from (5.4) we notice that

$$A \le \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{r-1} |h_k(u)|^2 \lambda_k(e^{-iu}) \ \mathrm{d}u \le B. \qquad \blacksquare$$

**Remark 5.4** A result analogous to Theorem 5.3 can also be found in [6], where more general scaling functions $g_k$ $(k = 0, \ldots, r-1)$ are considered. ♠

As a result of Theorems 5.2 and 5.3 we have

**Corollary 5.5** *Let $m \in \mathbb{N}_0$ and $1 \le r \le m+1$. Then the autocorrelation symbol $\boldsymbol{\Phi}_m^r(z)$ is positive definite for $z \in \mathbb{T}$ and in particular*

$$\det\ \boldsymbol{\Phi}_m^r(z) > 0 \quad (z \in \mathbb{T}).$$

We observe the following interesting equivalence between the Riesz basis property for the integer translates of $N_k^{m,r}$ $(k = 0, \ldots, r-1)$ and the unique solvability of the cardinal Hermite spline interpolation problem (5.1):

**Theorem 5.6** *Let $m \in \mathbb{N}_0$ and $1 \le r \le m+1$ be fixed. The following assertions are equivalent:*

(i) *The integer translates* $N_k^{m,r}(\cdot - l)$ $(l \in \mathbb{Z};\ \daleth = \nvdash, \ldots, \leftthreetimes - \nVdash)$ *form a Riesz basis of* $S_{m,r}(\mathbb{Z}) \bigcap \mathbb{L}_{\nvdash}(\mathbb{R})$.

(ii) *The autocorrelation symbol* $\mathbf{\Phi}_m^r(z)$ *is invertible on the unit circle* $z \in \mathbb{T}$.

(iii) *The Hermite interpolation problem* (5.1) *is uniquely solvable for any given data sequences* $(y_l^k)_{l=-\infty}^{\infty} \in l_1$ $(k = 0, \ldots, r-1)$.

(iv) *The Euler–Frobenius matrix* $\boldsymbol{H}_{2m+1}^r(z)$ *is invertible on the unit circle* $z \in \mathbb{T}$.

**Proof:** The equivalence of (i) and (ii) is shown in Theorem 5.3. The assertions (iii) and (iv) are equivalent by Theorem 5.1. Finally, from Corollary 4.5 it follows that the autocorrelation symbol $\mathbf{\Phi}_m^r(z)$ is invertible on $z \in \mathbb{T}$ if and only if the Euler–Frobenius matrix $\boldsymbol{H}_{2m+1}^r(z)$ is invertible on $z \in \mathbb{T}$. ∎

Theorem 5.6 and Corollary 5.5 imply the following important

**Corollary 5.7** *Let* $m \in \mathbb{N}_{\nvdash}$ *and* $1 \le r \le m+1$ *be fixed. Then the Hermite interpolation problem* (5.1) *is uniquely solvable for any given data sequences* $(y_l^k)_{l=-\infty}^{\infty} \in l_1$ $(k = 0, \ldots, r-1)$.

Finally, we want to compare our result with known results in the literature. A similar cardinal Hermite spline interpolation problem in $S_{m,r}(\mathbb{Z})$, but for $l_p$ data sequences and data sequences with power growth, was firstly considered by P.R. Lipow and I.J. Schoenberg [9]. The unique solvability of the cardinal Hermite interpolation problem was proved in [9] by showing that the reciprocal polynomial $\Omega_{2m+1}^r(z)$ defined by the determinant of a $(2m+1-r, 2m+1-r)$-matrix

$$\Omega_{2m+1}^r(z) := \det \begin{pmatrix} \binom{r}{0} & \binom{r}{1} & \ldots & \binom{r}{r-1} & 1-z & 0 & \ldots & 0 \\ \binom{r+1}{0} & \binom{r+1}{1} & \ldots & \binom{r+1}{r-1} & \binom{r+1}{r} & 1-z & \ddots & \vdots \\ \vdots & \vdots & & & & & \ddots & 0 \\ \binom{2m+1-r}{0} & \binom{2m+1-r}{1} & \ldots & & & \ldots & \binom{2m+1-r}{2m-r} & 1-z \\ \binom{2m+2-r}{0} & \binom{2m+2-r}{1} & \ldots & & & \ldots & \binom{2m+2-r}{2m-r} & \binom{2m+2-r}{2m+1-r} \\ \vdots & \vdots & & & & & \vdots & \vdots \\ \binom{2m+1}{0} & \binom{2m+1}{1} & \ldots & \ldots & \ldots & \ldots & \binom{2m+1}{2m-r} & \binom{2m+1}{2m+1-r} \end{pmatrix}$$

only possesses real, simple zeros. In particular, it follows the needed assertion that $\Omega_{2m+1}^r(z)$ does not vanish on the unit circle $z \in \mathbb{T}$. The proof in [9] is rather difficult and uses the theory of oscillating matrices.

