Optimal shift parameters for periodic spline interpolation

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Dedicated to Professor M. Tasche on the occasion of his 50th birthday

Using the exponential Euler spline, restricted on the unit circle, we sketch a unified approach to the periodic spline interpolation with shifted interpolation nodes. Mainly we are interested in the optimal choice of the shift parameter τ such that the corresponding interpolatory matrix possesses minimal condition or such that the related interpolation operator has minimal norm. We show that $\tau = 0$ is optimal in both cases. This improves known results of G. Merz, M. Reimer – D. Siepmann and F.B. Richards.

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1 Introduction

Let the centered cardinal B-spline M_m of degree m be defined as the m-fold convolution of

$$M_0(x) := \begin{cases} 1 & |x| < 1/2, \\ 1/2 & |x| = 1/2, \\ 0 & \text{otherwise} \end{cases}$$

with itself, i.e., for $m \geq 1$ we have

$$M_m(x) := \int_{-1/2}^{1/2} M_{m-1}(x+t)\ \mathrm{d}t \qquad (x \in \mathbb{R}).$$

Further let $S_m(\mathbb{Z})$ be the set of all linear combinations of shifts of M_m, i.e.,

$$S_m(\mathbb{Z}) := \text{span}\ \{M_m(\cdot - j);\ j \in \mathbb{Z}\}.$$

For fixed $N \in \mathbb{N}$ we introduce the subset $S_m^N(\mathbb{Z})$ of all N-periodic spline functions

$$S_m^N(\mathbb{Z}) := \{s \in S_m(\mathbb{Z});\ s(\cdot + N) = s\}.$$

We shall denote the Banach space of all p-integrable N-periodic functions by L_p^N ($1 \le p < \infty$). For any $f \in L_p^N$ we have

$$\|f\|_p := \Big(\int_0^N |f(t)|^p \, \mathrm{d}t \Big)^{1/p} \quad (1 \le p < \infty).$$

Now the following N-periodic spline interpolation problem is considered:
For a given N-periodic real data sequence $\{y_j\}_{j=-\infty}^{\infty}$ with $y_j = y_{j+N}$ ($j \in \mathbb{Z}$) and a fixed shift parameter $\tau \in (-1/2, 1/2]$ we try to find a spline function $s \in S_m^N(\mathbb{Z})$ such that

$$s(k+\tau) = y_k \qquad (k \in \mathbb{Z}). \tag{1}$$

The data sequence can be completely described by the column vector

$$\boldsymbol{y} := (y_j)_{j=0}^{N-1} \in \mathbb{R}^N.$$

Using fast Fourier transform we shall give a simple algorithm for the computation of the spline interpolant. Furthermore, the condition of the interpolatory matrix is investigated. We are mainly interested in the dependence of the matrix condition on the shift parameter τ. Finally, the norm of the spline interpolation operator $\mathcal{L}_{m,\tau}^N : \mathbb{R}^N \to L_1^N$, given by

$$\mathcal{L}_{m,\tau}^N \boldsymbol{y} := s,$$

where $s \in S_m^N(\mathbb{Z}) \subset L_1^N$ satisfies the interpolation conditions (1), is considered. It is shown that for $\tau = 0$ the condition of the interpolatory matrix as well as the norm of the corresponding spline interpolation operator are minimal.

In the following we use standard notations. First we recall some facts concerning circulant matrices, which form the backround in Section 3 (cf. [2]). For $\boldsymbol{a} := (a_j)_{j=0}^{N-1} \in \mathbb{R}^N$ the corresponding circulant matrix of the order N is defined by

$$\text{circ } \boldsymbol{a} := (a_{j-k})_{j,k=0}^{N-1} = \begin{pmatrix} a_0 & a_{N-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{pmatrix}.$$

Note that the subscripts have to be calculated modulo N here and in the following. Let $\boldsymbol{e}_k := (\delta_{jk})_{j=0}^{N-1} (k = 0, ..., N-1)$, where δ_{jk} denotes the Kronecker symbol. The matrix

$$\boldsymbol{V} := \text{circ } \boldsymbol{e}_1$$

is called the *fundamental circulant matrix*. Then we have

$$\boldsymbol{V}^k = \text{circ } \boldsymbol{e}_k \; (k = 0, ..., N-1), \quad \boldsymbol{V}^N = \boldsymbol{I}, \quad \boldsymbol{V}^T = \boldsymbol{V}^{-1} = \boldsymbol{V}^{N-1}.$$

Introducing the *representing polynomial* $a(z) := \sum_{j=0}^{N-1} a_j z^j$ ($z \in \mathbb{C}$) of the circulant matrix circ $\boldsymbol{a}$ it follows that

$$\text{circ } \boldsymbol{a} = \sum_{j=0}^{N-1} a_j \boldsymbol{V}^j = a(\boldsymbol{V}).$$

Note that all circulant matrices of order N commute. We have

$$\text{circ}\ \boldsymbol{a} = a(\boldsymbol{V}) = \boldsymbol{F}_N^{-1}\boldsymbol{D}_a\boldsymbol{F}_N = \frac{1}{N}\overline{\boldsymbol{F}}_N\boldsymbol{D}_a\boldsymbol{F}_N \tag{2}$$

with

$$\boldsymbol{D}_a = \text{diag}\ \left(a(w_N^j)\right)_{j=0}^{N-1} \quad (w_N := \exp(-2\pi i/N)),$$

where $\boldsymbol{F}_N := (w_N^{jk})_{j,k=0}^{N-1}$ denotes the Fourier matrix of order N. Therefore the eigenvalues λ_j $(j = 0, ..., N-1)$ of the circulant matrix circ $\boldsymbol{a}$ are given by

$$\lambda_j = a(w_N^j) \qquad (j = 0, ..., N-1).$$

We introduce the *j-th Bernoulli polynomial* $B_j(\cdot)$ on $[0, 1]$ recursively by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}B_{j+1}(t) &= (j+1)B_j(t) \quad (j \in \mathbb{N}_0), \qquad B_0(t) \equiv 1 \quad (t \in [0,1]), \\ \int_0^1 B_j(t)\ \mathrm{d}t &= 0 \quad (j \in \mathbb{N}) \end{aligned}$$

and the *j-th Euler polynomial* $E_j(\cdot)$ on $[0, 1]$ by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_{j+1}(t) &= (j+1)\ E_j(t) \quad (j \in \mathbb{N}_0), \qquad E_0(t) \equiv 1 \quad (t \in [0,1]), \\ E_j(0) + E_j(1) &= 0 \quad (j \in \mathbb{N}). \end{aligned}$$

The behaviour of these polynomials is well-known. The Euler numbers E_j and Bernoulli numbers B_j are given by

$$E_j := 2^j E_j(1/2) \qquad B_j := B_j(0) \qquad (j \in \mathbb{N}_0). \tag{3}$$

The following identity holds (cf. [1]):

$$E_j(0) = -E_j(1) = -\frac{2(2^{j+1}-1)}{(j+1)}B_{j+1} \qquad (j \in \mathbb{N}). \tag{4}$$

2 Solution of the interpolation problem by discrete Fourier transform

The symbol of the considered spline interpolation problem (1) has been investigated in [3]. We shall summarize some important properties of this function.

