CONSTRUCTION OF MULTI-SCALING FUNCTIONS WITH APPROXIMATION AND SYMMETRY

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Abstract. This paper presents a new and efficient way to create multi-scaling functions with given approximation order, regularity, symmetry and short support. Previous techniques were operating in time domain and required the solution of large systems of nonlinear equations. By switching to the frequency domain and employing the latest results of the multiwavelet theory we are able to elaborate a simple and efficient method of construction of multi-scaling functions. Our algorithm is based on a recently found factorization of the refinement mask through the two-scale similarity transform (TST). Theoretical results and new examples are presented.

Key words. approximation order, symmetry, multi-scaling functions, multiwavelets

AMS subject classifications. 41A25, 42A38, 39B62

1. Introduction. This paper discusses the construction of *multi-scaling functions* which generate a *multiresolution analysis* (MRA) and lead to *multiwavelets*. A standard (*scalar*) MRA assumes that there is only one scaling function $\phi(t)$ whose translates $\phi(t-k)$ $(k \in \mathbb{Z})$ constitute an L^2–stable basis of their span V_0 [D2, SN]. We move a step forward and allow several functions $\phi_0(t), \ldots, \phi_{r-1}(t)$. The vector $\boldsymbol{\phi}(t) = [\phi_0(t) \ldots \phi_{r-1}(t)]^{\mathrm{T}}$ is called *multi-scaling function* if the integer translates $\phi_\nu(\cdot - k)$ $(k \in \mathbb{Z}, \nu = 0, \ldots, r-1)$ form an L^2–stable basis of V_0 and if $\boldsymbol{\phi}(t)$ satisfies a *dilation equation*,

$$\boldsymbol{\phi}(t) = \sum_{n=0}^{N} \boldsymbol{P}_n\, \boldsymbol{\phi}(2t-n). \tag{1.1}$$

Here the coefficients $\boldsymbol{P}_n$ are $r \times r$ matrices instead of usual scalars. The multi-scaling function $\boldsymbol{\phi}$ generates a multiresolution analysis (MRA) $\{V_j : j \in \mathbb{Z}\}$ of multiplicity r. The corresponding wavelet spaces W_j can be generated by a *multiwavelet* $\boldsymbol{w}(t) = [w_0(t) \ldots w_{r-1}(t)]^{\mathrm{T}}$ associated with $\boldsymbol{\phi}(t)$, satisfying a *wavelet equation*

$$\boldsymbol{w}(t) = \sum_{n=0}^{K} \boldsymbol{D}_n\, \boldsymbol{\phi}(2t-n). \tag{1.2}$$

Again, $\boldsymbol{D}_n$ are $r \times r$ matrices.

Multiwavelets naturally generalize the scalar wavelets. For $r = 1$, (1.1) is the well-studied refinement equation (see e.g. [CDM, DL1, DL2]). However, multiwavelets have some completely new features arising from the matrix nature ($r > 1$) of the equation (1.1). They can simultaneously possess symmetry, orthogonality, and high approximation order which is not possible in the scalar case [SB, D2]. This suggests that in some applications multiwavelets may behave better than the scalar ones. The results of first experiments [SHSTH, XGHS] confirm this conjecture and show that the multiwavelets are definitely worth studying.

One of the first multiwavelet constructions is due to Alpert and Rokhlin [AR]. They considered a multi-scaling function whose components are polynomials of degree

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$r-1$ supported on $[0,1]$. The general theory of multiwavelets, based on the MRA of multiplicity r, is discussed in [GLT, GL].

Using fractal interpolation, Geronimo, Hardin, and Massopust succeeded to construct a continuous multi-scaling function $\boldsymbol{\phi}(t) = [\phi_0(t)\ \phi_1(t)]^{\mathrm{T}}$ with short support, symmetry and second approximation order [GHM]. The plot of this pair $\phi_0(t)$, $\phi_1(t)$ is presented in Figure 1.

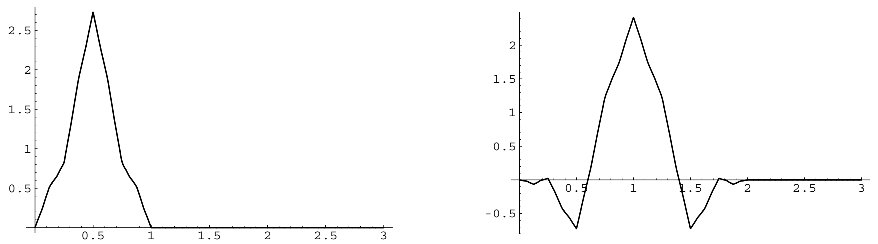

FIG.1 *GHM symmetric orthogonal multi-scaling function with approximation order* 2

The results of [GHM] triggered many attempts to construct more examples ([SS1, CL, DGHM]) as well as the systematic study of multi-scaling functions.

Properties of a refinable function can be formulated either in *time* or in *frequency* domain. In [SS2, HSS, L], conditions of orthogonality and approximation were established in the time domain. Also, a way to construct multi-scaling functions with short support and low approximation order was found [SS1, HSS, CL]. Unfortunately, this method required solution of a large system of nonlinear equations. We therefore switch to the frequency domain.

Working in the frequency domain, one faces the necessity to deal with the Fourier transformation of (1.1),

$$\widehat{\boldsymbol{\phi}}(\omega) = \boldsymbol{P}\left(\frac{\omega}{2}\right)\, \widehat{\boldsymbol{\phi}}\left(\frac{\omega}{2}\right), \tag{1.3}$$

where $\widehat{\boldsymbol{\phi}} := (\widehat{\phi}_0, \ldots, \widehat{\phi}_{r-1})^{\mathrm{T}}$, $\widehat{\phi}_\nu := \int_{-\infty}^{\infty} \phi_\nu(t)\, e^{-it\cdot}\, \mathrm{d}t$, and $\boldsymbol{P}(\omega)$ is the *refinement mask* corresponding to $\boldsymbol{\phi}(t)$,

$$\boldsymbol{P}(\omega) := \frac{1}{2}\sum_{n=0}^{N} \boldsymbol{P}_n\, e^{-i\omega n}. \tag{1.4}$$

In the scalar case, $P(\omega)$ is a *trigonometric polynomial.* In the vector case, $\boldsymbol{P}(\omega)$ becomes a *matrix of trigonometric polynomials.* To ensure certain approximation order, $\boldsymbol{P}(\omega)$ must satisfy necessary and sufficient conditions in the frequency domain. Those conditions were formulated and proved in [HSS, P3]. In [P3], it was also shown that the vector $\boldsymbol{\phi}(t)$ can only provide approximation order m if the refinement mask $\boldsymbol{P}(\omega)$ can be factorized in the form:

$$\boldsymbol{P}(\omega) = \frac{1}{2^m}\, \boldsymbol{C}_{m-1}(2\omega)\ldots\boldsymbol{C}_0(2\omega)\, \boldsymbol{P}^{(0)}(\omega)\, \boldsymbol{C}_0(\omega)^{-1}\ldots\boldsymbol{C}_{m-1}(\omega)^{-1}, \tag{1.5}$$

where $\boldsymbol{P}^{(0)}(\omega)$ is well-defined and $\boldsymbol{C}_0(\omega), \ldots, \boldsymbol{C}_{m-1}(\omega)$ are matrices of a special form. The factorization (1.5) is not unique. With the help of the *two-scale similarity transformation* (TST), the whole set of possible factorizations can be described [S1].

The factorization (1.5) naturally generalizes the scalar case $r = 1$. As known, one scaling function with compact support and linearly independent integer translates provides approximation order m if and only if its refinement mask $P(\omega)$ has m zeros at $\omega = \pi$:

$$P(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^m q(\omega) \tag{1.6}$$

with $q(0) = 1$ and $q(\pi) \neq 0$. For $r = 1$, (1.6) coincides with (1.5) taking $\boldsymbol{P}^{(0)}(\omega) = q(\omega)$ and $\boldsymbol{C}_0(\omega) = \ldots = \boldsymbol{C}_{m-1}(\omega) = (1 - e^{i\omega})$. Daubechies connected the behaviour of $q(\omega)$ in (1.6) with the decay properties of $\widehat{\phi}(\omega)$, and hence, she obtained estimates of smoothness of $\phi(t)$ [D1]. The factorization (1.5) plays the same role for a multi-scaling function as (1.6) for a scalar one. In [CDP] and independently in [S1], it was shown how the factorization of the refinement mask $\boldsymbol{P}(\omega)$ leads to decay of $\widehat{\boldsymbol{\phi}}(\omega)$. Similar results on regularity of refinable function vectors are presented in [Sh].

However, up to now, the factorization (1.5) has been shown to be necessary only. For the construction of multi-scaling functions we need the sufficiency of a factorization (1.5) for approximation order m. In this paper, we show how, under mild conditions, the factorization of the refinement mask $\boldsymbol{P}(\omega)$ yields a solution of (1.1) with desired approximation properties. Using this result and the TST, a construction of multi-scaling functions providing an arbitrary, fixed approximation order becomes simple. Description of corresponding algorithm is our main purpose.

The outline of the paper is as follows. In §2 we summarize previously known and new theoretical results on the symmetry of $\boldsymbol{\phi}(t)$, its approximation order, the factorization of the refinement mask $\boldsymbol{P}(\omega)$ and the TST. The main novelty of §2 is the observation that the factorization of the refinement mask leads to approximation order of the multi-scaling functions (Theorem 2.6). Other remarkable new results are given in Theorems 2.7, 2.9 and Lemma 2.5.

In §3, we present a new algorithm for the construction of a refinement mask $\boldsymbol{P}(\omega)$ with any given approximation order. We intensively study, how the inner matrix $\boldsymbol{P}^{(0)}(\omega)$ and the transformation matrices $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ should be chosen in order to obtain a smooth, symmetric multi-scaling function with compact support. Several examples are given.

§4 contains the proof of Theorem 2.6.

2. Old and new theoretical results. In this section, we are going to present the results needed for the construction of symmetric multi-scaling functions with given approximation order.

Let us start with definitions and notation. For a measurable function f over $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\|f\|_2 := (\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t)^{1/2}, \qquad \|f\|_{m,2} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_2.$$

Here and below $\mathrm{D} := \mathrm{d}/\mathrm{d}\omega$ denotes the differentiation operator with respect to ω.

Let $W_2^m(\mathbb{R})$ be the usual Sobolev space with norm $\|\cdot\|_{m,2}$. For a vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ of compactly supported functions let $\mathcal{S} = \mathcal{S}(\boldsymbol{\phi})$ be the shift-invariant space spanned by the integer translates $\phi_\nu(t-k)$ $(\nu = 0,\ldots,r-1,\ k \in \mathbf{Z})$. We say that $\boldsymbol{\phi}(t)$ provides *approximation order* m if for every $f \in W_2^m(\mathbb{R})$

$$\min\{\|f-s\| : s \in \mathcal{S}^h\} \le \mathrm{const}_{\mathcal{S}}\ h^m\, \|f\|_{W_2^m(\mathbb{R})},$$

where $\mathcal{S}^h$ is the scaled space $\mathcal{S}^h := \{s(\cdot/h) : s \in \mathcal{S}\}$.

A vector $\boldsymbol{v}$ of length r is said to be in $C_{2\pi}^m(\mathbb{R}^r)$, and analogously, an $r\times r$ matrix $\boldsymbol{V}$ is in $C_{2\pi}^m(\mathbb{R}^{r\times r})$, if all its entries are 2π–periodic m times continuously differentiable functions.

2.1. Conditions of approximation. Assume that $\phi_\nu \in C(\mathbb{R}) \cap BV(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$ are compactly supported functions. Here $BV(\mathbb{R})$ denotes the set of functions of bounded variation. If the integer translates $\phi_\nu(\cdot - l)$ form a Riesz basis of $\mathcal{S}(\boldsymbol{\phi})$, then the following statements are equivalent (see [JL, P3]):

(i) The function vector $\boldsymbol{\phi}(t)$ provides approximation order m $(m \in \mathbb{N})$.

(ii) All algebraic polynomials of degree up to $m-1$ can be exactly reproduced by a linear combination of integer translates of $\phi_\nu(t)$.

(iii) $\boldsymbol{\phi}(t)$ satisfies the Strang–Fix conditions [SF] of order m, in other words, there is a finitely supported sequence of vectors $\{\boldsymbol{a}_l\}_{l\in\mathbb{Z}}$ such that $f(t) := \sum_{l\in\mathbb{Z}} \boldsymbol{a}_l^{\mathrm{T}}\, \boldsymbol{\phi}(t-l)$ satisfies the following conditions:

$$\widehat{f}(0) \ne 0; \qquad \mathrm{D}^n \widehat{f}(2\pi l) = 0 \quad (l \in \mathbb{Z}\setminus\{0\};\, n = 0,\ldots,m-1).$$

Since condition (b) yields vanishing moments for the corresponding multiwavelets it is often used in applications.

The approximation order of a refinable function vector $\boldsymbol{\phi}(t)$ satisfying (1.1) is intimately related with the properties of the refinement mask $\boldsymbol{P}(\omega)$ defined by (1.4). In the scalar case $(r = 1)$, when there is only one scaling function, $\boldsymbol{P}_n$ are real numbers and $P(\omega)$ is a scalar trigonometric polynomial. Then m-th approximation order implies m zeros of $P(\omega)$ at $\omega = \pi$ [D2]. In the vector case, $\boldsymbol{P}(\omega)$ is a matrix, and the situation becomes more complicated. But still, similar conditions at the point $\omega = \pi$ hold.