In [8] (Theorem 4) it could be shown that the determinant of the matrix

$$\mathbf{\Delta}_{2m+1}^r(z) := \left( \mathrm{D}^\nu\, H_{2m+1-\mu}^1(z) \right)_{\nu,\mu=0}^{r-1} \quad (z \in \mathbb{T})$$

defined by the Euler–Frobenius polynomials and their derivatives

$$\mathrm{D}^\nu H_n^1(z) := \sum_{l=-\infty}^{\infty} \mathrm{D}^\nu N_0^{n,1}(l)\, z^l \quad (n = 2m+2-r, \ldots, 2m+1;\ \nu = 0, \ldots, r-1)$$

satisfies for $1 \le r \le m+1$ the relation

$$\det \boldsymbol{\Delta}^r_{2m+1}(z) = c_r\, z^r\, (1-z)^{(r-1)(4m+4-r)/2}\, \Omega^r_{2m+1}(z)$$

with

$$c_r := (-1)^{\lfloor r/2 \rfloor}\, 1!\, 2!\, 3!\, \ldots\, (r-1)!\,.$$

The connection between $\boldsymbol{\Delta}^r_{2m+1}$ and the Euler–Frobenius matrix $\boldsymbol{H}^r_{2m+1}$ can easily be shown.

**Lemma 5.8** *Let $m \in \mathbb{N}_{\not{r}}$ and $1 \le r \le m+1$ be fixed. Then we have*

$$\det \boldsymbol{\Delta}^r_{2m+1}(z) = c_{m,r}\, (1-z)^{(r-1)(4m+4-r)/2} \det \boldsymbol{H}^r_{2m+1}(z) \quad (z \in \mathbb{T}), \tag{5.6}$$

*where the constant $c_{m,r}$ does not depend on $z$.*

**Proof:**
1. The Poisson summation formula yields for $u \in \mathbb{R}$

$$\mathrm{D}^\nu H^1_n(e^{-iu}) = \sum_{l=-\infty}^{\infty} \mathrm{D}^\nu N^{n,1}_0(l)\, e^{-iul} \;=\; \sum_{l=-\infty}^{\infty} (i(u+2\pi l))^\nu\, \hat{N}^{n,1}_0(u+2\pi l)$$
$$(\nu = 0, \ldots, r-1;\; n = 2m+2-r, \ldots, 2m+1).$$

Thus, the matrix $\boldsymbol{\Delta}^r_{2m+1}(e^{-iu})$ can be written as

$$\boldsymbol{\Delta}^r_{2m+1}(e^{-iu}) = \sum_{l=-\infty}^{\infty} \Big((i(u+2\pi l))^k\Big)_{k=0}^{r-1} \hat{\boldsymbol{M}}^r_{2m+1}(u+2\pi l)^T$$

with the vector of normalized B-splines of defect 1

$$\begin{aligned}
\hat{\boldsymbol{M}}^r_{2m+1}(u+2\pi l) &:= \Big(\hat{N}^{2m+1,1}_0(u+2\pi l), \ldots, \hat{N}^{2m+2-r,1}_0(u+2\pi l)\Big)^T \\
&= \begin{pmatrix} (1-z)^{2m+2} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & (1-z)^{2m+3-r} \end{pmatrix} \begin{pmatrix} (i(u+2\pi l))^{-2m-2} \\ \vdots \\ (i(u+2\pi l))^{-2m-3+r} \end{pmatrix}
\end{aligned}$$

with $z = e^{-iu}$. For the determinant of $\boldsymbol{\Delta}^r_{2m+1}(z)$ it follows

$$\begin{aligned}
&\det \boldsymbol{\Delta}^r_{2m+1}(z) \\
&= (1-z)^{(4m+5-r)r/2} \det \left( \sum_{l=-\infty}^{\infty} \Big((i(u+2\pi l))^k\Big)_{k=0}^{r-1} \Big((i(u+2\pi l))^k\Big)_{k=-2m-2}^{-2m-3+r\,T} \right).
\end{aligned} \tag{5.7}$$

2. Using the recursion formula (2.5) we obtain with $z = e^{-iu}$ that

$$\begin{aligned}
\hat{\boldsymbol{N}}^r_{2m+1}(u+2\pi l) &= \frac{1}{(i(u+2\pi l))^{2m+2}} \boldsymbol{A}^r_{2m+1}(z) \ldots \boldsymbol{A}^r_0(z)\, \hat{\boldsymbol{N}}^r_{-1}(u+2\pi l) \\
&= \boldsymbol{A}^r_{2m+1}(z) \ldots \boldsymbol{A}^r_0(z) \begin{pmatrix} 0 & \ldots & \frac{1}{r-1} \\ \vdots & & \vdots \\ 1 & \ldots & 0 \end{pmatrix} \begin{pmatrix} (i(u+2\pi l))^{-2m-2} \\ \vdots \\ (i(u+2\pi l))^{-2m-3+r} \end{pmatrix}
\end{aligned}$$