The exponential Euler spline (cf. [13]) is defined for $z \in \mathbb{C}$ by

$$\Phi_m(x, z) := \sum_{j=-\infty}^{\infty} M_m(x+j)\, z^j \qquad (x \in \mathbb{R}).$$

We are especially interested in $\Phi_m(x,\cdot)$ on the unit circle. The function

$$\varphi_m(x,\cdot) := \Phi_m(x, e^{-i\,\cdot})$$

is called the *symbol of cardinal spline interpolation with shift parameter x*. Then the following properties of the symbol hold:

THEOREM 2.1
For $m \in \mathbb{N}$, $x \in \mathbb{R}$ and $u \in (-\pi,\pi]$ we have
(i) $$|\varphi_m(x,u)| \le 1, \qquad \varphi_m(x,0) = 1,$$
(ii)

$$\begin{aligned} \varphi_m(x, 2\pi k + u) &= \varphi_m(x,u) \qquad (k \in \mathbb{Z}), \\ \varphi_m(x \pm 1, u) &= e^{\pm iu} \varphi_m(x,u), \\ \varphi_m(x,-u) &= \varphi_m(-x,u) = \overline{\varphi_m(x,u)}, \\ e^{-iu}\varphi_m(1/2 + x, u) &= \overline{\varphi_m(1/2 - x, u)}, \end{aligned}$$

(iii)
$$\frac{\partial}{\partial x}\varphi_m(x,u) = (1 - e^{-iu})\,\varphi_{m-1}(x + 1/2, u) \qquad (m \ge 2).$$
Now let $m \in \mathbb{N}$ and $u_0 \in (0,\pi)$ be fixed. Then

(iv) *The function* $\arg\varphi_m(\cdot, u_0)$ *is strictly increasing on* $[0,1]$*. In particular,*

$$\arg\varphi_m(0,u_0) = 0, \qquad \arg\varphi_m(1/2, u_0) = u_0/2, \qquad \arg\varphi_m(1,u_0) = u_0.$$

(v) *The function* $|\varphi_m(\cdot,u_0)|$ *is strictly decreasing on* $[0,1/2]$*. In particular, we have* $|\varphi_m(x,u_0)| > 0$ *for* $x \in \mathbb{R}$*.*

(vi) *The following inequalities hold:*

$$0 < \arg\varphi_m(x,u_0) < xu_0 \quad \textit{for} \quad 0 < x < 1/2,$$

$$xu_0 < \arg\varphi_m(x,u_0) < u_0 \quad \textit{for} \quad 1/2 < x < 1.$$

Further we have:

(vii) *Let $m \in \mathbb{N}$ and $x_0 \in \mathbb{R}$ be fixed. Then* $|\varphi_m(x_0,\cdot)|$ *is strictly decreasing on* $[0,\pi]$*.*

(viii) *Let $x_0 \in [0,1/2]$ and $u_0 \in (0,\pi)$ be fixed. Then for $m \in \mathbb{N}$,*

$$|\varphi_m(x_0,u_0)| \le |\varphi_{m-1}(x_0,u_0)|.$$

(ix) *The function* $\varphi_m(\cdot,\pi)$ *is real–valued and strictly decreasing on* $[0,1]$ *with* $\varphi_m(1/2,\pi) = 0$*. We have for* $x \in [0,1/2]$

$$\varphi_m(x,\pi) = \begin{cases} \dfrac{(-1)^{(m+1)/2}\,2^m}{m!} E_m(x) & m \textit{ odd}, \\ \dfrac{(-1)^{m/2}\,2^m}{m!} E_m(x+1/2) & m \textit{ even}. \end{cases}$$

Proof

The identities (i) – (iii) are immediate consequences of the definition of φ_m. For a proof of (iv) – (viii) we refer to [3]. The connection between $\varphi_m(\cdot,\pi)$ and the Euler polynomials E_m has already been pointed out in [3] in a less precise fashion. Therefore we shall give a short proof of (ix) here.
For $x \in [0,1/2]$ let

$$h_m(x,\pi) := \begin{cases} (-1)^{(m+1)/2}\varphi_m(x,\pi) & m \text{ odd}, \\ (-1)^{m/2}\varphi_m(x-1/2,\pi) & m \text{ even}. \end{cases}$$

Then the assertion reads $h_m(x,\pi) = 2^m\, E_m(x)/m!$. From Theorem 2.1 (ii) and (iii) it follows

$$\frac{\mathrm{d}}{\mathrm{d}x} h_m(x,\pi) = 2\, h_{m-1}(x,\pi) \quad (m \in \mathbb{N},\ m \geq 2)$$

and $h_m(1,\pi) = -h_m(0,\pi)$.
For $m = 1$ we have with $h_1(x,\pi) = 2x-1$ the identity $h_1(x,\pi) = 2\,E_m(x)$. Now we assume $h_k(x,\pi) = 2^k\, E_k(x)/k!$ for $k \geq 1$. Then we find for $h_{k+1}(x,\pi)$:

$$\frac{\mathrm{d}}{\mathrm{d}x} h_{k+1}(x,\pi) = 2\, h_k(x,\pi) = \frac{2^{k+1}}{k!} E_k(x) = \frac{2^{k+1}}{(k+1)!}\frac{\mathrm{d}}{\mathrm{d}x} E_{k+1}(x),$$

i.e., there is a constant $c \in \mathbb{R}$ with

$$c := h_{k+1}(x,\pi) - \frac{2^{k+1}}{(k+1)!} E_{k+1}(x) \quad (x \in [0,1]).$$

Replacing x by 0 and 1 we find by

$$h_{k+1}(1,\pi) + h_{k+1}(0,\pi) = E_{k+1}(1) + E_{k+1}(0) = 0$$

$c = -c$ and thus $c = 0$. ■

With the help of the symbol $\varphi_m(\tau,\cdot)$ the periodic fundamental spline can be represented as follows.