Theorem 2.1. [HSS, P3] *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of compactly supported functions ϕ_ν. Further, assume that the integer translates $\phi_\nu(t-l)$ $(l \in \mathbb{Z})$ form a Riesz basis of $\mathcal{S}(\boldsymbol{\phi})$. Then $\boldsymbol{\phi}(t)$ provides approximation order m if and only if the refinement mask $\boldsymbol{P}(\omega)$ of $\boldsymbol{\phi}$ satisfies the following conditions: There are vectors $\boldsymbol{y}_k \in \mathbb{R}^r$; $\boldsymbol{y}_0 \ne \mathbf{0}$ $(k = 0,\ldots,m-1)$ such that for $n = 0,\ldots,m-1$,*

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^T\, (2i)^{k-n}\, (D^{n-k}\boldsymbol{P})(0) = 2^{-n}\, (\boldsymbol{y}_n)^T, \tag{2.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^T\, (2i)^{k-n}\, (D^{n-k}\boldsymbol{P})(\pi) = \mathbf{0}^T. \tag{2.2}$$

Here $\mathbf{0}$ *denotes the zero vector.*

If a matrix $\boldsymbol{P}(\omega) \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ satisfies (2.1) and (2.2) for $n = 0,\ldots,m-1$ with vectors $\boldsymbol{y}_0,\ldots,\boldsymbol{y}_{m-1}$ $(\boldsymbol{y}_0 \ne \mathbf{0})$, then we shortly say that $\boldsymbol{P}(\omega)$ provides approximation order m with $\boldsymbol{y}_0,\ldots,\boldsymbol{y}_{m-1}$. In order to prove that the relations (2.1) and (2.2) imply approximation order m, one only needs to assume that $\boldsymbol{y}_0^{\mathrm{T}}\, \widehat{\boldsymbol{\phi}}(0) \ne 0$. Riesz stability of integer translates $\phi_\nu(t-l)$ is *not* needed.

Remark. The result of Theorem 2.1 is a natural generalization of the scalar case. For $r = 1$, equations (2.1), (2.2) can be simplified to

$$P(0) = 1; \qquad \mathrm{D}^n P(\pi) = 0 \quad (n = 0, \ldots, m-1) \tag{2.3}$$

implying m zeros of $P(\omega)$ at $\omega = \pi$. Note that in the vector case, we need conditions in *two* points, $\omega = 0$ and $\omega = \pi$. Also, both eigenvalues and eigenvectors of $\boldsymbol{P}(0)$ and $\boldsymbol{P}(\pi)$ are important.

2.2. Two-scale similarity transform. A very useful research and construction tool in the theory of multiwavelets is the two-scale similarity transform (TST) [S1]. We say that $\boldsymbol{Q}(\omega)$ is a TST of $\boldsymbol{P}(\omega)$ with the *transformation matrix* $\boldsymbol{M}(\omega) \in C_{2\pi}(\mathbb{R}^{r\times r})$ if

$$\boldsymbol{Q}(\omega) = \boldsymbol{M}(2\omega)\, \boldsymbol{P}(\omega)\, \boldsymbol{M}(\omega)^{-1}.$$

If $\boldsymbol{M}(\omega)$ is invertible for all $\omega \in \mathbb{R}$, then the TST is *non-degenerate*. It is easy to see that if $\boldsymbol{P}(\omega) \in C_{2\pi}(\mathbb{R}^{r\times r})$ is the refinement mask of $\boldsymbol{\phi} \in L^2(\mathbb{R}^r)$, then a non-degenerate TST with transformation matrix $\boldsymbol{M}(\omega) \in C_{2\pi}(\mathbb{R}^{r\times r})$ yields a matrix $\boldsymbol{Q}(\omega)$ which itself is a refinement mask of a refinable function vector $\boldsymbol{\psi} \in L^2(\mathbb{R}^r)$ such that $\widehat{\boldsymbol{\psi}}(\omega) = \boldsymbol{M}(\omega)\, \widehat{\boldsymbol{\phi}}(\omega)$:

$$\begin{aligned} \widehat{\boldsymbol{\psi}}(\omega) &= \boldsymbol{M}(\omega)\, \widehat{\boldsymbol{\phi}}(\omega) = \boldsymbol{M}(\omega)\, \boldsymbol{P}\left(\frac{\omega}{2}\right) \widehat{\boldsymbol{\phi}}\left(\frac{\omega}{2}\right) \\ &= \boldsymbol{M}(\omega)\, \boldsymbol{P}\left(\frac{\omega}{2}\right) \boldsymbol{M}\left(\frac{\omega}{2}\right)^{-1} \widehat{\boldsymbol{\psi}}\left(\frac{\omega}{2}\right) = \boldsymbol{Q}\left(\frac{\omega}{2}\right) \widehat{\boldsymbol{\psi}}\left(\frac{\omega}{2}\right). \end{aligned}$$

The following theorem shows, that a non-degenerate TST preserves approximation properties of a refinement mask.

THEOREM 2.2. [S1] *Let a transformation matrix $\boldsymbol{M}(\omega) \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ be invertible for all $\omega \in \mathbb{R}$. Assume that $\boldsymbol{P} \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ provides approximation order m with vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$. Then $\boldsymbol{Q}(\omega) = \boldsymbol{M}(2\omega)\, \boldsymbol{P}(\omega)\, \boldsymbol{M}(\omega)^{-1}$ also provides approximation order m with vectors $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{m-1}$, given by*

$$\boldsymbol{u}_k^T := \sum_{l=0}^{k} \binom{k}{l} i^{l-k}\, \boldsymbol{y}_l^T\, (D^{k-l}\boldsymbol{M}^{-1})(0) \qquad (k = 0, \ldots, m-1).$$

For more properties of the TST and the proof of Theorem 2.2 see [S1, S2].

2.3. Factorizations of the refinement mask. In the scalar case, the conditions of approximation (2.3) lead to a factorization of $P(\omega)$. A zero at $\omega = \pi$ means that $P(\omega)$ has a factor $(1 + e^{-i\omega})$. So $P(\omega)$ factorizes as in (1.6). This factorization plays the key role in the construction of regular scalar scaling functions [D2].

In the vector case, the conditions of approximation (2.1), (2.2) are more complicated but still they imply a factorization of the *matrix* refinement mask $\boldsymbol{P}(\omega)$. This factorization opens a constructive way toward the creation of new multi-scaling functions. But before starting with the factorization, we need to review some notation.

Let $r \in \mathbb{N}$ be fixed, and let $\boldsymbol{y} \in \mathbb{R}^r$ be a vector of length r. To start, assume that $\boldsymbol{y}$ is of the form

$$\boldsymbol{y} = [y_0 \ldots y_{l-1}\ 0 \ldots 0]^{\mathrm{T}} \tag{2.4}$$

with $1 \le l \le r$ and $y_\nu \ne 0$ for $\nu = 0, \dots, l-1$. We introduce the direct sum of square matrices $\boldsymbol{A} \oplus \boldsymbol{B} := \mathrm{diag}(\boldsymbol{A}, \boldsymbol{B})$ and define the matrix $\boldsymbol{C}_{\boldsymbol{y}}$ by

$$\boldsymbol{C}_{\boldsymbol{y}}(\omega) := \widetilde{\boldsymbol{C}}_{\boldsymbol{y}}(\omega) \oplus \boldsymbol{I}_{r-l}. \tag{2.5}$$

Here $\boldsymbol{I}_{r-l}$ denotes the $(r-l) \times (r-l)$ unit matrix, and for $l > 1$, $\widetilde{\boldsymbol{C}}_{\boldsymbol{y}}(\omega)$ is defined by

$$\widetilde{\boldsymbol{C}}_{\boldsymbol{y}}(\omega) := \begin{bmatrix} y_0^{-1} & -y_0^{-1} & 0 & \dots & 0 \\ 0 & y_1^{-1} & -y_1^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & y_{l-2}^{-1} & -y_{l-2}^{-1} \\ -e^{-i\omega}/y_{l-1} & 0 & \dots & 0 & y_{l-1}^{-1} \end{bmatrix} .$$

If $l = 1$, let $\widetilde{\boldsymbol{C}}_{\boldsymbol{y}}(\omega) = (1 - e^{-i\omega})/y_0$.

For general $\boldsymbol{y} = [y_0 \dots y_{r-1}]^{\mathrm{T}} \in \mathbb{R}^r$, $\boldsymbol{y} \ne \boldsymbol{0}$ we define $\boldsymbol{C}_{\boldsymbol{y}} := (C_{j,k})_{j,k=0}^{r-1}$ by reshuffling of rows and columns. More exactly, let $j_0 := \min\{j;\ y_j \ne 0\}$ and $j_1 := \max\{j; y_j \ne 0\}$. For all $j < j_1$ with $y_j \ne 0$ let $d_j := \min\{\mu : \mu > j,\ y_\mu \ne 0\}$. For $j_0 < j_1$, the entries of $\boldsymbol{C}_{\boldsymbol{y}}$ are defined by

$$C_{j,k}(\omega) := \begin{cases} y_j^{-1} & y_j \ne 0 \quad \text{and} \quad j = k, \\ 1 & y_j = 0 \quad \text{and} \quad j = k, \\ -y_j^{-1} & y_j \ne 0 \quad \text{and} \quad d_j = k, \\ -e^{-i\omega}/y_{j_1} & j = j_1 \quad \text{and} \quad k = j_0, \\ 0 & \text{otherwise} \end{cases} \qquad (j,\, k = 0, \dots, r-1). \tag{2.6}$$

For $j_0 = j_1$, $\boldsymbol{C}_{\boldsymbol{y}}$ is a diagonal matrix of the form

$$\boldsymbol{C}_{\boldsymbol{y}}(\omega) := \mathrm{diag}(\underbrace{1, \dots, 1}_{j_0}, (1 - e^{-i\omega})/y_{j_0}, \underbrace{1, \dots, 1}_{r-1-j_0}). \tag{2.7}$$

It is easy to observe that $\boldsymbol{C}_{\boldsymbol{y}}(\omega)$ is invertible for $\omega \ne 0$. In particular,

$$\det \boldsymbol{C}_{\boldsymbol{y}}(\omega) = \left(\prod_{\substack{\nu=0 \\ y_\nu \ne 0}}^{r-1} y_\nu^{-1} \right) (1 - e^{-i\omega}). \tag{2.8}$$

Furthermore, $\boldsymbol{C}_{\boldsymbol{y}}$ is chosen so that $\boldsymbol{y}^{\mathrm{T}}\, \boldsymbol{C}_{\boldsymbol{y}}(0) = \boldsymbol{0}^{\mathrm{T}}$. We introduce

$$\boldsymbol{G}_{\boldsymbol{y}}(\omega) := (1 - e^{-i\omega})\, \boldsymbol{C}_{\boldsymbol{y}}^{-1}(\omega) . \tag{2.9}$$

If $\boldsymbol{y}$ is of the form (2.4), then $\boldsymbol{G}_{\boldsymbol{y}}(\omega) = \widetilde{\boldsymbol{G}}_{\boldsymbol{y}}(\omega) \oplus (1 - e^{-i\omega})\, \boldsymbol{I}_{r-l}$ with

$$\widetilde{\boldsymbol{G}}_{\boldsymbol{y}}(\omega) := \begin{bmatrix} y_0 & y_1 & y_2 & \dots & y_{l-1} \\ y_0 z & y_1 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & y_{l-1} \\ y_0 z & y_1 z & \ddots & y_{l-2} & y_{l-1} \\ y_0 z & y_1 z & \dots & y_{l-2} z & y_{l-1} \end{bmatrix} \qquad (z := e^{-i\omega}) .$$

Finally, let $\boldsymbol{e} = (e_\nu)_{\nu=0}^{r-1}$ corresponding to $\boldsymbol{y} = (y_\nu)_{\nu=0}^{r-1}$ be defined by

$$e_\nu := \begin{cases} 1 & \text{for } y_\nu \neq 0 \\ 0 & \text{for } y_\nu = 0 \end{cases} \qquad (\nu = 0, \ldots, r-1). \tag{2.10}$$

Now we can proceed with the factorization of $\boldsymbol{P}(\omega)$.

THEOREM 2.3. [P3] *Let $m > 1$ be fixed. Assume that $\boldsymbol{P} \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ provides approximation order m with vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$ ($\boldsymbol{y}_0 \neq \boldsymbol{0}$). Then*

$$\widetilde{\boldsymbol{P}}(\omega) := 2\, \boldsymbol{C}_{\boldsymbol{y}_0}(2\omega)^{-1}\, \boldsymbol{P}(\omega)\, \boldsymbol{C}_{\boldsymbol{y}_0}(\omega), \tag{2.11}$$

with $\boldsymbol{C}_{\boldsymbol{y}_0}(\omega)$ defined by $\boldsymbol{y}_0$ via (2.6)–(2.7), provides approximation order at least $m-1$ with vectors $\widetilde{\boldsymbol{y}}_0, \ldots, \widetilde{\boldsymbol{y}}_{m-2}$, given by

$$(\widetilde{\boldsymbol{y}}_k)^T := \frac{1}{k+1} \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} i^{\nu-k-1}\, (\boldsymbol{y}_\nu)^T\, (D^{k+1-\nu} \boldsymbol{C}_{\boldsymbol{y}_0})(0) \tag{2.12}$$

for $k = 0, \ldots, m-2$. In particular $\widetilde{\boldsymbol{y}}_0 \neq \boldsymbol{0}$.

Moreover, if $\boldsymbol{e}$ corresponds to $\boldsymbol{y}_0$ in the sence of (2.10), then $\widetilde{\boldsymbol{P}}(\omega)$ in (2.11) satisfies $\widetilde{\boldsymbol{P}}(0)\, \boldsymbol{e} = \boldsymbol{e}$.