with matrices $\boldsymbol{A}_n^r$ $(n = 0, \ldots, 2m+1)$ defined in (2.6) – (2.8). By definition (4.6) of the Euler–Frobenius matrix $\boldsymbol{H}_{2m+1}^r$ and Remark 2.4 we find for the determinant with $z = e^{-iu}$

$$\begin{aligned} &\det \boldsymbol{H}_{2m+1}^r(z) \\ &= c'_{m,r}\,(1-z)^{2m+2} \det \left( \sum_{l=-\infty}^{\infty} \Big((i(u+2\pi l))^k\Big)_{k=0}^{r-1} \Big((i(u+2\pi l))^k\Big)_{k=-2m-2}^{-2m-3+r\;T} \right), \quad (5.8) \end{aligned}$$

where the constant $c'_{m,r}$ does not depend on $z$. Comparing (5.7) and (5.8) it follows the assertion. ■

Lemma 5.8 yields that

$$\det \boldsymbol{H}_{2m+1}^r(z) = c_{m,r}\, z^r\, \Omega_{2m+1}^r(z) \quad (z \in \mathbb{T})$$

with a constant $c_{m,r}$ not depending on $z$. That is, the invertibility of $\boldsymbol{H}_{2m+1}^r(z)$ on the unit circle $z \in \mathbb{T}$ is equivalent to the assertion that $\Omega_{2m+1}^r(z)$ does not vanish on the unit circle $z \in \mathbb{T}$, proved by P.R. Lipow and I.J. Schoenberg.

**Remark 5.9** It is well-known that the invertibility of the Euler–Frobenius matrix $\boldsymbol{H}_{2m+1}^r(z)$ for $z \in \mathbb{T}$ also causes the existence and uniqueness of solutions in the case of periodic Hermite spline interpolation (cf. [10, 11, 12]). Thus, from the Riesz basis property of $N_k^{m,r}(\cdot - l)$ $(l \in \mathbb{Z};\ \urcorner = \nvdash, \ldots, \diagdown - \nVdash)$ in $S_{m,r}(\mathbb{Z}) \bigcap \mathbb{L}_{\nvDash}(\mathbb{R})$ it also follows that the periodic Hermite spline interpolation problem is uniquely solvable. ♠

# Acknowledgement

This research was supported by the Deutsche Forschungsgemeinschaft.

# References

[1] C. de Boor, *Splines as linear combinations of B-splines. A survey*, in Approximation Theory II (G. Lorentz, C.K. Chui, L. Schumaker, eds.), Academic Press, New York, 1976, 1 – 47.

[2] C.K. Chui, "An Introduction to Wavelets", Academic Press, Boston, 1992.

[3] C.K. Chui and J.Z. Wang, *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc. **330** (1992), 903 – 915.

[4] T.N.T. Goodman, *Interpolatory Hermite spline wavelets*, J. Approx. Theory, to appear.

[5] T.N.T. Goodman and S.L. Lee, *Wavelets of multiplicity* $r$, Applied Analysis Report AA/921, University of Dundee, 1992.

[6] T.N.T. Goodman, S.L. Lee and W.S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc. **338** (1993), 639 – 654.

[7] S.L. Lee, *Fourier transforms of B-splines and fundamental splines for cardinal Hermite interpolation*, Proc. Amer. Math. Soc. **57** (1976), 291 – 296.

[8] S.L. Lee and A. Sharma, *Cardinal lacunary interpolation by g-splines I. The characteristic polynomials*, J. Approx. Theory **16** (1976), 85 – 96.

[9] P.R. Lipow and I.J. Schoenberg, *Cardinal interpolation and spline functions III. Cardinal Hermite interpolation*, Linear Algebra Appl. **6** (1973), 273 – 304.

[10] G. Meinardus and G. Merz, *Hermite–Interpolation mit periodischen Spline–Funktionen*, Numerical Methods of Approximation Theory (L. Collatz, G. Meinardus and H. Werner, eds.), Birkhäuser, Basel (1980), 200 – 210.

[11] G. Merz and W. Sippel, *Zur Konstruktion periodischer Hermite–Interpolationssplines bei äquidistanter Knotenverteilung*, J. Approx. Theory **54** (1988), 92 – 106.

[12] G. Plonka and M. Tasche, *Efficient algorithms for periodic Hermite spline interpolation*, Math. Comp. **58** (1992), 693 – 703.

[13] I.J. Schoenberg, *Cardinal interpolation and spline functions*, J. Approx. Theory **2** (1969), 288 – 297.

[14] I.J. Schoenberg and A. Sharma, *Cardinal interpolation and spline functions V. The B-splines for cardinal Hermite interpolation*, Linear Algebra Appl. **7** (1973), 1 – 42.