THEOREM 2.2
Let $m, N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2]$ be given such that

$$\varphi_m(\tau, 2\pi j/N) \neq 0 \qquad (j = 0, ..., N-1). \tag{5}$$

Then $L_m^N \in S_m^N(\mathbb{Z})$,

$$L_m^N(x) := \frac{1}{N}\sum_{j=0}^{N-1} \frac{\varphi_m(x, 2\pi j/N)}{\varphi_m(\tau, 2\pi j/N)} \quad (x \in \mathbb{R}), \tag{6}$$

is a periodic fundamental spline with the shift parameter τ, i.e.,

$$L_m^N(k+\tau) = \delta_{0k}^N \qquad (k \in \mathbb{Z})$$

with

$$\delta_{0k}^N := \begin{cases} 1 & k \equiv 0 \,(\bmod N) \\ 0 & k \not\equiv 0 \,(\bmod N). \end{cases}$$

Proof

Since $\varphi_m(\tau, 2\pi j/N) \neq 0\,(j = 0, ..., N-1)$, the function L_m^N is well-defined. By Theorem 2.1 (ii) we have for $k \in \mathbb{Z}$ and $w_N := \exp(-2\pi i/N)$

$$L_m^N(k+\tau) = \frac{1}{N} \sum_{j=0}^{N-1} \frac{\varphi_m(\tau, 2\pi j/N)\, w_N^{-jk}}{\varphi_m(\tau, 2\pi j/N)} = \frac{1}{N} \sum_{j=0}^{N-1} w_N^{-jk} = \delta_{0k}^N. \quad \blacksquare$$

Now let $m, N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2]$ with $\varphi_m(\tau, 2\pi j/N) \neq 0$ $(j = 0, ..., N-1)$ be given. Then a continuous solution $s \in S_m^N(\mathbb{Z})$ of the periodic spline interpolation problem (1) reads

$$s(x) = \sum_{j=0}^{N-1} y_j L_m^N(x-j) \qquad (x \in \mathbb{R}).$$

Let $\boldsymbol{y} = (y_j)_{j=0}^{N-1}$ be the given real data vector. With

$$\boldsymbol{s}(t) := \Big(s(t+k)\Big)_{k=0}^{N-1}, \quad \boldsymbol{l}_m(t) := \Big(L_m^N(t+k)\Big)_{k=0}^{N-1} \quad (t \in (0,1])$$

the convolution equation

$$\boldsymbol{s}(t) = \boldsymbol{y} * \boldsymbol{l}_m(t)$$

is obtained. Using discrete Fourier transform we find

$$\boldsymbol{F}_N \boldsymbol{s}(t) = \hat{\boldsymbol{y}} \circ \Big(\boldsymbol{F}_N \boldsymbol{l}_m(t)\Big) = \hat{\boldsymbol{y}} \circ \left(\frac{\varphi_m(t, 2\pi j/N)}{\varphi_m(\tau, 2\pi j/N)}\right)_{j=0}^{N-1}, \tag{7}$$

where $\hat{\boldsymbol{y}} := \boldsymbol{F}_N \boldsymbol{y} = (\hat{y}_j)_{j=0}^{N-1}$. Here $\circ$ denotes componentwise multiplication. Let P_m be the N-periodization of the B-spline M_m, i.e.

$$P_m := \sum_{l=-\infty}^{\infty} M_m(\cdot + lN).$$

Then it follows that

$$\varphi_m(x, 2\pi j/N) = \sum_{k=0}^{N-1} P_m(x+k) w_N^{jk} \quad (x \in \mathbb{R}).$$

With $\boldsymbol{p}_m(x) := \Big(P_m(x+k)\Big)_{k=0}^{N-1}$ we find

$$\Big(\varphi_m(x, 2\pi j/N)\Big)_{j=0}^{N-1} = \boldsymbol{F}_N \boldsymbol{p}_m(x) \quad (x \in \mathbb{R}).$$

Thus we obtain the following algorithm:

ALGORITHM 2.3 (*Computation of $s(t+k)$ for $t \in (0,1]$ and $k = 0, ..., N-1$):*

Input: $m, N \in \mathbb{N}$, (N *power of* 2),
$\tau \in (-1/2, 1/2]$ *with* (5),
$\boldsymbol{y} \in \mathbb{R}^N$,
$t \in (0,1]$.

1. *Compute* $\boldsymbol{p}_m(t)$ *and* $\boldsymbol{p}_m(\tau)$ *by B-spline recursion formula.*

2. *Compute*

$$\Big(\varphi_m(t, 2\pi j/N)\Big)_{j=0}^{N-1} := \boldsymbol{F}_N \boldsymbol{p}_m(t), \quad \Big(\varphi_m(\tau, 2\pi j/N)\Big)_{j=0}^{N-1} := \boldsymbol{F}_N \boldsymbol{p}_m(\tau).$$

3. *Compute by fast Fourier transform*

$$\hat{\boldsymbol{y}} := \boldsymbol{F}_N \boldsymbol{y} \qquad (\hat{\boldsymbol{y}} = (\hat{y}_j)_{j=0}^{N-1}).$$

4. *Form for* $j = 0, ..., N-1$,

$$\hat{s}_j := \frac{\hat{y}_j \, \varphi_m(t, 2\pi j/N)}{\varphi_m(\tau, 2\pi j/N)}.$$

5. *Compute by fast Fourier transform*

$$\boldsymbol{s} := \boldsymbol{F}_N^{-1} \hat{\boldsymbol{s}} \qquad (\hat{\boldsymbol{s}} := (\hat{s}_j)_{j=0}^{N-1}).$$

Output: $\boldsymbol{s} = \Big(s(t+k)\Big)_{k=0}^{N-1}$.

The vector $\Big(\varphi_m(\tau, 2\pi j/N)\Big)_{j=0}^{N-1}$ can be precomputed. The vector $\Big(\varphi_m(t, 2\pi j/N)\Big)_{j=0}^{N-1}$ can be directly obtained in $O(mN)$ arithmetic operations. In step 3 and step 5 we have to perform a discrete Fourier transform of length N.

Figure 1

Figure 1: Periodic fundamental spline L_5^{16} for the shift parameter $\tau = 0$.

Figure 2

Figure 2: Periodic fundamental spline L_5^{16} for the shift parmeter $\tau = 0.3$.