Assume that $\boldsymbol{P} \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ provides approximation order m, then repeated application of Theorem 2.3 yields the desired factorization of $\boldsymbol{P}(\omega)$:

$$\boldsymbol{P}(\omega) = \frac{1}{2^m} \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2\omega) \ldots \boldsymbol{C}_{\boldsymbol{x}_0}(2\omega)\, \boldsymbol{P}^{(0)}(\omega)\, \boldsymbol{C}_{\boldsymbol{x}_0}(\omega)^{-1} \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(\omega)^{-1}. \tag{2.13}$$

Here $\boldsymbol{P}^{(0)}(\omega) \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ and $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1} \in \mathbb{R}^r$ are defined recursively by (2.12) [P3]. In particular, $\boldsymbol{x}_{m-1} = \boldsymbol{y}_0$ and, by (2.8),

$$\det \boldsymbol{P}(\omega) = \left(\frac{1 + e^{-i\omega}}{2^r} \right)^m \det \boldsymbol{P}^{(0)}(\omega).$$

However, the factorization (2.13) is not unique. The following theorem is a generalization of Theorem 2.3.

THEOREM 2.4. [S1] *Let $m \geq 1$ be fixed, and let $\boldsymbol{P} \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ provide approximation order m with vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$. Further, let $\boldsymbol{M} \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ satisfy the following conditions:*

1. *$\boldsymbol{M}(\omega)$ is invertible for all $\omega \neq 0$.*
2. *$\boldsymbol{M}(0)$ has a simple eigenvalue 0 with a corresponding left eigenvector $\boldsymbol{y}_0$ and $D(\det \boldsymbol{M})(0) \neq 0$.*

Then,

$$\widetilde{\boldsymbol{P}}(\omega) = 2\, \boldsymbol{M}(2\omega)^{-1}\, \boldsymbol{P}(\omega)\, \boldsymbol{M}(\omega) \tag{2.14}$$

provides approximation order at least $m-1$ with vectors $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{m-2}$ ($m > 1$) defined by

$$\boldsymbol{u}_k^T := \frac{1}{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} i^{j-k-1}\, \boldsymbol{y}_j^T\, (D^{k+1-j} \boldsymbol{M})(0) \qquad (k = 0, \ldots, m-2).$$

In particular, $\boldsymbol{u}_0 \neq 0$. *If* $\boldsymbol{P}$ *exactly provides approximation order* $m = 1$, *then* $\widetilde{\boldsymbol{P}}(0)$ *has the eigenvalue* 1 *but there exists no vector* $\boldsymbol{y} \neq \boldsymbol{0}$ *such that* $\widetilde{\boldsymbol{P}}(\omega)$ *satisfies* (2.1), (2.2) *for* $n = 0$.

In [S1], this result was obtained directly, using similar ideas as in the proof of Theorem 2.3 in [P3]. Here we would like to give another proof which clearly shows the connection between the particular factorization matrix $\boldsymbol{C}_{\boldsymbol{y}_0}$ and general factorization matrices $\boldsymbol{M}$.

LEMMA 2.5. *Let* $\boldsymbol{y} \in \mathbb{R}^r$ *be a fixed nonzero vector, and let* $\boldsymbol{M} \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ *satisfy the assumptions* 1 *and* 2 *of Theorem* 2.4 *(with* $\boldsymbol{y}$ *instead of* $\boldsymbol{y}_0$*). Further, let* $\boldsymbol{C}_{\boldsymbol{y}}$ *be an* $r \times r$ *matrix defined by* $\boldsymbol{y}$ *via* (2.6)–(2.7). *Then, there exists a matrix* $\boldsymbol{M}_0(\omega) \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ *which is invertible for all* $\omega \in \mathbb{R}$, *and*

$$\boldsymbol{C}_{\boldsymbol{y}}(\omega)\, \boldsymbol{M}_0(\omega) = \boldsymbol{M}(\omega).$$

Proof. Let $\boldsymbol{G}_{\boldsymbol{y}}$ be the $r \times r$ matrix defined by $\boldsymbol{C}_{\boldsymbol{y}}$ via (2.9). Define $\boldsymbol{M}_0(\omega)$ as follows:

$$\boldsymbol{M}_0(\omega) := \begin{cases} \boldsymbol{C}_{\boldsymbol{y}}(\omega)^{-1}\, \boldsymbol{M}(\omega) & \text{for } \omega \neq 0, \\ (-i)\left((\mathrm{D}\boldsymbol{G}_{\boldsymbol{y}})(0)\, \boldsymbol{M}(0) + \boldsymbol{G}_{\boldsymbol{y}}(0)\, (\mathrm{D}\boldsymbol{M})(0)\right) & \text{for } \omega = 0. \end{cases}$$

Here, $\boldsymbol{M}_0(0)$ is found by the rule of l'Hospital from

$$M_0(0) = \lim_{\omega\to 0} \boldsymbol{C}_{\boldsymbol{y}}(\omega)^{-1}\, \boldsymbol{M}(\omega) = \lim_{\omega\to 0} \frac{1}{1 - e^{-i\omega}}\, \boldsymbol{G}_{\boldsymbol{y}}(\omega)\, \boldsymbol{M}(\omega).$$

Since $\boldsymbol{C}_{\boldsymbol{y}}(\omega)$ is invertible for $\omega \neq 0$, the relation $\boldsymbol{C}_{\boldsymbol{y}}(\omega)\, \boldsymbol{M}_0(\omega) = \boldsymbol{M}(\omega)$ easily follows for $\omega \neq 0$. For $\omega = 0$, we find

$$\boldsymbol{C}_{\boldsymbol{y}}(0)\, \boldsymbol{M}_0(0) = (-i)\ \left(\boldsymbol{C}_{\boldsymbol{y}}(0)\, (\mathrm{D}\boldsymbol{G}_{\boldsymbol{y}})(0)\, \boldsymbol{M}(0) + \boldsymbol{C}_{\boldsymbol{y}}(0)\, \boldsymbol{G}_{\boldsymbol{y}}(0)\, (\mathrm{D}\boldsymbol{M})(0)\right).$$

Observe that, by definition,

$$\boldsymbol{G}_{\boldsymbol{y}}(\omega)\, \boldsymbol{C}_{\boldsymbol{y}}(\omega) = \boldsymbol{C}_{\boldsymbol{y}}(\omega)\, \boldsymbol{G}_{\boldsymbol{y}}(\omega) = (1 - e^{-i\omega})\, \boldsymbol{I}_r.$$

Hence,

$$\begin{aligned} \boldsymbol{C}_{\boldsymbol{y}}(0)\, \boldsymbol{G}_{\boldsymbol{y}}(0) &= \boldsymbol{0}_r, \\ (\mathrm{D}\boldsymbol{C}_{\boldsymbol{y}})(0)\, \boldsymbol{G}_{\boldsymbol{y}}(0) + \boldsymbol{C}_{\boldsymbol{y}}(0)\, (\mathrm{D}\boldsymbol{G}_{\boldsymbol{y}})(0) &= i\boldsymbol{I}_r. \end{aligned}$$

From the assumption $\boldsymbol{y}^{\mathrm{T}}\, \boldsymbol{M}(0) = \boldsymbol{0}^{\mathrm{T}}$ we have $\boldsymbol{G}_{\boldsymbol{y}}(0)\boldsymbol{M}(0) = \boldsymbol{0}_r$. Thus,

$$\boldsymbol{C}_{\boldsymbol{y}}(0)\, \boldsymbol{M}_0(0) = (-i)\ \left(i\boldsymbol{I}_r - (\mathrm{D}\boldsymbol{C}_{\boldsymbol{y}})(0)\, \boldsymbol{G}_{\boldsymbol{y}}(0)\right)\ \boldsymbol{M}(0) = \boldsymbol{M}(0).$$

We see that $\boldsymbol{C}_{\boldsymbol{y}}(\omega)\, \boldsymbol{M}_0(\omega) = \boldsymbol{M}(\omega)$ for all $\omega \in \mathbb{R}$. Since $\boldsymbol{C}_{\boldsymbol{y}}(\omega)$ and $\boldsymbol{M}(\omega)$ are invertible for $\omega \neq 0$, $\boldsymbol{M}_0(\omega)$ is also invertible for $\omega \neq 0$. Further, since $\mathrm{D}(\det \boldsymbol{C}_{\boldsymbol{y}})(0) \neq 0$ and $\mathrm{D}(\det \boldsymbol{M})(0) \neq 0$, it follows that

$$\det \boldsymbol{M}_0(0) = \lim_{\omega\to 0} \frac{\det \boldsymbol{M}(\omega)}{\det \boldsymbol{C}_{\boldsymbol{y}}(\omega)} = \frac{\mathrm{D}(\det \boldsymbol{M}(0))}{\mathrm{D}(\det \boldsymbol{C}_{\boldsymbol{y}}(0))} \neq 0.$$

Thus, $\boldsymbol{M}_0(0)$ is invertible. $\square$

Proof. (*of Theorem* 2.4) Recall that by Theorem 2.2, a TST with an invertible transformation matrix does not change the approximation order of a refinement mask. Using the result of Lemma 2.5, we simply observe that a factorization step (2.14) with a matrix $\boldsymbol{M}(\omega)$ can be considered as a combination of factorization step (2.11) with $\boldsymbol{C}_{\boldsymbol{y}_0}(\omega)$ and a non-degenerate TST with the transform matrix $\boldsymbol{M}_0(\omega)$. ∎

While the matrices $\boldsymbol{C}_{\boldsymbol{y}}$ are determined by a left eigenvector $\boldsymbol{y}$ of $\boldsymbol{C}_{\boldsymbol{y}}(0)$ to the eigenvalue 0, we want to identify the matrices $\boldsymbol{M}$ with the help of right eigenvectors of $\boldsymbol{M}(0)$ to the eigenvalue 0. Letting $\boldsymbol{r}_0$ be a right eigenvector of $\boldsymbol{M}(0)$ in Theorem 2.4, we then have $\boldsymbol{M}_{\boldsymbol{r}_0} := \boldsymbol{M}$. Hence, similarly to (2.13), repeated application of Theorem 2.4 gives a general factorization of $\boldsymbol{P}(\omega)$:

$$\boldsymbol{P}(\omega) = \frac{1}{2^m} \boldsymbol{M}_{\boldsymbol{r}_{m-1}}(2\omega) \, \ldots \, \boldsymbol{M}_{\boldsymbol{r}_0}(2\omega) \, \boldsymbol{P}^{(0)}(\omega) \, \boldsymbol{M}_{\boldsymbol{r}_0}(\omega)^{-1} \, \ldots \, \boldsymbol{M}_{\boldsymbol{r}_{m-1}}(\omega)^{-1}. \tag{2.15}$$

2.4. Factorization implies approximation order. In this subsection we state the main theoretical results of the paper. First, let us again return for a moment to the scalar case ($r = 1$). In [St1], it was shown that the approximation order defines the number of factors $(1+e^{-i\omega})$ in $P(\omega)$, and on the other hand each such factor increases the approximation order by one. Therefore, our next step is to prove the reverse of Theorem 2.3 and Theorem 2.4, or in other words, to show that the factorization (2.15) of the refinement mask yields approximation order m for the corresponding refinable function vector.

To this end, we need to introduce the "modified" Bernoulli numbers $\widetilde{B}_n$ ($n \in \mathbb{N}$), defined by the following relations:

$$\widetilde{B}_0 = 1, \qquad \sum_{l=0}^{n} \binom{n+1}{l} (-1)^l \, \widetilde{B}_l = 0, \tag{2.16}$$

or

$$\widetilde{B}_0 = 1, \qquad \widetilde{B}_n = \frac{(-1)^{n+1}}{n+1} \sum_{l=0}^{n-1} \binom{n+1}{l} (-1)^l \, \widetilde{B}_l \qquad (n \ge 1).$$

In particular,

$$\widetilde{B}_1 = \frac{1}{2}, \quad \widetilde{B}_2 = \frac{1}{6}, \quad \widetilde{B}_4 = -\frac{1}{30}.$$

Note that, apart from $\widetilde{B}_1$, the modified Bernoulli numbers coincide with the usual Bernoulli numbers B_n:

$$\widetilde{B}_n = B_n \quad (n \in \mathbb{N} \setminus \{1\}), \qquad \widetilde{B}_1 = -B_1.$$

This means that $\widetilde{B}_{2n+1} = B_{2n+1} = 0$ ($n \ge 1$), and we have

$$\sum_{l=0}^{n} \binom{n+1}{l} (-2)^l \, \widetilde{B}_l = \sum_{l=0}^{n} \binom{n+1}{l} 2^l \, B_l = \begin{cases} 1 & n = 0, \\ 2(-2^{n+1}+1) \, \widetilde{B}_{n+1} & n \ge 1 \end{cases} \tag{2.17}$$

(see [AS]).

Now we are ready to state the main results of this section.