3 Condition of the interpolatory matrix

Now we are interested in the optimal choice of the shift parameter τ such that the related interpolatory matrix has minimal condition. First let us recall the known result on the existence and uniqueness of solutions of the spline interpolation problem (1) (cf. [9]).

THEOREM 3.1
Let $m, N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2]$ be fixed. Then the N-periodic spline interpolation problem (1) *is uniquely solvable for any given N-periodic real data $\{y_j\}_{j=-\infty}^{\infty}$ if and only if*

$$\varphi_m(\tau, 2\pi j/N) \neq 0 \qquad (j = 0, ..., N-1). \tag{8}$$

The property (8) *is satisfied if and only if one of the following conditions hold:*
(i) *N is odd;*
(ii) *N is even and $\tau \neq 1/2$.*

Proof
The assertion directly follows from Theorem 2.1 (ii), (v) and (ix). ■

We assume that $\tau \in (-1/2, 1/2)$. By (7) the solution vector can be represented as follows:

$$\begin{aligned} \boldsymbol{s}(t) &= \boldsymbol{F}_N^{-1}\left[\left(\frac{\varphi_m(t, 2\pi j/N)}{\varphi_m(\tau, 2\pi j/N)}\right)_{j=0}^{N-1} \circ \boldsymbol{F}_N \boldsymbol{y}\right] \\ &= \boldsymbol{F}_N^{-1} \boldsymbol{D}_m(t,\tau) \boldsymbol{F}_N \boldsymbol{y} \end{aligned}$$

with

$$\boldsymbol{D}_m(t,\tau) := \operatorname{diag}\left(\frac{\varphi_m(t, 2\pi j/N)}{\varphi_m(\tau, 2\pi j/N)}\right)_{j=0}^{N-1}.$$

By (2) we find

$$\boldsymbol{F}_N^{-1} \boldsymbol{D}_m(t,\tau) \boldsymbol{F}_N = \Phi_m(\tau, \boldsymbol{V})^{-1} \Phi_m(t, \boldsymbol{V}),$$

where

$$\Phi_m(t, \boldsymbol{V}) := \sum_{j=0}^{N-1} P_m(j+t) \boldsymbol{V}^j = \sum_{j=-\infty}^{\infty} M_m(j+t) \boldsymbol{V}^j$$

is the exponential Euler spline with the argument $\boldsymbol{V}$. Thus the following system of linear equations has to be solved:

$$\boldsymbol{s}(t) = \Phi_m(\tau, \boldsymbol{V})^{-1} \Phi_m(t, \boldsymbol{V}) \boldsymbol{y},$$

where $\Phi_m(\tau, \boldsymbol{V})^{-1}$ and $\Phi_m(t, \boldsymbol{V})$ are circulant matrices. The stability of the algorithm is mainly determined by the condition of $\Phi_m(\tau, \boldsymbol{V})$.

In the following let $\|\boldsymbol{A}\|_\infty$ be the matrix ∞-norm, $\|\boldsymbol{A}\|_1$ the matrix 1-norm and $\|\boldsymbol{A}\|_2$ the matrix 2-norm of the (N, N)-matrix $\boldsymbol{A} := (a_{ij})_{i,j=0}^{N-1}$, i.e.,

$$\|\boldsymbol{A}\|_\infty \ := \ \max\{\sum_{j=0}^{N-1} |a_{ij}|;\ i = 0, ..., N-1\},$$

$$\begin{aligned}
\|\boldsymbol{A}\|_1 &:= \max\{\sum_{i=0}^{N-1} |a_{ij}|;\ j=0,...,N-1\},\\
\|\boldsymbol{A}\|_2 &:= \sqrt{\rho(\boldsymbol{A}^T\boldsymbol{A})} \quad \text{with} \quad \rho(\boldsymbol{A}) := \max\{|\alpha|;\ \boldsymbol{A}\boldsymbol{y} = \alpha\boldsymbol{y};\ \boldsymbol{y} \neq \boldsymbol{o}\}.
\end{aligned}$$

The condition of $\boldsymbol{A}$ is defined by

$$\mathrm{cond}_p\boldsymbol{A} := \|\boldsymbol{A}^{-1}\|_p\|\boldsymbol{A}\|_p \qquad (p = 1, 2, \infty).$$

We shall investigate for which $\tau \in (-1/2, 1/2)$ the condition of $\Phi_m(\tau, \boldsymbol{V})$ is minimal. First we show how to estimate the norm $\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_p$ ($p = 1, 2, \infty$) with the help of the symbol $\varphi_m(\tau, \cdot)$.

LEMMA 3.2
Let $m, N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2)$. Then for even N we have

$$\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_\infty = \|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_1 = \|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_2 = |\varphi_m(\tau, \pi)|^{-1}$$

and for odd N,

$$\begin{aligned}
\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_\infty &= \|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_1 < |\varphi_m(\tau, \pi)|^{-1},\\
\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_2 &= |\varphi_m(\tau, \pi(N-1)/N)|^{-1} < |\varphi_m(\tau, \pi)|^{-1}.
\end{aligned}$$

Proof

1. By Theorem 3.1 the matrix $\Phi_m(\tau, \boldsymbol{V})$ is regular for $\tau \in (-1/2, 1/2)$. The inverse matrix $\Phi_m(\tau, \boldsymbol{V})^{-1}$ is circulant. Thus $\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_\infty = \|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_1$.
Observe that $\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_p = \|\Phi_m(-\tau, \boldsymbol{V})^{-1}\|_p$ for $p = 1, 2, \infty$, since

$$\Phi_m(-\tau, \boldsymbol{V}) = \sum_{j=-\infty}^{\infty} M_m(j - \tau)\boldsymbol{V}^j = \sum_{k=-\infty}^{\infty} M_m(\tau + k)\boldsymbol{V}^{-k} = \Phi_m(\tau, \boldsymbol{V})^T. \tag{9}$$

Hence we can restrict ourselves to $\tau \in [0, 1/2)$. In [9] it is shown that for $\tau \in [0, 1)$ the generalized Euler–Frobenius polynomial

$$H_m(\tau, z) := \sum_{j=-\infty}^{\infty} M_m(\tau + j - (m+1)/2)z^j = \begin{cases} z^{(m+1)/2}\Phi_m(\tau, z) & m \text{ odd},\\ z^{m/2}\Phi_m(\tau - 1/2, z) & m \text{ even}\end{cases} \tag{10}$$

possesses exactly m simple zeros in $(-\infty, 0]$, i.e.