THEOREM 2.6. *Let $m \ge 1$ be a fixed integer and let $\widetilde{\boldsymbol{P}} \in C_{2\pi}^m(\mathbb{R}^{r \times r})$ be a refinement mask providing the approximation order m with $\widetilde{\boldsymbol{y}}_0, \ldots, \widetilde{\boldsymbol{y}}_{m-1} \in \mathbb{R}^r$ ($\widetilde{\boldsymbol{y}}_0 \ne \boldsymbol{0}$).*

Further, assume that there is a vector $\boldsymbol{e} \in \mathbb{R}^r$ $(\boldsymbol{e} \neq \boldsymbol{0})$*, containing only the entries* 0 *or* 1*, such that* $\widetilde{\boldsymbol{P}}(0)\,\boldsymbol{e} = \boldsymbol{e}$*. Let* $\boldsymbol{y} = (y_\nu)_{\nu=0}^{r-1} \in \mathbb{R}^r$ $(\boldsymbol{y} \neq \boldsymbol{0})$ *be an arbitrary vector such that* $\boldsymbol{e}$ *corresponds to* $\boldsymbol{y}$ *in the sense of* (2.10)*. Then the matrix* $\boldsymbol{P}(\omega)$*,*

$$\boldsymbol{P}(\omega) := \frac{1}{2}\,\boldsymbol{C_y}(2\omega)\,\widetilde{\boldsymbol{P}}(\omega)\,\boldsymbol{C_y}(\omega)^{-1}$$

with $\boldsymbol{C_y}$ *defined by* $\boldsymbol{y}$ *via* (2.6)–(2.7)*, provides approximation order at least* $m+1$ *with vectors* $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_m$*,*

$$\boldsymbol{y}_k^T := (-ik)\,\widetilde{\boldsymbol{y}}_{k-1}^T\,(D\boldsymbol{G_y})(0) + \sum_{l=0}^{k} \binom{k}{l}\,\widetilde{B}_{k-l}\,\widetilde{\boldsymbol{y}}_l^{\,T}\,\boldsymbol{G_y}(0) \quad (k = 0, \ldots, m-1), \tag{2.18}$$

$$\begin{aligned} \boldsymbol{y}_m^T := {} & (-im)\,\widetilde{\boldsymbol{y}}_{m-1}^T\,(D\boldsymbol{G_y})(0) + \sum_{l=0}^{m-1} \binom{m}{l}\,\widetilde{B}_{m-l}\,\widetilde{\boldsymbol{y}}_l^{\,T}\,\boldsymbol{G_y}(0) \\ & - \frac{2^m}{2^m-1} \sum_{k=0}^{m-1} \binom{m}{k}\,(2i)^{k-m}\,\widetilde{\boldsymbol{y}}_k^{\,T}\,(D^{m-k}\widetilde{\boldsymbol{P}})(0)\,\boldsymbol{G_y}(0), \end{aligned} \tag{2.19}$$

where $\widetilde{\boldsymbol{y}}_{-1} := \boldsymbol{0}$*.*

The proof of Theorem 2.6 is presented in §4. In particular, we obtain from (2.18) that $\boldsymbol{y}_0^{\mathrm{T}} = \widetilde{\boldsymbol{y}}_0^{\mathrm{T}}\,\boldsymbol{G_y}(0) = \left(\sum_{\nu=0}^{r-1} \widetilde{y}_{0,\nu}\right)\,\boldsymbol{y}^{\mathrm{T}}$ with $\widetilde{\boldsymbol{y}}_0 = (\widetilde{y}_{0,\nu})_{\nu=0}^{r-1}$. Observe that the technical assumption $\widetilde{\boldsymbol{P}}(0)\boldsymbol{e} = \boldsymbol{e}$ ensures, that $\boldsymbol{C_y}$ has the same right eigenvector $\boldsymbol{e}$ to the eigenvalue 0 as $\widetilde{\boldsymbol{P}}(0)$ to the eigenvalue 1.

Again, we can generalize this result using the TST.

THEOREM 2.7. *Let* $m \geq 1$ *be a fixed integer and let* $\widetilde{\boldsymbol{P}} \in C_{2\pi}^m(\mathbb{R}^{r\times r})$ *be a refinement mask providing approximation order* m *with vectors* $\widetilde{\boldsymbol{y}}_0, \ldots, \widetilde{\boldsymbol{y}}_{m-1} \in \mathbb{R}^r$ $(\boldsymbol{y}_0 \neq \boldsymbol{0})$*. Further, let* $\boldsymbol{r}$ *be a right eigenvector of* $\widetilde{\boldsymbol{P}}(0)$ *to the eigenvalue* 1*.*
Choose a matrix $\boldsymbol{M_r}(\omega) \in C_{2\pi}^m(\mathbb{R}^{r\times r})$*, such that*

1. $\boldsymbol{M_r}(\omega)$ *is invertible for all* $\omega \in \mathbb{R}$*,* $\omega \neq 0$*.*
2. $\boldsymbol{M_r}(0)$ *has a simple eigenvalue* 0 *with* $\boldsymbol{M_r}(0)\,\boldsymbol{r} = \boldsymbol{0}$*.*
3. $D(\det \boldsymbol{M_r})(0) \neq 0$*.*

Let $\boldsymbol{u}$ *be a left eigenvector of* $\boldsymbol{M_r}(0)$ *corresponding to the eigenvalue* 0*. Then the matrix*

$$\boldsymbol{P}(\omega) := \frac{1}{2}\,\boldsymbol{M_r}(2\omega)\,\widetilde{\boldsymbol{P}}(\omega)\,\boldsymbol{M_r}(\omega)^{-1} \tag{2.20}$$

provides approximation order at least $m+1$ *with vectors* $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_m$*, given by*

$$\boldsymbol{u}_k^T := (-ik)\,\widetilde{\boldsymbol{u}}_{k-1}^T\,(D\boldsymbol{G_u})(0) + \sum_{l=0}^{k} \binom{k}{l}\,\widetilde{B}_{k-l}\,\widetilde{\boldsymbol{u}}_l^{\,T}\,\boldsymbol{G_u}(0) \qquad (k = 0, \ldots, m-1),$$

$$\begin{aligned} \boldsymbol{u}_m^T := {} & (-im)\,\widetilde{\boldsymbol{u}}_{m-1}^T\,(D\boldsymbol{G_u})(0) + \sum_{l=0}^{m-1} \binom{m}{l}\,\widetilde{B}_{m-l}\,\widetilde{\boldsymbol{u}}_l^{\,T}\,\boldsymbol{G_u}(0) \\ & - \frac{2^m}{2^m-1} \sum_{k=0}^{m-1} \binom{m}{k}\,(2i)^{k-m}\,\widetilde{\boldsymbol{u}}_k^{\,T}\,(D^{m-k}(\boldsymbol{M}_0(2\cdot)^{-1}\,\widetilde{\boldsymbol{P}}\,\boldsymbol{M}_0))(0)\,\boldsymbol{G_u}(0), \end{aligned}$$

where $\boldsymbol{M}_0(\omega)$ *is an invertible matrix such that* $\boldsymbol{C}_{\boldsymbol{u}}(\omega)\, \boldsymbol{M}_0(\omega) = \boldsymbol{M}_{\boldsymbol{r}}(\omega)$, $\widetilde{\boldsymbol{u}}_{-1} := \boldsymbol{0}$, *and* $\widetilde{\boldsymbol{u}}_k^T := \sum_{l=0}^{k} \binom{k}{l}\, i^{(l-k)}\, \widetilde{\boldsymbol{y}}_l^T\, (D^{k-l} \boldsymbol{M}_0^{-1})(0)$ *for* $k = 0, \ldots, m-1$.

Proof. In [S1], it is shown that $\boldsymbol{P}(\omega)$ defined by (2.20) is in $C_{2\pi}^m(\mathbb{R}^{r\times r})$ and $\boldsymbol{P}(0)$ has a left eigenvector $\boldsymbol{u}$, corresponding to the eigenvalue 1:

$$\boldsymbol{u}^{\mathrm{T}}\, \boldsymbol{P}(0) = \boldsymbol{u}^{\mathrm{T}}.$$

Let $\boldsymbol{C}_{\boldsymbol{u}}$ be defined by $\boldsymbol{u}$ via (2.6)–(2.7), then $\boldsymbol{u}^{\mathrm{T}}\, \boldsymbol{C}_{\boldsymbol{u}}(0) = \boldsymbol{0}^{\mathrm{T}}$. By Lemma 2.5, there exists a regular matrix $\boldsymbol{M}_0 \in \boldsymbol{C}_{2\pi}^m(\mathbb{R}^{r\times r})$ such that

$$\boldsymbol{C}_{\boldsymbol{u}}(\omega)\, \boldsymbol{M}_0(\omega) = \boldsymbol{M}_{\boldsymbol{r}}(\omega).$$

Recall that the eigenvalue 0 of $\boldsymbol{C}_{\boldsymbol{u}}(0)$ is simple, and we have $\boldsymbol{C}_{\boldsymbol{u}}(0)\, \boldsymbol{e} = \boldsymbol{0}$, where $\boldsymbol{e}$ is connected with $\boldsymbol{u}$ via (2.10). Hence, from $\boldsymbol{M}_{\boldsymbol{r}}(0)\, \boldsymbol{r} = \boldsymbol{0}$, it follows that $\boldsymbol{M}_0(0)\, \boldsymbol{r} = c\, \boldsymbol{e}$ with some constant $c \neq 0$. Since $\boldsymbol{M}_0(\omega)$ is invertible for all $\omega \in \mathbb{R}$, Theorem 2.2 implies that the matrix $\boldsymbol{M}_0(2\omega)\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{M}_0(\omega)^{-1}$ also provides approximation order m. Furthermore,

$$\boldsymbol{M}_0(0)\, \widetilde{\boldsymbol{P}}(0)\, \boldsymbol{M}_0(0)^{-1}\, \boldsymbol{e} = \frac{1}{c}\, \boldsymbol{M}_0(0)\, \widetilde{\boldsymbol{P}}(0) \boldsymbol{r} = \frac{1}{c}\, \boldsymbol{M}_0(0) \boldsymbol{r} = \boldsymbol{e}.$$

Now, we are ready to apply Theorem 2.6 to the matrix $\boldsymbol{M}_0(2\omega)\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{M}_0(\omega)^{-1}$, yielding that

$$\begin{aligned}\boldsymbol{P}(\omega) &= \frac{1}{2}\, \boldsymbol{M}_{\boldsymbol{r}}(2\omega)\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{M}_{\boldsymbol{r}}(\omega)^{-1} \\ &= \frac{1}{2}\, \boldsymbol{C}_{\boldsymbol{u}}(2\omega)\, \boldsymbol{M}_0(2\omega)\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{M}_0(\omega)^{-1}\, \boldsymbol{C}_{\boldsymbol{u}}(\omega)^{-1}\end{aligned}$$

provides approximation order at least $m+1$. The construction of $\boldsymbol{u}_k$ $(k = 0, \ldots, m)$ follows from Theorems 2.6 and 2.2. $\square$

Remarks. 1. Let us mention that a degenerate TST with $\mathrm{D}(\det \boldsymbol{M})(0) \neq 0$ can change the approximation order only by one. This fact does not follow directly from Theorem 2.4 or from Theorem 2.7. Only together these theorems imply it.

2. In particular, we obtain that, in Theorem 2.7, the vector $\boldsymbol{u}_0$ is a multiple of $\boldsymbol{u}$, since $\boldsymbol{u}_0^{\mathrm{T}} = \widetilde{\boldsymbol{u}}_0^{\mathrm{T}}\, \boldsymbol{G}_{\boldsymbol{u}}(0)$.

Repeated application of Theorem 2.7 yields the following Corollary.

COROLLARY 2.8. *Suppose that a matrix* $\boldsymbol{P}^{(0)}(\omega) \in C_{2\pi}^{m-1}(\mathbb{R}^{r\times r})$ *is given. Moreover, let*

$$\boldsymbol{P}^{(0)}(0) \boldsymbol{r}_0 = \boldsymbol{r}_0, \quad \boldsymbol{x}_0^T \boldsymbol{P}^{(0)}(0) = \boldsymbol{x}_0^T, \quad \boldsymbol{x}_0^T \boldsymbol{P}^{(0)}(\pi) \neq 0,$$

for some $\boldsymbol{x}_0, \boldsymbol{r}_0 \in \mathbb{R}^r$. *For* $n = 1, \ldots, m$, *construct the matrices*

$$\boldsymbol{P}^{(n)}(\omega) := \frac{1}{2} \boldsymbol{M}_{\boldsymbol{r}_{n-1}}(2\omega) \boldsymbol{P}^{(n-1)}(\omega) \boldsymbol{M}_{\boldsymbol{r}_{n-1}}^{-1}(\omega).$$

Here $\boldsymbol{M}_{\boldsymbol{r}_{n-1}}(\omega)$ *are chosen such that*

1. $\boldsymbol{M}_{\boldsymbol{r}_{n-1}}(\omega)$ *is invertible for all* $\omega \neq 0$ *and* $D(\det \boldsymbol{M}_{\boldsymbol{r}_{n-1}})(0) \neq 0$;
2. $\boldsymbol{M}(0)$ *has a simple eigenvalue* 0 *with a right eigenvector* $\boldsymbol{r}_{n-1}$,

$$\boldsymbol{M}_{\boldsymbol{r}_{n-1}}(0) \boldsymbol{r}_{n-1} = \boldsymbol{0},$$

where $\boldsymbol{r}_{n-1}$ *is the* 1*–eigenvector of* $\boldsymbol{P}^{(n-1)}(0)$, *i.e.,* $\boldsymbol{P}^{(n-1)}(0) \boldsymbol{r}_{n-1} = \boldsymbol{r}_{n-1}$.

Then there exist vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$ $(\boldsymbol{y}_0 \neq \boldsymbol{0})$ *such that the matrix* $\boldsymbol{P}^{(m)}$

$$\boldsymbol{P}^{(m)}(\omega) := \frac{1}{2^m}\, \boldsymbol{M}_{\boldsymbol{r}_{m-1}}(2\omega) \ldots \boldsymbol{M}_{\boldsymbol{r}_0}(2\omega)\, \boldsymbol{P}^{(0)}(\omega)\, \boldsymbol{M}_{\boldsymbol{r}_0}(\omega)^{-1} \ldots \boldsymbol{M}_{\boldsymbol{r}_{m-1}}(\omega)^{-1}$$

provides approximation order m *with* $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$.

Corollary 2.8 opens an easy way to construction of multi-scaling functions with given approximation order. We discuss it in §3. Note that the bulky formulas in Theorems 2.6 and 2.7 for $\boldsymbol{y}_k$ and $\boldsymbol{u}_k$ are only of theoretical interest. They will be used for the proof, but they need not be computed during the construction.