$$H_m(\tau, z) := (z + z_1)\dots(z + z_m)$$

with $0 \le z_1 < z_2 < ... < z_j < 1 < z_{j+1} < ... < z_m$. We consider the Laurent series of $H_m(\tau, -z)^{-1}$ for $|z| = 1$ and obtain

$$\begin{aligned}
H_m(\tau, -z)^{-1} &= \prod_{\nu=1}^{j} \frac{1}{(z_\nu - z)} \prod_{\nu=j+1}^{m} \frac{1}{(z_\nu - z)}\\
&= \frac{(-1)^j}{z^j z_{j+1}...z_m} \prod_{\nu=1}^{j} \frac{1}{1 - (z_\nu/z)} \prod_{\nu=j+1}^{m} \frac{1}{1 - (z/z_\nu)}\\
&= \frac{(-1)^j}{z^j z_{j+1}...z_m} \prod_{\nu=1}^{j} \left(\sum_{l=0}^{\infty} \left(\frac{z_\nu}{z}\right)^l\right) \prod_{\nu=j+1}^{m} \left(\sum_{l=0}^{\infty} \left(\frac{z}{z_\nu}\right)^l\right) = \sum_{k=-\infty}^{\infty} b_k z^k.
\end{aligned}$$

Observe that all real coefficients b_k ($k \in \mathbb{Z}$) have the same sign. Thus by (10) for $\Phi_m(\tau, -z)$ ($\tau \in [0, 1/2)$), there are real coefficients c_k with

$$\Phi_m(\tau, -z)^{-1} = \sum_{k=-\infty}^{\infty} c_k z^k, \tag{11}$$

where all c_k ($k \in \mathbb{Z}$) have the same sign. It follows that

$$\begin{aligned}\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_\infty &= \|\sum_{l=-\infty}^{\infty} c_l (-1)^l \boldsymbol{V}^l\|_\infty \le \sum_{l=-\infty}^{\infty} |c_l| \|\boldsymbol{V}^l\|_\infty \\ &= \sum_{l=-\infty}^{\infty} |c_l| = |\sum_{l=-\infty}^{\infty} c_l| = |\varphi_m(\tau, \pi)|^{-1}.\end{aligned}$$

2. Let

$$G_m(\tau, z) := \sum_{j=0}^{N-1} g_j z^j$$

be the representing polynomial of $\Phi_m(\tau, \boldsymbol{V})^{-1}$. Then by $\boldsymbol{V}^N = \boldsymbol{I}$ we find

$$g_j = \sum_{l=-\infty}^{\infty} c_{lN+j} (-1)^{lN+j}.$$

For even N it follows

$$|g_j| = |\sum_{l=-\infty}^{\infty} c_{lN+j} (-1)^j| = \sum_{l=-\infty}^{\infty} |c_{lN+j}| \quad (j = 0, ..., N-1).$$

Hence,

$$\|G_m^1(\tau, \boldsymbol{V})\|_\infty = \sum_{j=0}^{N-1} |g_j| = \sum_{l=-\infty}^{\infty} |c_l| = |\varphi_m(\tau, \pi)|^{-1}.$$

For odd N by

$$|g_j| = |\sum_{l=-\infty}^{\infty} c_{lN+j} (-1)^{l+j}| = |\sum_{l=-\infty}^{\infty} c_{lN+j} (-1)^l| < \sum_{l=-\infty}^{\infty} |c_{lN+j}|$$

we obtain

$$\|G_m^1(\tau, \boldsymbol{V})\|_\infty = \sum_{j=0}^{N-1} |g_j| < |\varphi_m(\tau, \pi)|^{-1}\,.$$

3. The function $|\Phi_m(\tau, e^{-i\,\cdot})|$ is strictly decreasing on $[0, \pi]$ (cf. Theorem 2.1 (vii)). Thus for $\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_2$ we find by

$$\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_2 = \Big(\min\{|\Phi_m(\tau, w_N^j)|;\ j = 0, ..., N-1\}\Big)^{-1}$$

that

$$\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_2 = \begin{cases} |\varphi_m(\tau, \pi)|^{-1} & N \text{ even}, \\ |\varphi_m(\tau, \pi(N-1)/N)|^{-1} < |\varphi_m(\tau, \pi)|^{-1} & N \text{ odd}. \end{cases} \quad \blacksquare$$

REMARK 3.3
1. Lemma 3.2, taken in conjunction with the Riesz convexity theorem, guarantees that for all $1 \le p \le \infty$

$$\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_p = |\varphi_m(\tau, \pi)|^{-1}$$

for even N and

$$\|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_p < |\varphi_m(\tau, \pi)|^{-1}$$

for odd N.
2. By (11) we find with $z = e^{iu}$

$$\Phi_m(\tau, z)^{-1} = \varphi_m(\tau, u)^{-1} = \sum_{k=-\infty}^{\infty} c_k (-1)^k e^{iuk}.$$

The coefficients $(-1)^k c_k$ can be considered as Fourier coefficients of $1/\varphi_m(\tau, \cdot)$. Thus we have

$$(-1)^k c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-iuk}}{\varphi_m(\tau, u)} \, \mathrm{d}u.$$

The function

$$L_m := \sum_{j=-\infty}^{\infty} (-1)^k c_k M_m(\cdot - j)$$

is the cardinal fundamental spline, i.e. $L_m(k + \tau) = \delta_{0k}$ $(k \in \mathbb{Z})$. By (11) the de Boor-points $(-1)^k c_k$ alternate in sign:

$$\text{sign } (-1)^k c_k = (-1)^k \text{sign } c_0 \quad (k \in \mathbb{Z}). \quad \spadesuit$$

By definition the matrix $\Phi_m(\tau, \boldsymbol{V})$ possesses positive elements only. The partition of unity property yields

$$\|\Phi_m(\tau, \boldsymbol{V})\|_\infty = \|\Phi_m(\tau, \boldsymbol{V})\|_1 = \Phi_m(\tau, 1) = 1.$$

For the matrix 2-norm we find

$$\|\Phi_m(\tau, \boldsymbol{V})\|_2 = \Phi_m(\tau, 1) = 1,$$

since

$$\Phi_m(\tau, 1) = \varphi_m(\tau, 0) = \max\{|\varphi_m(\tau, 2\pi j/N)|;\ j = 0, ..., N-1\}.$$

Thus we have

$$\text{cond}_p\Big(\Phi_m(\tau, \boldsymbol{V})\Big) = \|\Phi_m(\tau, \boldsymbol{V})^{-1}\|_p \le |\varphi_m(\tau, \pi)|^{-1} \quad (p = 1, 2, \infty), \tag{12}$$

where the equality is satisfied for even N.
For fixed $m \in \mathbb{N}$ the function $|\varphi_m(\tau, \pi)|^{-1}$ only depends on the shift parameter τ but not on the period N. Now we can show

THEOREM 3.4
Let $m, N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2)$ be given. Then we have
(i) $\mathrm{cond}_2\Big(\Phi_m(\tau, \boldsymbol{V})\Big) = \mathrm{cond}_2\Big(\Phi_m(-\tau, \boldsymbol{V})\Big)$.
(ii) *The function* $\mathrm{cond}_2\Big(\Phi_m(\tau, \boldsymbol{V})\Big)$ *is strictly increasing for* $\tau \in [0, 1/2)$.
Moreover,

$$\min\Big\{\ \mathrm{cond}_2\Big(\Phi_m(\tau, \boldsymbol{V})\Big);\ \tau \in (-1/2, 1/2)\Big\} =\ \mathrm{cond}_2\Big(\Phi_m(0, \boldsymbol{V})\Big)$$
$$\leq \begin{cases} \dfrac{(m+1)!}{2^{m+1}(2^{m+1}-1)|B_{m+1}|} & m \text{ odd,} \\ \dfrac{m!}{|E_m|} & m \text{ even,} \end{cases} \tag{13}$$

where B_{m+1} denotes the $(m+1)$-th Bernoulli number and E_m the m-th Euler number. For even N equality holds in (13).
(iii) *Let $N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2)$ be fixed. Then* $\mathrm{cond}_2\Big(\Phi_m(\tau, \boldsymbol{V})\Big)$ *is increasing with respect to m.*

Proof
1. The assertion (i) follows from (9). It is sufficient to consider $\tau \in [0, 1/2)$.
2. Let N be even. By Theorem 2.1 (ix) the function $\varphi_m(\cdot, \pi)$ is strictly decreasing on $[0, 1/2]$ and $\varphi_m(1/2, \pi) = 0$. Thus by (12) $\mathrm{cond}_2\Big(\Phi_m(\tau, \boldsymbol{V})\Big)$ is minimal for $\tau = 0$. With

$$|\varphi_m(0, \pi)| = \begin{cases} \dfrac{2^m}{m!}\,|E_m(0)| & m \text{ odd,} \\ \dfrac{2^m}{m!}\,|E_m(1/2)| & m \text{ even,} \end{cases}$$

the assertion (ii) is obtained by (3) and (4).
Let N be odd. Then (ii) immediately follows from Lemma 3.2 and Theorem 2.1 (v) and (vii).
3. For odd N the statement (iii) is satisfied by Lemma 3.2 and Theorem 2.1 (viii). For even N and $m \in \mathbb{N}_0$ we only have to prove

$$|\varphi_{m+1}(\tau, \pi)| \leq |\varphi_m(\tau, \pi)| \qquad (\tau \in [0, 1/2)).$$

Thus we inductively show the relation

$$|E_{m+1}(\tau)| < \frac{m+1}{2}|E_m(1/2-\tau)| \qquad (\tau \in (0, 1/2)). \tag{14}$$

For $m = 0$, (14) is satisfied by $1/2 - \tau < 1/2$ $(\tau \in (0, 1/2))$. Note that $|E_m(\tau)| = (-1)^n E_m(\tau)$ $(\tau \in (0, 1/2))$ with $n := \lfloor (m+1)/2 \rfloor$. Assume that (14) holds for even $m \in \mathbb{N}_0$. Then we find by integration for $\tau \in (0, 1/2)$

$$\begin{aligned} \int_0^\tau (-1)^{(m+2)/2} E_{m+1}(t)\ \mathrm{d}t \ &< \ \frac{m+1}{2}\int_0^\tau (-1)^{m/2} E_m(1/2-t)\ \mathrm{d}t \\ &= \ \frac{m+1}{2}\int_{1/2-\tau}^{1/2} (-1)^{m/2} E_m(s)\ \mathrm{d}s, \end{aligned}$$

i.e.

$$\frac{(-1)^{(m+2)/2}}{(m+2)}\Big(E_{m+2}(\tau)-E_{m+2}(0)\Big)<\frac{(m+1)(-1)^{m/2}}{2\,(m+1)}\Big(E_{m+1}(1/2)-E_{m+1}(1/2-\tau)\Big).$$

Thus (14) follows for $m+1$ by $E_{m+2}(0)=E_{m+1}(1/2)=0$.
If m is odd, then (14) can be derived in the same manner by integrating over $[1/2-\tau, 1/2]$.
■

REMARK 3.5
Another approach to this problem is given in [10]. ♠

Figure 3

Figure 3: Condition of the interpolatory matrix for even N and $\tau \in [-0.4, 0.4]$.

EXAMPLE 3.6
By Theorem 3.4 we have for even N:

$$\begin{aligned}
\mathrm{cond}_2\Big(\Phi_1(0,\boldsymbol{V})\Big) &= 1, &\quad \mathrm{cond}_2\Big(\Phi_4(0,\boldsymbol{V})\Big) &= 4.8,\\
\mathrm{cond}_2\Big(\Phi_2(0,\boldsymbol{V})\Big) &= 2, &\quad \mathrm{cond}_2\Big(\Phi_5(0,\boldsymbol{V})\Big) &= 7.5,\\
\mathrm{cond}_2\Big(\Phi_3(0,\boldsymbol{V})\Big) &= 3, &\quad \mathrm{cond}_2\Big(\Phi_6(0,\boldsymbol{V})\Big) &= \frac{720}{61}\approx 11.80328.
\end{aligned}$$ ♠

We summarize:
The choice of the shift parameter $\tau=0$ is most favourable, since for $\tau=0$ the condition $\mathrm{cond}_2\Big(\Phi_m(\tau,\boldsymbol{V})\Big)$ is minimal.
Spline functions with a high degree m are unfavourable from the numerical point of view, since the condition of $\Phi_m(\tau,\boldsymbol{V})$ is strictly increasing with respect to m.

4 Norm of the interpolation operator

Let $m, N \in \mathbb{N}$ and $\tau \in (-1/2, 1/2)$ be given. We want to investigate the norm $\|\mathcal{L}^N_{m,\tau}\|_1$ of the spline interpolation operator $\mathcal{L}^N_{m,\tau}: \mathbb{R}^N \to L^N_1$. Here the norm in $\mathbb{R}^N$ is defined by

$$\|\boldsymbol{y}\|_1 := \sum_{j=0}^{N-1}|y_j| \quad (\boldsymbol{y} := (y_j)_{j=0}^{N-1} \in \mathbb{R}^N).$$

We are especially interested in the dependence of $\|\mathcal{L}_{m,\tau}^N\|_1$ on the shift parameter τ.