2.5. Regularity of multi-scaling functions. In the scalar case, the approximation properties of the refinement mask are closely related with regularity of the scaling function. What happens in the vector case? To give an answer to this question we recall results from [CDP, S1].

Let $\boldsymbol{v}$ be a right eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. We introduce the spectral radius of $\boldsymbol{P}(0)$,

$$\rho(\boldsymbol{P}(0)) := \max\{|\lambda| :\ \boldsymbol{P}(0)\,\boldsymbol{x} = \lambda\,\boldsymbol{x},\ \boldsymbol{x} \neq \boldsymbol{0}\}.$$

Suppose that $\rho(\boldsymbol{P}(0)) < 2$. Then

$$\widehat{\boldsymbol{\Upsilon}}(\omega) := \lim_{n\to\infty} \Pi_{j=1}^{n} \boldsymbol{P}\left(\frac{\omega}{2^j}\right) \boldsymbol{v} \tag{2.21}$$

converges pointwise for all ω and the convergence is uniform on compact sets (see [CDP]). Moreover, the following theorem holds.

THEOREM 2.9. [CDP] *Let* $\boldsymbol{P}$ *be an* $r \times r$ *matrix of the form*

$$\boldsymbol{P}(\omega) = \frac{1}{2^m}\, \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2\omega) \ldots \boldsymbol{C}_{\boldsymbol{x}_0}(2\omega)\, \boldsymbol{P}^{(0)}(\omega)\, \boldsymbol{C}_{\boldsymbol{x}_0}(\omega)^{-1} \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(\omega)^{-1},$$

where $\boldsymbol{C}_{\boldsymbol{x}_k}$ *are defined by the vectors* $\boldsymbol{x}_k \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ *via* (2.6)–(2.7) *and* $\boldsymbol{P}^{(0)}(\omega)$ *is an* $r \times r$ *matrix with trigonometric polynomials as entries. Suppose that* $\boldsymbol{P}^{(0)}(0)\boldsymbol{e}_0 = \boldsymbol{e}_0$, *where* $\boldsymbol{e}_0$ *is defined by* $\boldsymbol{x}_0$ *via* (2.10). *Further, suppose that* $\rho(\boldsymbol{P}^{(0)}(0)) < 2$, *and let, for* $k \geq 1$,

$$\gamma_k := \frac{1}{k} \log_2 \sup_{\omega} \|\boldsymbol{P}^{(0)}\left(\frac{\omega}{2}\right) \ldots \boldsymbol{P}^{(0)}\left(\frac{\omega}{2^k}\right)\|. \tag{2.22}$$

Then there exists a constant $C > 0$ *such that for all* $\omega \in \mathbb{R}$

$$\|\widehat{\boldsymbol{\Upsilon}}(\omega)\| \leq C\,(1 + |\omega|)^{-m+\gamma_k},$$

where $\|\widehat{\boldsymbol{\Upsilon}}(\omega)\|$ *denotes the Euklidian norm of* $\widehat{\boldsymbol{\Upsilon}}(\omega) := (\widehat{\Upsilon}_\nu(\omega))_{\nu=0}^{r-1}$. *Hence, if* $\gamma_k < m - d$ $(d \in \mathbb{N})$, *then* Υ_ν $(\nu = 0, \ldots, r-1)$ *are* $d-1$ *times continuously differentiable.*

If the conditions of Theorem 2.9 are satisfied and $\inf_{k\geq 1} \gamma_k < m - 1$, then a compactly supported continuous solution $\boldsymbol{\Upsilon}(t)$ of (1.1) is unique in a wide class of functions. Further, the uniform convergence of the cascade algorithm (in time domain) is ensured (see [CDP, Sh]). Using the techniques from [S1] we obtain the following result:

COROLLARY 2.10. *Assume that for* $n = 1, \ldots, m$, $\boldsymbol{P}^{(n)}(\omega)$ *is of the form*

$$\boldsymbol{P}^{(n)}(\omega) = \frac{1}{2^n}\, \boldsymbol{M}_{\boldsymbol{r}_{n-1}}(2\omega) \ldots \boldsymbol{M}_{\boldsymbol{r}_0}(2\omega)\, \boldsymbol{P}^{(0)}(\omega)\, \boldsymbol{M}_{\boldsymbol{r}_0}(\omega)^{-1} \ldots \boldsymbol{M}_{\boldsymbol{r}_{n-1}}(\omega)^{-1}.$$

Let $\boldsymbol{P}^{(0)}(\omega)$, $\boldsymbol{P}^{(n)}(\omega)$ and $\boldsymbol{M}_{\boldsymbol{r}_{n-1}}(\omega)$ $(n = 1, \dots, m)$ satisfy the assumptions of Corollary 2.8. *Further, suppose that $\rho(\boldsymbol{P}^{(0)}(0)) < 2$ and $\inf_{k\geq 1} \gamma_k < m - d$ $(d \in \mathbb{N})$, where γ_k is defined in* (2.22). *Then, $\boldsymbol{\Upsilon}(t)$ is a compactly supported $d-1$ times continuously differentiable solution of* (1.1) *with refinement mask $\boldsymbol{P}^{(m)}(\omega)$ providing approximation order at least m.*

Similarly to the scalar case, the regularity of multi-scaling functions depends both on the approximation order and the behavior of the residual $\boldsymbol{P}^{(0)}(\omega)$. Roughly speaking, each approximation order adds one more derivative to the corresponding function vector, but the starting number of the derivatives depends on the $\boldsymbol{P}^{(0)}(\omega)$:

LEMMA 2.11. *Let $\boldsymbol{P}(\omega)$ be the refinement mask of a compactly supported continuously differentiable function vector $\boldsymbol{\phi} \in C^1(\mathbb{R}^r)$ providing approximation order at least* 1, *i.e., there exists a vector $\boldsymbol{y} \in \mathbb{R}^r$, $\boldsymbol{y} \neq \boldsymbol{0}$, such that*

$$\boldsymbol{y}^T \boldsymbol{P}(0) = \boldsymbol{y}^T, \qquad \boldsymbol{y}^T \boldsymbol{P}(\pi) = \boldsymbol{0}^T.$$

Further, assume, that $\boldsymbol{P}(0)$ has a spectrum of the form $\{1, \mu_1, \dots, \mu_{r-1}\}$ with each $\mu_\nu < 1/2$. Let $\boldsymbol{M}(\omega) \in C^1_{2\pi}(\mathbb{R})$ be an $r \times r$ matrix satisfying the assumptions 1, 2 *of Theorem* 2.4 *(with $\boldsymbol{y}$ instead of $\boldsymbol{y}_0$). Then*

$$\widetilde{\boldsymbol{P}}(\omega) := 2\,\boldsymbol{M}(2\omega)^{-1}\,\boldsymbol{P}(\omega)\,\boldsymbol{M}(\omega) \tag{2.23}$$

is the refinement mask of a continuous function vector $\boldsymbol{\psi} = (\psi_\nu)_{\nu=0}^{r-1} \in C(\mathbb{R}^r) \cap L^1(\mathbb{R})$ and there is a constant $c_0 \in \mathbb{R}$ such that

$$\widehat{\boldsymbol{\phi}}(\omega) = \frac{c_0}{i\omega}\,\boldsymbol{M}(\omega)\widehat{\boldsymbol{\psi}}(\omega).$$

In particular, if $\boldsymbol{M} = \boldsymbol{C}_{\boldsymbol{y}}\boldsymbol{M}_0$, with $\boldsymbol{C}_{\boldsymbol{y}}$ defined by $\boldsymbol{y}$ as in (2.6)–(2.7) *and a constant, invertible matrix $\boldsymbol{M}_0$, then $\boldsymbol{\psi}$ is also compactly supported.*

Proof. 1. Let us start with the case when $\widetilde{\boldsymbol{P}}(\omega) := 2\,\boldsymbol{C}_{\boldsymbol{y}}(2\omega)^{-1}\,\boldsymbol{P}(\omega)\,\boldsymbol{C}_{\boldsymbol{y}}(\omega)$ and $\boldsymbol{C}_{\boldsymbol{y}}$ is defined by $\boldsymbol{y}$ as in (2.6)–(2.7). The assumptions on the spectrum of $\boldsymbol{P}(0)$ and results of [CDP, S1] imply that $\rho(\widetilde{\boldsymbol{P}}(0)) = 1$, and 1 is a simple eigenvalue of $\widetilde{\boldsymbol{P}}(0)$. Hence, we can represent $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\psi}}$ in the form

$$\widehat{\boldsymbol{\phi}}(\omega) := \prod_{j=1}^{\infty} \boldsymbol{P}\left(\frac{\omega}{2^j}\right)\boldsymbol{a}, \qquad \widehat{\boldsymbol{\psi}}(\omega) := \prod_{j=1}^{\infty} \widetilde{\boldsymbol{P}}\left(\frac{\omega}{2^j}\right)\boldsymbol{b}, \tag{2.24}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are right eigenvectors of $\boldsymbol{P}(0)$ and $\widetilde{\boldsymbol{P}}(0)$, respectively. The convergence of the products in (2.24) is ensured by Theorem 3.2 in [CDP]. The observations in [P3] imply that $\widetilde{\boldsymbol{P}}(\omega)$ is a matrix of trigonometric polynomials ensuring a compactly supported solution $\boldsymbol{\psi}(t)$ of (1.1). The solutions $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are uniquely determined by (2.24) up to a constant factor [CDP, HC].

By the repeated substitution of (2.23) into (2.24) we get

$$\begin{aligned}
\widehat{\boldsymbol{\phi}}(\omega) &= \lim_{n\to\infty}\left(\prod_{j=1}^{n} \frac{1}{2}\,\boldsymbol{C}_{\boldsymbol{y}}\left(\frac{2\omega}{2^j}\right)\widetilde{\boldsymbol{P}}\left(\frac{\omega}{2^j}\right)\boldsymbol{C}_{\boldsymbol{y}}\left(\frac{\omega}{2^j}\right)^{-1}\right)\boldsymbol{a}\\
&= \boldsymbol{C}_{\boldsymbol{y}}(\omega)\lim_{n\to\infty}\frac{1}{2^n}\left(\prod_{j=1}^{n}\widetilde{\boldsymbol{P}}\left(\frac{\omega}{2^j}\right)\right)\boldsymbol{C}_{\boldsymbol{y}}\left(\frac{\omega}{2^n}\right)^{-1}\boldsymbol{a}.
\end{aligned}$$

Formula (2.9) gives

$$\widehat{\boldsymbol{\phi}}(\omega) = \lim_{n\to\infty} \frac{1}{2^n\,(1-e^{-i\omega/2^n})}\, \boldsymbol{C}_{\boldsymbol{y}}(\omega) \prod_{j=1}^{n} \widetilde{\boldsymbol{P}}\left(\frac{\omega}{2^j}\right)\, \boldsymbol{G}_{\boldsymbol{y}}\left(\frac{\omega}{2^n}\right)\, \boldsymbol{a}.$$

2. Replacing $\boldsymbol{C}_{\boldsymbol{y}}(\omega)$ and $\boldsymbol{C}_{\boldsymbol{y}}(2\omega)^{-1}$ by $(1-e^{-i\omega})\,\boldsymbol{G}_{\boldsymbol{y}}(\omega)^{-1}$ and $(1-e^{-2i\omega})^{-1}\,\boldsymbol{G}_{\boldsymbol{y}}(2\omega)$, respectively, we obtain from (2.23) (with $\boldsymbol{M} = \boldsymbol{C}_{\boldsymbol{y}}$) that

$$\frac{1}{2}\,(1+e^{-i\omega})\,\widetilde{\boldsymbol{P}}(\omega)\,\boldsymbol{G}_{\boldsymbol{y}}(\omega) = \boldsymbol{G}_{\boldsymbol{y}}(2\omega)\,\boldsymbol{P}(\omega).$$

In particular, for $\omega = 0$, it follows that $\widetilde{\boldsymbol{P}}(0)\,\boldsymbol{G}_{\boldsymbol{y}}(0) = \boldsymbol{G}_{\boldsymbol{y}}(0)\,\boldsymbol{P}(0)$. Hence, $\boldsymbol{G}_{\boldsymbol{y}}(0)\,\boldsymbol{a}$ is a right eigenvector of $\widetilde{\boldsymbol{P}}(0)$, and there is a constant c_0 such that

$$\boldsymbol{G}_{\boldsymbol{y}}(0)\,\boldsymbol{a} = c_0\,\boldsymbol{b}.$$

Observing that $\lim_{n\to\infty} 2^{-n}(1-e^{-i\omega/2^n})^{-1} = (i\omega)^{-1}$, we get

$$\widehat{\boldsymbol{\phi}}(\omega) = \frac{c_0}{i\omega}\,\boldsymbol{C}_{\boldsymbol{y}}(\omega) \prod_{j=1}^{\infty} \widetilde{\boldsymbol{P}}\left(\frac{\omega}{2^j}\right)\,\boldsymbol{b} = \frac{c_0}{i\omega}\,\boldsymbol{C}_{\boldsymbol{y}}(\omega)\,\widehat{\boldsymbol{\psi}}(\omega).$$

Take now a refinement mask $\boldsymbol{P}(\omega)$ of a compactly supported function vector $\boldsymbol{\phi} \in C^1(\mathbb{R}^r)$ and an arbitrary matrix $\boldsymbol{M} \in \boldsymbol{C}^1_{2\pi}(\mathbb{R}^{r\times r})$ corresponding to $\boldsymbol{P}$ such that $\boldsymbol{M}$ satisfies the conditions 1, 2 of Theorem 2.4 (with $\boldsymbol{y}$ instead of $\boldsymbol{y}_0$). Then, by Corollary 2.10,

$$\widetilde{\boldsymbol{P}}(\omega) = 2\,\boldsymbol{M}_{\boldsymbol{y}}^{-1}(2\omega)\,\boldsymbol{P}(\omega)\,\boldsymbol{M}_{\boldsymbol{y}}(\omega)$$

is a refinement mask of a continuous function vector $\boldsymbol{\psi} \in C(\mathbb{R}^r)$. Using Lemma 2.5 we can prove the relation

$$\widehat{\boldsymbol{\phi}}(\omega) = \frac{c_0}{i\omega}\,\boldsymbol{M}_{\boldsymbol{y}}(\omega)\,\widehat{\boldsymbol{\psi}}(\omega)$$

with an arbitrary chosen constant c_0 in the same manner as above. $\square$

Using the spectral properties of transition operators, more results on regularity can be obtained [CDP, Sh, J].