For the L_1^N-norm of the image of $\mathcal{L}_{m,\tau}^N$ we find

$$\begin{aligned}\|\mathcal{L}_{m,\tau}^N \boldsymbol{y}\|_1 &= \int_0^N |\sum_{j=0}^{N-1} y_j L_m^N(x-j)| \ \mathrm{d}x \le \sum_{j=0}^{N-1} |y_j| \int_0^N |L_m^N(x-j)| \ \mathrm{d}x \\ &= \Big(\int_0^N |L_m^N(x)| \ \mathrm{d}x\Big) \, \|\boldsymbol{y}\|_1.\end{aligned}$$

Further, for $\boldsymbol{e}_0 := (\delta_{0k})_{k=0}^{N-1}$ we have

$$\|\mathcal{L}_{m,\tau}^N \boldsymbol{e}_0\|_1 = \int_0^N |L_m^N(x)| \ \mathrm{d}x = \Big(\int_0^N |L_m^N(x)| \ \mathrm{d}x\Big) \, \|\boldsymbol{e}_0\|_1.$$

Thus,

$$\|\mathcal{L}_{m,\tau}^N\|_1 = \int_0^N |L_m^N(x)| \ \mathrm{d}x.$$

Assume that N is odd. The N-periodic fundamental spline possesses exactly $(N-1)$ zeros in $[0, N]$ (cf. [7, 11]). By

$$L_m^N(k+\tau) = \delta_{0k} \quad (k = 0, ..., N-1)$$

it follows that

$$\operatorname{sign} L_m^N(x) = (-1)^k \qquad (x \in (k+\tau, k+1+\tau), \ k = 0, ..., N-1).$$

From (6) we obtain with $w_N := \exp(-2\pi i/N)$:

$$\begin{aligned}\|\mathcal{L}_{m,\tau}^N\|_1 &= \int_\tau^{N+\tau} |L_m^N(x)| \ \mathrm{d}x = \sum_{k=0}^{N-1} \int_{k+\tau}^{k+1+\tau} (-1)^k L_m^N(x) \ \mathrm{d}x \\ &= \sum_{k=0}^{N-1} (-1)^k \Big(\int_\tau^1 L_m^N(k+t) \ \mathrm{d}t + \int_0^\tau L_m^N(k+1+t) \ \mathrm{d}t \Big)\end{aligned}$$

$$\begin{aligned}&= \sum_{k=0}^{N-1} \frac{(-1)^k}{N} \left(\sum_{j=0}^{N-1} \frac{\int_\tau^1 \varphi_m(t, 2\pi j/N) \ \mathrm{d}t}{\varphi_m(\tau, 2\pi j/N)} w_N^{-kj} + \sum_{j=0}^{N-1} \frac{\int_0^\tau \varphi_m(t, 2\pi j/N) \ \mathrm{d}t}{\varphi_m(\tau, 2\pi j/N)} w_N^{-(k+1)j} \right) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \frac{2}{\varphi_m(\tau, 2\pi j/N)(1+w_N^{-j})} \Big(\int_\tau^1 \varphi_m(t, 2\pi j/N) \ \mathrm{d}t + w_N^{-j} \int_0^\tau \varphi_m(t, 2\pi j/N) \ \mathrm{d}t \Big).\end{aligned}$$

Using Theorem 2.1 (ii) and (iii), i.e.

$$\begin{aligned}\frac{\partial}{\partial x} \varphi_{m+1}(x, 2\pi j/N) &= (1-w_N^j) \, \varphi_m(x+1/2, 2\pi j/N) \qquad (x \in \mathbb{R}, \ j = 0, ..., N-1) \\ \varphi_m(-1/2, 2\pi j/N) &= w_N^j \, \varphi_m(1/2, 2\pi j/N),\end{aligned}$$

it follows by integration that

$$\int_0^\tau \varphi_m(t, 2\pi j/N)\ \mathrm{d}t = \frac{1}{(1-w_N^j)}\Big(\varphi_{m+1}(\tau-1/2, 2\pi j/N) - w_N^j \varphi_{m+1}(1/2, 2\pi j/N)\Big)$$

and

$$\int_\tau^1 \varphi_m(t, 2\pi j/N)\ \mathrm{d}t = \frac{1}{(1-w_N^j)}\Big(\varphi_{m+1}(1/2, 2\pi j/N) - \varphi_{m+1}(\tau-1/2, 2\pi j/N)\Big).$$

Thus for $j = 1, ..., N-1$ we have

$$\int_\tau^1 \varphi_m(t, 2\pi j/N)\ \mathrm{d}t + w_N^{-j}\int_0^\tau \varphi_m(t, 2\pi j/N)\ \mathrm{d}t = w_N^{-j}\ \varphi_{m+1}(\tau-1/2, 2\pi j/N).$$

Since $\varphi_m(x, 0) = 1$ ($x \in \mathbb{R}$, $m \in \mathbb{N}$) we arrive at

$$\|\mathcal{L}_{m,\tau}^N\|_1 = \frac{2}{N}\sum_{j=0}^{N-1}\frac{\varphi_{m+1}(\tau-1/2, 2\pi j/N)}{\varphi_m(\tau, 2\pi j/N)(1+w_N^j)} \quad (\tau \in (-1/2, 1/2]). \tag{15}$$

REMARK 4.1

1. The formula (15) for $\|\mathcal{L}_{m,\tau}^N\|_1$ was already found by G. Merz (cf. [7]). Furthermore, in [7] some special results on $\|\mathcal{L}_{2,\tau}^N\|_1$ are given.
2. For even N, an explicit representation of $\|\mathcal{L}_{m,\tau}^N\|_1$ like in (15) is unknown. In general, for even N the fundamental spline L_m^N possesses apart from the $N-1$ zeros, determined by the interpolation conditions, an additional zero, which depends on the spline degree m, the period N and the shift parameter τ. The explicit computation of this additional zero seems to be difficult. ♠

Using the properties of the symbol φ_m we are able to describe the dependence of $\|\mathcal{L}_{m,\tau}^N\|_1$ on τ. The following Theorem improves the results in [7] and [11].