2.6. Symmetry of multi-scaling functions. In many applications, symmetry of the scaling functions is very desirable. Unfortunately, this property is very restrictive, and in the scalar case symmetry cannot be combined with orthogonality. In the vector case, there is more freedom, and the components of a refinable function vector can be symmetric and orthogonal at the same time. One such example was constructed in [GHM] and is shown in Figure 1. In this subsection we are going to discuss some results on symmetry of multi-scaling functions. All details can be found in [S1].

We say that a refinable function vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ *is symmetric* if all its components $\phi_\nu(t)$ are symmetric or antisymmetric. Symmetry implies some restrictions on a refinement mask $\boldsymbol{P}(\omega)$.

THEOREM 2.12. [S1] *If there is a diagonal matrix*

$$\boldsymbol{E}(\omega) := \operatorname{diag}\left(\pm e^{-i2T_0\omega}, \ldots, \pm e^{-i2T_{r-1}\omega}\right)$$

such that the refinement mask $\boldsymbol{P}(\omega)$ *of a refinable function vector* $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ *satisfies*

$$\boldsymbol{P}(\omega) = \boldsymbol{E}(2\omega)\boldsymbol{P}(-\omega)\boldsymbol{E}(\omega)^{-1}, \tag{2.25}$$

then $\boldsymbol{\phi}$ *is symmetric. The constants* T_ν *occuring in* $\boldsymbol{E}(\omega)$ *are points of symmetry of the components* $\phi_\nu(t)$, *i.e.,* $\phi_\nu(T_\nu - t) = \pm\phi_\nu(T_\nu + t)$.

While constructing a vector of multi-scaling functions using Corollary 2.8, it is reasonable to start with a symmetric one and try to preserve the symmetry at each step (see §3). The following theorem specifies the factorization matrices $\boldsymbol{M}(\omega)$ which preserve the symmetry.

THEOREM 2.13. [S1] *Suppose that all components* $\widetilde{\phi}_\nu(t)$ *of a refinable function vector* $\widetilde{\boldsymbol{\phi}} = (\widetilde{\phi}_\nu)_{\nu=0}^{r-1}$ *are symmetric (or antisymmetric) with points of symmetry* $\widetilde{T}_\nu$ *determining*

$$\widetilde{\boldsymbol{E}}(\omega) := \operatorname{diag}\left(\pm e^{-i2\widetilde{T}_0\omega}, \ldots, \pm e^{-i2\widetilde{T}_{r-1}\omega}\right). \tag{2.26}$$

Take a matrix $\boldsymbol{M}(\omega) \in C_{2\pi}(\mathbb{R}^{r\times r})$ *satisfying assumptions* 1, 2 *of Theorem* 2.4, *and a matrix*

$$\boldsymbol{E}(\omega) := \operatorname{diag}\left(\pm e^{-i2T_0\omega}, \ldots, \pm e^{-i2T_{r-1}\omega}\right) \tag{2.27}$$

such that

$$\boldsymbol{M}(\omega) = -\boldsymbol{E}(\omega)\boldsymbol{M}(-\omega)\widetilde{\boldsymbol{E}}^{-1}(\omega). \tag{2.28}$$

Then the new vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$, *determined by* $\widehat{\boldsymbol{\phi}}(\omega) = \frac{c_0}{i\omega}\boldsymbol{M}(\omega)\widehat{\widetilde{\boldsymbol{\phi}}}(\omega)$, *is also symmetric and* T_ν, $\nu = 0, \ldots r-1$ *are points of symmetry of its components.*

Remark. Let us mention that if ϕ_ν has finite support l_ν, starting at point $t_1 \geq 0$ and T_ν is the point of symmetry of ϕ_ν, then $l_\nu \leq 2T_\nu$.

3. Construction of multi-scaling functions. Finally we have reached the point where we can show how to construct refinement masks which yield multi-scaling functions with desirable properties.

In the scalar case, there is no problem to find a mask providing any given order of accuracy. One can start with a trigonometric polynomial $P(\omega)$ such that $P(0) = 1$, and multiply by a power of $\frac{1}{2}(1 + e^{-i\omega})$ (see e.g. [St1]). In the vector case, a TST with transformation matrix $\boldsymbol{M}(\omega)$ (as described in Theorem 2.7) plays the role of the factor $(1 + e^{-i\omega})$.

An algorithm for the construction of refinement masks, yielding multi-scaling functions with given approximation order, can be obtained as a consequence of Corollary 2.8.

ALGORITHM 3.1. *Start with a matrix trigonometric polynomial* $\boldsymbol{P}^{(n)}(\omega)$ *providing approximation order* $n \in \mathbb{N}_0$, *such that* $\rho(\boldsymbol{P}^{(n)}(0)) < 2$. *Further, let* $\boldsymbol{P}^{(n)}(0)$ *possess an eigenvalue* 1 *with corresponding right eigenvector* $\boldsymbol{r}_n$, *i.e.,* $\boldsymbol{P}^{(n)}(0)\boldsymbol{r}_n = \boldsymbol{r}_n$.

1. *Choose* $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ *such that:*

(a) $\det \boldsymbol{M}_{\boldsymbol{r}_n}(\omega) \neq 0$ *for* $\omega \neq 0$,
(b) $D(\det \boldsymbol{M}_{\boldsymbol{r}_n})(0) \neq 0$,
(c) $\boldsymbol{M}_{\boldsymbol{r}_n}(0)\boldsymbol{r}_n = \boldsymbol{0}$.

2. *Construct the matrix* $\boldsymbol{P}^{(n+1)}(\omega)$:

$$\boldsymbol{P}^{(n+1)}(\omega) := \frac{1}{2}\boldsymbol{M}_{\boldsymbol{r}_n}(2\omega)\boldsymbol{P}^{(n)}(\omega)\boldsymbol{M}_{\boldsymbol{r}_n}^{-1}(\omega).$$

3. *Find a right eigenvector* $\boldsymbol{r}_{n+1}$ *corresponding to the eigenvalue* 1 *of* $\boldsymbol{P}^{(n+1)}(0)$.
4. *Repeat steps* 1, 2, 3 *as many times as needed.*

By Theorem 2.7, the approximation order of $\boldsymbol{P}^{(n+1)}(\omega)$ is $n+1$, and $m-n$ cycles of Algorithm 3.1 are needed to get a refinement mask $\boldsymbol{P}^{(m)}$ providing approximation order m. In [S1], it was proved that $\boldsymbol{P}^{(n+1)}(0)$ has eigenvalue 1, so step 4 is consistent.

One can see that there are two matrices to be chosen in Algorithm 3.1, the starting matrix $\boldsymbol{P}^{(n)}(\omega)$ (only once in the beginning) and the transformation matrix $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ (one on each cycle of the algorithm).

Corollary 2.10 shows that the regularity of the final function vector (determined by the refinement mask $\boldsymbol{P}^{(m)}(\omega)$) is governed by its approximation order m and by the properties of the starting matrix $\boldsymbol{P}^{(n)}(\omega)$.

The approximation order n implies that $\boldsymbol{P}^{(n)}$ can be factored:

$$\boldsymbol{P}^{(n)}(\omega) = \frac{1}{2^n}\,\boldsymbol{C}_{\boldsymbol{x}_{n-1}}(2\omega)\ldots\boldsymbol{C}_{\boldsymbol{x}_0}(2\omega)\,\boldsymbol{P}^{(0)}(\omega)\,\boldsymbol{C}_{\boldsymbol{x}_0}(\omega)^{-1}\ldots\boldsymbol{C}_{\boldsymbol{x}_{n-1}}(\omega)^{-1},$$

where $\boldsymbol{C}_{\boldsymbol{x}_k}$ are defined by vectors $\boldsymbol{x}_k \neq \boldsymbol{0}$ via (2.6)–(2.7). Further, the spectral radii of $\boldsymbol{P}^{(0)}(\omega)$ and $\boldsymbol{P}^{(k)}(0)$,

$$\boldsymbol{P}^{(k)}(\omega) := \frac{1}{2^k}\,\boldsymbol{C}_{\boldsymbol{x}_{k-1}}(2\omega)\ldots\boldsymbol{C}_{\boldsymbol{x}_0}(2\omega)\,\boldsymbol{P}^{(0)}(\omega)\,\boldsymbol{C}_{\boldsymbol{x}_0}(\omega)^{-1}\ldots\boldsymbol{C}_{\boldsymbol{x}_{k-1}}(\omega)^{-1} \quad (k \le n),$$

are related as follows [CDP, S1]:

$$\rho(\boldsymbol{P}^{(k)}(0)) = \max\{1,\, 2^{-k}\,\rho(\boldsymbol{P}^{(0)}(0))\} \quad (k = 0, \ldots, n).$$

Let k_0 $(0 \le k_0 \le n)$ be the smallest integer such that $\rho(\boldsymbol{P}^{(k_0)}(0)) < 2$. Then by Theorem 2.9, it follows that the Fourier transformed solution vector $\widehat{\boldsymbol{\phi}}_n$ of (1.3) determined by $\boldsymbol{P}^{(n)}$, satisfies

$$\|\widehat{\boldsymbol{\phi}}_n(\omega)\| \le C\,(1+|\omega|)^{-n+k_0+K_0},$$

where $K_0 := \inf_{l\ge 1}\gamma_l$, $\gamma_l = \frac{1}{l}\log_2\sup_\omega \|\boldsymbol{P}^{(k_0)}(\frac{\omega}{2})\ldots\boldsymbol{P}^{(k_0)}(\frac{\omega}{2^l})\|$. Thus, $m-n$ cycles of Algorithm 3.1 yield $\boldsymbol{P}^{(m)}$ providing a solution vector $\widehat{\boldsymbol{\phi}}_m(\omega)$ such that

$$\|\hat{\boldsymbol{\phi}}_m(\omega)\| \le C\,(1+|\omega|)^{-m+k_0+K_0}.$$

So, if we want to get a multi-scaling function with approximation order at least m and p derivatives, we need to apply $m_0 - n$ cycles of Algorithm 3.1, where m_0 is chosen such that $m_0 \ge \max\{m,\, k_0+K_0+p+1\}$.

3.1. How to choose the transformation matrices $\mathbf{M}_{\mathbf{r}_n}(\omega)$. In the scalar case, $\boldsymbol{M}_{\boldsymbol{r}_k}(\omega) = (1-e^{-i\omega})$ is fixed. In the vector case, we are flexible in the choice of $\boldsymbol{M}_{\boldsymbol{r}_k} \in C_{2\pi}(\mathbb{R}^{r\times r})$. Actually, only one eigenvalue and one eigenvector are restricted in $\boldsymbol{M}_{\boldsymbol{r}_k}(\omega)$. We can use this freedom to obtain multi-scaling functions with desired properties.

Finite support. A refinement mask $\boldsymbol{P}^{(n+1)}(\omega)$ corresponds to a finitely supported scaling vector if and only if all components of $\boldsymbol{P}^{(n+1)}(\omega)$ are trigonometric polynomials (algebraic polynomials in $z = e^{-i\omega}$). But

$$\boldsymbol{P}^{(n+1)}(\omega) = \frac{1}{2}\boldsymbol{M}_{\boldsymbol{r}_n}(2\omega)\boldsymbol{P}^{(n)}(\omega)\boldsymbol{M}_{\boldsymbol{r}_n}^{-1}(\omega) \tag{3.1}$$

contains $\boldsymbol{M}_{\boldsymbol{r}_n}(2\omega)$ and $\boldsymbol{M}^{-1}_{\boldsymbol{r}_n}(\omega)$ which generally are not matrices of trigonometric polynomials at the same time.

LEMMA 3.2. *Assume that $\boldsymbol{P}^{(n)}(\omega)$ is a matrix of trigonometric polynomials. If $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ satisfies conditions* (a)–(c) *of Algorithm* 3.1*, $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ is a matrix of trigonometric polynomials, and* $\det \boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ *is linear in $z = e^{-i\omega}$, then the components of $\boldsymbol{P}^{(n+1)}(\omega)$ in* (3.1) *are trigonometric polynomials.*

Proof. Let us use a well-known formula for an inverse matrix:

$$\boldsymbol{M}^{-1}_{\boldsymbol{r}_n}(\omega) = \frac{1}{\det \boldsymbol{M}_{\boldsymbol{r}_n}(\omega)} \boldsymbol{N}_{\boldsymbol{r}_n}(\omega). \tag{3.2}$$

Here the (i,j) element of the matrix $\boldsymbol{N}_{\boldsymbol{r}_n}(\omega)$ is the minor for the (j,i) element of $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ (see [St2], page 225). In particular, $\boldsymbol{N}_{\boldsymbol{r}_n}(\omega)$ contains only trigonometric polynomials.

Since $\det \boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ is linear in z, and $\det \boldsymbol{M}_{\boldsymbol{r}_n}(0) = 0$, we have

$$\det \boldsymbol{M}_{\boldsymbol{r}_n}(\omega) = c_0(1 - e^{-i\omega}),$$

with a constant $c_0 \neq 0$, and according to (3.2),

$$\boldsymbol{P}^{(n+1)}(\omega) = \frac{1}{2\, c_0\, (1 - e^{-i\omega})} \boldsymbol{M}_{\boldsymbol{r}_n}(2\omega) \boldsymbol{P}^{(n)}(\omega) \boldsymbol{N}_{\boldsymbol{r}_n}(\omega). \tag{3.3}$$

It is easy to see that the components of $\boldsymbol{M}_{\boldsymbol{r}_n}(2\omega)\boldsymbol{P}^{(n)}(\omega)\boldsymbol{N}_{\boldsymbol{r}_n}(\omega)$ are trigonometric polynomials. In [S1], it was proved that $\boldsymbol{P}^{(n+1)}(0)$ is bounded. On the other hand, $(1 - e^{-i\omega})^{-1}$ is infinite at $\omega = 0$. Thus, all components of $\boldsymbol{M}_{\boldsymbol{r}_n}(2\omega)\boldsymbol{P}^{(n)}(\omega)\boldsymbol{N}_{\boldsymbol{r}_n}(\omega)$ must possess a root at $\omega = 0$ or, in other words, must be divisible by $(1 - e^{-i\omega})$. Hence, reducing $\boldsymbol{M}_{\boldsymbol{r}_n}(2\omega)\boldsymbol{P}^{(n)}(\omega)\boldsymbol{N}_{\boldsymbol{r}_n}(\omega)$ by $(1-e^{-i\omega})$ we get a matrix trigonometric polynomial $\boldsymbol{P}^{(n+1)}(\omega)$. $\square$

One way to choose $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ satisfying the conditions of Algorithm 3.1 and Lemma 3.2 is given by Lemma 2.5. Take an arbitrary vector $\boldsymbol{y}_n = (y_{n,\nu})_{\nu=0}^{r-1}$ corresponding to $\boldsymbol{r}_n = (r_{n,\nu})_{\nu=0}^{r-1}$ in the sense that $y_{n,\nu} \neq 0$ if and only if $r_{n,\nu} \neq 0$ for $\nu = 0, \ldots, r-1$. Put

$$\boldsymbol{M}_{\boldsymbol{r}_n}(\omega) := \boldsymbol{C}_{\boldsymbol{y}_n}(\omega)\, \boldsymbol{R}_n$$

with $\boldsymbol{C}_{\boldsymbol{y}_n}(\omega)$ defined by $\boldsymbol{y}_n$ as in (2.6)–(2.7) and an arbitrary constant $r \times r$ matrix $\boldsymbol{R}_n$ with the only restriction

$$\boldsymbol{R}_n\, \boldsymbol{r}_n = \boldsymbol{e}_n\,,$$

where $\boldsymbol{e}_n$ corresponds to $\boldsymbol{r}_n$ via (2.10). Then $\boldsymbol{M}_{\boldsymbol{r}_n}(\omega)$ is linear in $z = e^{-i\omega}$ by construction and, by (2.8), $\det \boldsymbol{M}_{\boldsymbol{r}_n}$ is of the desired form. Moreover, we have $\boldsymbol{M}_{\boldsymbol{r}_n}(0)\, \boldsymbol{r}_n = \boldsymbol{C}_{\boldsymbol{y}_n}(0)\, \boldsymbol{R}_n\, \boldsymbol{r}_n = \boldsymbol{C}_{\boldsymbol{y}_n}(0)\, \boldsymbol{e}_n = \boldsymbol{0}$. A simple $\boldsymbol{R}_n$ satisfying the relation above is $\boldsymbol{R}_n := \operatorname{diag}(\widetilde{r}_{n,0}, \ldots, \widetilde{r}_{n,r-1})$, where

$$\widetilde{r}_{n,\nu} := \begin{cases} 1/r_{n,\nu} & r_{n,\nu} \neq 0, \\ 1 & r_{n,\nu} = 0. \end{cases}$$

Symmetry A reasonable way to get symmetric multi-scaling functions with high approximation order is to start with $\boldsymbol{P}^{(n)}(\omega)$, yielding a symmetric function vector with low approximation order, and to preserve symmetry on each cycle of Algorithm 3.1. It is remarkable that after each cycle, the number of symmetric and antisymmetric components of the multi-scaling function changes, independent of the choice of $\boldsymbol{M}_{\boldsymbol{r}_n}$:

LEMMA 3.3. *Suppose that* $\boldsymbol{M}(\omega)$ *satisfies the conditions* (a)–(c) *of Algorithm* 3.1, *and a* TST *with transformation matrix* $\boldsymbol{M}(\omega)$ *preserves the symmetry; i.e.,* $\boldsymbol{\phi} = (\phi_\nu)_{\nu=1}^{r-1}$, $\widetilde{\boldsymbol{\phi}} = (\widetilde{\phi}_\nu)_{\nu=1}^{r-1}$ *are two symmetric multi-scaling functions connected by the relation:*

$$\widehat{\boldsymbol{\phi}}(\omega) = \frac{c_0}{i\omega}\, \boldsymbol{M}(\omega)\, \widehat{\widetilde{\boldsymbol{\phi}}}(\omega). \tag{3.4}$$

Then, for even r, *the difference in the number of antisymmetric components in* $\widetilde{\boldsymbol{\phi}}$ *and* $\boldsymbol{\phi}$ *is odd, and for odd* r, *this difference is even.*

Proof. Let $\boldsymbol{P}^{(n)}(\omega)$ be the refinement mask of $\widetilde{\boldsymbol{\phi}}$ and $\boldsymbol{P}^{(n+1)}(\omega)$ the refinement mask of $\boldsymbol{\phi}$, and let $\boldsymbol{P}^{(n)}$ and $\boldsymbol{P}^{(n+1)}$ be related as in Algorithm 3.1, with $\boldsymbol{M}_{\boldsymbol{r}_n} := \boldsymbol{M}$. Then (3.4) is a consequence of Lemma 2.11. By Theorem 2.13 we have

$$\boldsymbol{M}(\omega) = -\boldsymbol{E}(\omega)\, \boldsymbol{M}(-\omega)\, \widetilde{\boldsymbol{E}}^{-1}(\omega), \tag{3.5}$$

where $\boldsymbol{E}(\omega)$, $\widetilde{\boldsymbol{E}}(\omega)$ are defined by the points of symmetry T_ν, $\widetilde{T}_\nu$ of ϕ_ν, $\widetilde{\phi}_\nu$ ($\nu = 0, \ldots, r-1$) via (2.26), (2.27). Since $\boldsymbol{M}(\omega)$ satisfies the conditions of Algorithm 3.1, $\det \boldsymbol{M}(\omega)$ has a simple zero at $\omega = 0$, such that

$$f(e^{-i\omega}) := \det \boldsymbol{M}(\omega) = (1 - e^{-i\omega}) f_0(e^{-i\omega}), \quad f_0(1) \neq 0. \tag{3.6}$$

From (3.5), (2.26) and (2.27), it follows that

$$\begin{aligned} \det \boldsymbol{M}(\omega) &= f(e^{-i\omega}) = (-1)^r \det \boldsymbol{E}(\omega) \cdot \det \widetilde{\boldsymbol{E}}^{-1}(\omega) \cdot \det \boldsymbol{M}(-\omega) \\ &= e^{-2iT\omega} f(e^{i\omega})(-1)^{N+r} \end{aligned} \tag{3.7}$$

where $T = \sum_{\nu=0}^{r-1}(T_\nu - \widetilde{T}_\nu)$, and N is the difference in the number of antisymmetric functions in $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$. Let $z := e^{-i\omega}$, then by (3.6)

$$f(z) = (1 - z) f_0(z)$$

and by (3.7)

$$f(z) = z^{2T}\, (-1)^{N+r}\, f(\frac{1}{z}) = z^{2T}\, (-1)^{N+r}\, (1 - \frac{1}{z})\, f_0(\frac{1}{z}).$$

Combining these two relations we find

$$(1 - z) f_0(z) = -(-1)^{N+r}\, z^{2T-1}\, (1 - z)\, f_0(\frac{1}{z})$$

and hence

$$f_0(z) = (-1)^{N+r+1}\, z^{2T-1}\, f_0(\frac{1}{z}). \tag{3.8}$$

But (3.8) implies that, if $N + r + 1$ is odd, then $f_0(1) = 0$ and thus $\mathrm{D}(\det \boldsymbol{M})(0) = 0$, which contradicts the assumptions. So $N + r + 1$ must be even and $N + r$ must be odd. ∎

3.2. Examples. In this final section, we employ Algorithm 3.1 for the construction of multi-scaling functions with high approximation order and other desirable properties.

Example 1. In the first example, we are going to increase the approximation order of the refinement mask $\boldsymbol{P}^{(2)}(\omega)$ corresponding to the Geronimo–Hardin–Massopust multi-scaling function $\boldsymbol{\phi} := [\phi_0\ \phi_1]^{\mathrm{T}}$ (see Figure 1):

$$\boldsymbol{P}^{(2)}(\omega) = \frac{1}{20}\begin{bmatrix} 6+6e^{-i\omega} & 8\sqrt{2} \\ (-1+9e^{-i\omega}+9e^{-2i\omega}-e^{-3i\omega})/\sqrt{2} & -3+10e^{-i\omega}-3e^{-2i\omega} \end{bmatrix}.$$

The functions $\phi_0(t)$, $\phi_1(t)$ are continuous, symmetric and provide second order approximation. The integer translates $\phi_0(t-l)$, $\phi_1(t-l)$ $(l \in \mathbb{Z})$ are orthogonal. It is easy to see that a 1–eigenvector of $\boldsymbol{P}^{(2)}(0)$ is $\boldsymbol{r}_2 = [\sqrt{2}\ \ 1]^{\mathrm{T}}$:

$$\boldsymbol{P}^{(2)}(0)\boldsymbol{r}_2 = \frac{1}{20}\begin{bmatrix} 12 & 8\sqrt{2} \\ 8\sqrt{2} & 4 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$$

Let us apply one cycle of Algorithm 3.1 to $\boldsymbol{P}^{(2)}(\omega)$ with transformation matrix $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ preserving symmetry and ensuring short support. Then, $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ must satisfy the assumptions of Lemma 3.2 and the following relation:

$$\boldsymbol{M}_{\boldsymbol{r}_2}(\omega) = -\boldsymbol{E}(\omega)\,\boldsymbol{M}_{\boldsymbol{r}_2}(-\omega)\,\widetilde{\boldsymbol{E}}^{-1}(\omega) \tag{3.9}$$

(cf. Theorem 2.13). The first GHM scaling function is symmetric about $\widetilde{T}_0 = 1/2$, and the second is symmetric about $\widetilde{T}_1 = 1$, hence $\widetilde{\boldsymbol{E}}(\omega) = \mathrm{diag}(e^{-i\omega},\, e^{-2i\omega})$. In order to get the supports of the new scaling functions as short as possible, we choose $T_0 = T_1 = 1$. Thus, let $\boldsymbol{E}(\omega) = \mathrm{diag}(-e^{-2i\omega},\, e^{-2i\omega})$. We put

$$\boldsymbol{M}_{\boldsymbol{r}_2}(\omega) := \begin{bmatrix} 1+e^{-i\omega} & -2\sqrt{2} \\ 1-e^{-i\omega} & 0 \end{bmatrix},$$

then (3.9) is satisfied. Moreover,

$$\boldsymbol{M}_{\boldsymbol{r}_2}(0)\boldsymbol{r}_2 = \begin{bmatrix} 2 & -2\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = \boldsymbol{0},$$

$\det \boldsymbol{M}_{\boldsymbol{r}_2}(\omega) = 2\sqrt{2}(1-e^{-i\omega}) \neq 0$ for $\omega \neq 0$, $\quad \mathrm{D}(\det \boldsymbol{M}_{\boldsymbol{r}_2})(0) = i2\sqrt{2} \neq 0$, so $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ satisfies all conditions of Algorithm 3.1. $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ is a matrix of trigonometric polynomials and $\det \boldsymbol{M}_{\boldsymbol{r}_2}(\omega) = 2\sqrt{2}(1-e^{-i\omega})$ is linear in $z = e^{-i\omega}$, so by Lemma 3.2, finite support for the new scaling functions is ensured.

Now we perform step 3 of Algorithm 3.1 and compute $\boldsymbol{P}^{(3)}(\omega)$:

$$\begin{aligned}\boldsymbol{P}^{(3)}(\omega) &= \frac{1}{2}\boldsymbol{M}_{\boldsymbol{r}_2}(2\omega)\boldsymbol{P}^{(2)}(\omega)\boldsymbol{M}_{\boldsymbol{r}_2}^{-1}(\omega) \\ &= \frac{1}{40}\begin{bmatrix} -7+10e^{-i\omega}-7e^{-2i\omega} & 15(1-e^{-2i\omega}) \\ -4(1-e^{-2i\omega}) & 10(1+e^{-i\omega})^2 \end{bmatrix}.\end{aligned}$$

The resulting scaling functions are continuously differentiable and provide approximation order 3. They are plotted in Figure 2.