THEOREM 4.2

Let $N, m \in \mathbb{N}$ (N odd) and $\tau \in (-1/2, 1/2]$ be given. Then for $\|\mathcal{L}_{m,\tau}^N\|_1$ we have

(i) $\|\mathcal{L}_{m,\tau}^N\|_1 = \|\mathcal{L}_{m,-\tau}^N\|_1 \qquad (\tau \in (-1/2, 0])$.

(ii) *The norm $\|\mathcal{L}_{m,\tau}^N\|_1$ is strictly increasing with respect to $\tau \in [0, 1/2]$. The following holds:*

$$\min\Big\{\|\mathcal{L}_{m,\tau}^N\|_1;\ \tau \in (-1/2, 1/2]\Big\} = \|\mathcal{L}_{m,0}^N\|_1.$$

Proof

Let m and $N = 2n+1$ be fixed and $h(\tau) := \|\mathcal{L}_{m,\tau}^N\|_1$ ($\tau \in (-1/2, 1/2]$).

1. By Theorem 2.1 (i) and (ii) we find

$$\begin{aligned}
h(\tau) &= \frac{2}{N}\sum_{k=0}^{N-1}\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)}{\varphi_m(\tau,2\pi k/N)(1+w_N^k)} \\
&= \frac{2}{2N}+\frac{2}{N}\sum_{k=1}^{n}\left(\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)}{(1+w_N^k)\varphi_m(\tau,2\pi k/N)}+\frac{\overline{\varphi_{m+1}(\tau-1/2,2\pi k/N)}}{(1+w_N^{-k})\overline{\varphi_m(\tau,2\pi k/N)}}\right) \\
&= \frac{1}{N}+\frac{4}{N}\sum_{k=1}^{n}\mathrm{Re}\left(\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)}{(1+w_N^k)\varphi_m(\tau,2\pi k/N)}\right) \\
&= \frac{1}{N}+\frac{4}{N}\sum_{k=1}^{n}\mathrm{Re}\left(\frac{\overline{\varphi_{m+1}(-1/2-\tau,2\pi k/N)}w_N^k}{(1+w_N^k)\overline{\varphi_m(-\tau,2\pi k/N)}}\right) \\
&= \frac{1}{N}+\frac{4}{N}\sum_{k=1}^{n}\mathrm{Re}\left(\frac{\overline{\varphi_{m+1}(-\tau-1/2,2\pi k/N)}}{(1+w_N^{-k})\overline{\varphi_m(-\tau,2\pi k/N)}}\right)=h(-\tau).
\end{aligned}$$

2. We consider the first derivative of $h(\tau)$ and obtain by Theorem 2.1 (iii)

$$h'(\tau)=\frac{4}{N}\sum_{k=1}^{n}\mathrm{Re}\left(\frac{1-w_N^k}{1+w_N^k}\Big(1-\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)\,\varphi_{m-1}(\tau+1/2,2\pi k/N)}{\varphi_m(\tau,2\pi k/N)^2}\Big)\right).$$

Since

$$\frac{1-w_N^k}{1+w_N^k}=i\,\frac{2\sin(2\pi k/N)}{|1+w_N^k|^2},$$

it follows that

$$h'(\tau)=\frac{8}{N}\sum_{k=1}^{n}\frac{\sin(2\pi k/N)}{|1+w_N^k|^2}\,\mathrm{Im}\left(\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)\varphi_{m-1}(\tau+1/2,2\pi k/N)}{\varphi_m(\tau,2\pi k/N)^2}\right). \qquad (16)$$

We consider

$$\arg\Big(\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)\,\varphi_{m-1}(\tau+1/2,2\pi k/N)}{\varphi_m(\tau,2\pi k/N)^2}\Big)$$

for $0<\tau<1/2$ and fixed k $(1\le k\le n)$. By Theorem 2.1 (ii) and (vi) we find for the numerator

$$2\tau\,\frac{2\pi k}{N}<\arg\Big(\varphi_{m+1}(\tau-1/2,2\pi k/N)\,\varphi_{m-1}(\tau+1/2,2\pi k/N)\Big)<\frac{2\pi k}{N}$$

and for the denominator

$$0<\arg\Big(\varphi_m(\tau,2\pi k/N)^2\Big)<2\tau\,\frac{2\pi k}{N}.$$

Therefore we have

$$0<\arg\Big(\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)\varphi_{m-1}(\tau+1/2,2\pi k/N)}{\varphi_m(\tau,2\pi k/N)^2}\Big)<\frac{2\pi k}{N},$$

i.e.,

$$\mathrm{Im}\Big(\frac{\varphi_{m+1}(\tau-1/2,2\pi k/N)\varphi_{m-1}(\tau+1/2,2\pi k/N)}{\varphi_m(\tau,2\pi k/N)^2}\Big)>0$$

for $\tau \in (0, 1/2)$ and $1 \leq k \leq n$. From (16) it follows the inequality

$$h'(\tau) > 0 \qquad (\tau \in (0, 1/2)),$$

i.e., the continuous function h is strictly increasing on $[0, 1/2]$. By (i) the function h is strictly decreasing on $(-1/2, 0]$, such that

$$h(0) = \min\{h(\tau);\ \tau \in (-1/2, 1/2]\}. \qquad \blacksquare$$

We summarize:
Considering the norm $\|\mathcal{L}_{m,\tau}^N\|_1$ we obtain the same result as above for the condition, namely that the shift parameter $\tau = 0$ is optimal.

Figure 4

Figure 4: Norm of the interpolation operator $\|\mathcal{L}_{m,\tau}^{15}\|_1$ for $m = 3, 4, 5, 6$ and $\tau \in [-0.5, 0.5]$.

REMARK 4.3
1. We conjecture that the norm $\|\mathcal{L}_{m,\tau}^N\|_\infty$ of the spline interpolation operator $\mathcal{L}_{m,\tau}^N : \mathbb{R}^N \to C^N$ is also strictly increasing for $\tau \in [0, 1/2)$. Here C^N denotes the Banach space of all continuous N-periodic functions with the Chebyshev norm.
2. For the norm $\|\mathcal{L}_{m,\tau}^N\|_2$ of $\mathcal{L}_{m,\tau}^N : R^N \to L_2^N$ and for odd period N the following holds (cf. [11, 8]):

$$\|\mathcal{L}_{m,0}^N\|_2 = 1.$$

3. Note that for $N \to \infty$ we obtain the cardinal spline interpolation operator $\mathcal{L}_{m,\tau}$ which has been investigated for $\tau = 0,\ 1/2$ in several papers, for instance in [12, 4, 5, 6].

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