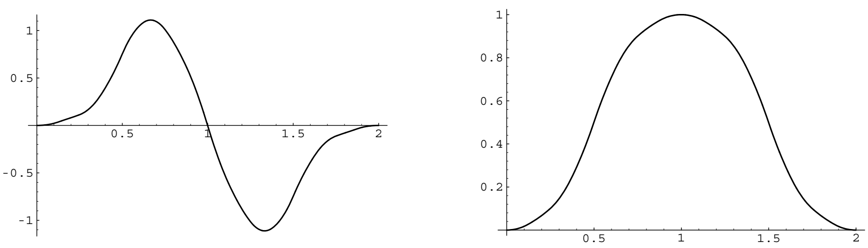

FIG. 2 *Symmetric multi-scaling function with approximation order 3*

The mask $\boldsymbol{P}^{(3)}(\omega)$ corresponds to a dilation equation (1.1) with 3 matrix coefficients

$$\boldsymbol{P}_0 = \frac{1}{40}\begin{bmatrix} -7 & 15 \\ -4 & 10 \end{bmatrix}, \quad \boldsymbol{P}_1 = \frac{1}{40}\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}, \quad \boldsymbol{P}_2 = \frac{1}{40}\begin{bmatrix} -7 & -15 \\ 4 & 10 \end{bmatrix}.$$

We mention that the GHM dilation equation has 4 coefficients since GHM functions ϕ_0, ϕ_1 have different supports.

Observe that, in accordance with Lemma 3.3, one scaling function is symmetric and the other is antisymmetric. Moreover, the sum of the supports grows exactly by 1.

Unfortunately, the new functions are not orthogonal and for practical applications a biorthogonal multi-scaling function should be constructed. This can be done using cofactor method described in [SS4].

Example 2. In the second example, we construct polynomial, symmetric multi-scaling functions with two components, short support and arbitrarily high approximation order. Let us start with the function vector $\boldsymbol{\phi}_2 := [\phi_{2,0}\ \phi_{2,1}]^{\mathrm{T}}$,

$$\phi_{2,0}(t) := \chi_{[0,1]}, \qquad \phi_{2,1}(t) := (1-2t)\,\chi_{[0,1]},$$

where $\chi_{[0,1]}$ denotes the characteristic function of $[0,\, 1]$. The index 2 in $\boldsymbol{\phi}_2$ denotes the approximation order 2 provided by $\boldsymbol{\phi}_2$. Observe, that both $\phi_{2,0}$ and $\phi_{2,1}$ are piecewise polynomials, but discontinuous, $\phi_{2,0}(1/2+t) = \phi_{2,0}(1/2-t)$ and $\phi_{2,1}(1/2+t) = -\phi_{2,1}(1/2-t)$. The vector $\boldsymbol{\phi}_2$ has the refinement mask

$$\boldsymbol{P}^{(2)}(\omega) := \frac{1}{4}\begin{bmatrix} 2+2z & 0 \\ 1-z & 1+z \end{bmatrix} \qquad (z := e^{-i\omega}),$$

with 1–eigenvector $\boldsymbol{r}_2 := [1 \quad 0]^{\mathrm{T}}$

$$\boldsymbol{P}^{(2)}(0)\,\boldsymbol{r}_2 = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We want to apply to $\boldsymbol{P}^{(2)}(\omega)$ one cycle of Algorithm 3.1 with a suitable transformation matrix $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ which preserves symmetry and short support. We try to find $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ satisfying the assumptions of Lemma 3.2 and such that

$$\boldsymbol{M}_{\boldsymbol{r}_2}(\omega) = -\,\mathrm{diag}(e^{-i\omega},\, e^{-2i\omega})\,\boldsymbol{M}_{\boldsymbol{r}_2}(-\omega)\ \mathrm{diag}(e^{-i\omega},\, -e^{-i\omega}).$$

Letting

$$\boldsymbol{M}_{\boldsymbol{r}_2}(\omega) := \begin{bmatrix} 0 & 2 \\ 1-z & -1-z \end{bmatrix} \quad (z = e^{-i\omega}),$$

we obtain by application of Algorithm 3.1,

$$\boldsymbol{P}^{(3)}(\omega) = \frac{1}{2}\,\boldsymbol{M}_{\boldsymbol{r}_2}(2\omega)\,\boldsymbol{P}_2(\omega)\,\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)^{-1} = \frac{1}{8}\begin{bmatrix} 2(1+z) & 2 \\ 2z(1+z) & 1+4z+z^2 \end{bmatrix}.$$

The corresponding compactly supported function vector $\boldsymbol{\phi}_3 = [\phi_{3,0}\ \phi_{3,1}]^{\mathrm{T}}$ provides approximation order 3, since $\boldsymbol{M}_{\boldsymbol{r}_2}(\omega)$ satisfies all assumptions of Theorem 2.7. We easily observe that

$$\phi_{3,0}(t) = \begin{cases} 2t(1-t) & t \in [0,1], \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_{3,1}(t) = \begin{cases} t^2 & t \in [0,1], \\ (2-t)^2 & t \in [1,2], \\ 0 & \text{otherwise}. \end{cases}$$

In particular, $\phi_{3,0}$ and $\phi_{3,1}$ are continuous functions. (This can also be seen by Corollary 2.10.)

Now we apply a second cycle of Algorithm 3.1 to $\boldsymbol{P}^{(3)}(\omega)$ in order to get a symmetric vector $\boldsymbol{\phi}_4$ of scaling functions $\phi_{4,0}$, $\phi_{4,1}$ with short support and approximation order 4. Observe that $\boldsymbol{P}^{(3)}(0)\,\boldsymbol{r}_3 = \boldsymbol{r}_3$ with $\boldsymbol{r}_3 := [1\ \ 2]^{\mathrm{T}}$, so the transformation matrix

$$\boldsymbol{M}_{\boldsymbol{r}_3}(\omega) := 3\begin{bmatrix} 1-z & 0 \\ 1+z & -1 \end{bmatrix} \quad (z = e^{-i\omega})$$

satisfies the assumptions of Lemma 3.2, and we have

$$\boldsymbol{M}_{\boldsymbol{r}_3}(\omega) = -\operatorname{diag}(e^{-2i\omega},\ -e^{-2i\omega})\,\boldsymbol{M}_{\boldsymbol{r}_3}(-\omega)\ \operatorname{diag}(e^{-i\omega},\ e^{-2i\omega}).$$

We construct

$$\begin{aligned} \boldsymbol{P}^{(4)}(\omega) &= \frac{1}{2}\,\boldsymbol{M}_{\boldsymbol{r}_3}(2\omega)\,\boldsymbol{P}_3(\omega)\,\boldsymbol{M}_{\boldsymbol{r}_3}(\omega)^{-1} \\ &= \frac{1}{16}\begin{bmatrix} 4(1+z)^2 & -2(1-z)(1+z) \\ 3(1-z)(1+z) & -1+4z-z^2 \end{bmatrix}. \end{aligned}$$

The corresponding (compactly supported) functions $\phi_{4,0}$ and $\phi_{4,1}$ are again piecewise polynomials:

$$\phi_{4,0}(t) = \begin{cases} (-2t^3+3t^2) & t \in [0,1), \\ (2-t)^2\,(2t-1) & t \in [1,2], \\ 0 & \text{otherwise}, \end{cases}$$

$$\phi_{4,1}(t) = \begin{cases} t^2(3t-3) & t \in [0,1) \\ (2-t)^2\,(-3t-3) & t \in [1,2] \\ 0 & \text{otherwise}. \end{cases}$$

The functions $\phi_{4,0}$ and $\phi_{4,1}$ are symmetric, continously differentiable functions. Note that $\phi_{4,0}$, $\phi_{4,1}$ are finite element functions studied in [SS3]. They are presented in

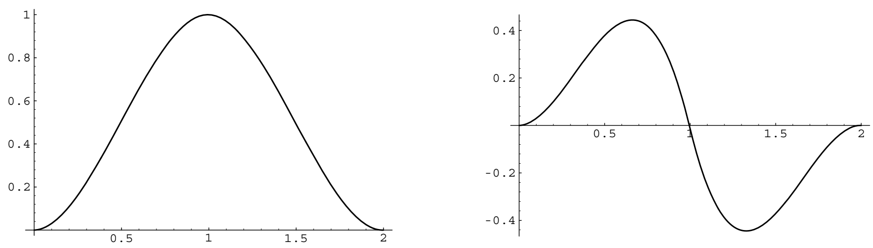

FIG. 3 *Polynomial multi-scaling function with approximation order 4*

Figure 3. Obviously, functions $\phi_{4,0}$ and $\phi_{4,1}$ are not orthogonal. For the construction of dual scaling functions and wavelets see [SS4].

The procedure can be repeated as follows. Take

$$\boldsymbol{M}_{\boldsymbol{r}_{2k}}(\omega) := \begin{bmatrix} 0 & 2 \\ 1-z & -1-z \end{bmatrix} \qquad (k \in \mathbb{N},\ z = e^{-i\omega})$$

and

$$\boldsymbol{M}_{\boldsymbol{r}_{2k+1}}(\omega) := (2k+1) \begin{bmatrix} (1-z)/k & 0 \\ (1+z)/k & -2/(k+1) \end{bmatrix} \qquad (k \in \mathbb{N},\ z = e^{-i\omega})$$

and apply Algorithm 3.1 repeatedly with these transformation matrices. The refinement mask $\boldsymbol{P}^{(n)}$ ($n \in \mathbb{N}$; $n \geq 3$) then provides approximation order n, the corresponding multi-scaling functions $\phi_{n,0}$ and $\phi_{n,1}$ are $(n-3)$–times continuously differentiable. If $n = 2k+1$ ($k \geq 1$), the corresponding multi-scaling functions $\phi_{2k+1,0}$ and $\phi_{2k+1,1}$ are nothing but polynomial B-splines of order $2k+1$ with double knots, defined by the spline knots $0, 0, 1, 1, \ldots, k, k$ and $0, 1, 1, 2, 2, \ldots, k, k, k+1$ respectively. (This follows from a comparison with known recursion formulas for the refinement mask of B-splines vectors with multiple knots [P1, P2]). In particular, $\operatorname{supp} \phi_{2k+1,0} = [0, k]$, $\operatorname{supp} \phi_{2k+1,1} = [0,\ k+1]$ and

$$\phi_{2k+1,0}(t) = \phi_{2k+1,0}(k-t), \qquad \phi_{2k+1,1}(t) = \phi_{2k+1,1}(k+1-t).$$

If $n = 2k$ ($k \geq 1$), the corresponding multi-scaling functions $\phi_{2k,0}$ and $\phi_{2k,1}$ are nothing but polynomial B-splines of order $2k$, defined as the sum and the difference of the B-splines $N_{2k,0}$, $N_{2k,1}$ of order $2k$ with double knots, respectively. In other words, if $N_{2k,0}$ and $N_{2k,1}$ are defined by the spline knots $0, 0, \ldots, k-1, k-1, k$ and $0, 1, 1, \ldots, k-1, k, k$, then $\phi_{2k,0} = N_{2k,0} + N_{2k,1}$ and $\phi_{2k,1} = N_{2k,0} - N_{2k,1}$. In particular, $\operatorname{supp} \phi_{2k,0} = \operatorname{supp} \phi_{2k,1} = [0,\ k]$ and

$$\phi_{2k,0}(t) = \phi_{2k,0}(k-t), \qquad \phi_{2k+1,1}(t) = -\phi_{2k+1,1}(k-t).$$

Remark. For $r = 1$, the refinement mask $\boldsymbol{P}(\omega) = 2^{-m}(1 + e^{-i\omega})^m$ determines the cardinal B-spline N_m of order m. Let $x_l := \lfloor l/r \rfloor$ ($l \in \mathbb{Z}$), where $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$. Then, the refinement mask

$$\boldsymbol{P}_m^r(\omega) := \frac{1}{2^m}\, \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2\omega) \ldots \boldsymbol{C}_{\boldsymbol{x}_0}(\omega)\, \boldsymbol{P}^{(0)}\, \boldsymbol{C}_{\boldsymbol{x}_0}(\omega)^{-1} \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(\omega)^{-1}$$

with $\boldsymbol{C}_{\boldsymbol{x}_k}$ defined by the vector $\boldsymbol{x}_k := (x_{k+1}, \ldots, x_{k+r})^{\mathrm{T}}$ $(k = 0, \ldots m-1)$ and

$$\boldsymbol{P}^{(0)} := \operatorname{diag}\left(2^{r-1}, \ldots, 2^0\right)$$

determines the vector of cardinal B-splines with r–fold knots.

4. Proof of Theorem 2.6. Before starting the proof of Theorem 2.6 let us show some preliminary assertions. For a given $\widetilde{\boldsymbol{P}} \in C_{2\pi}^m(\mathbb{R}^{r\times r})$ and a nonzero vector $\boldsymbol{y} \in \mathbb{R}$ let the $r \times r$ matrix $\boldsymbol{P} \in C_{2\pi}^m(\mathbb{R}^{r\times r})$ be defined by

$$\boldsymbol{P}(\omega) = \frac{1}{2}\boldsymbol{C}_{\boldsymbol{y}}(2\omega)\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{C}_{\boldsymbol{y}}(\omega)^{-1},$$

where $\boldsymbol{C}_{\boldsymbol{y}}(\omega)$ is defined by $\boldsymbol{y}$ via (2.6)–(2.7). Hence, we have by (2.9)

$$(1 - e^{-i\omega})\, \boldsymbol{G}_{\boldsymbol{y}}(2\omega)\, \boldsymbol{P}(\omega) = \frac{1}{2}\,(1 - e^{-2i\omega})\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{G}_{\boldsymbol{y}}(\omega),$$

i.e.,

$$\boldsymbol{G}_{\boldsymbol{y}}(2\omega)\, \boldsymbol{P}(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)\, \widetilde{\boldsymbol{P}}(\omega)\, \boldsymbol{G}_{\boldsymbol{y}}(\omega). \tag{4.1}$$

In the next lemma we compute $\boldsymbol{G}_{\boldsymbol{y}}(2\omega)\,(\mathrm{D}^k\boldsymbol{P})(\omega)$ in terms of derivatives of $\widetilde{\boldsymbol{P}}(\omega)$ and lower derivatives of $\boldsymbol{P}(\omega)$:

LEMMA 4.1. *We have for $k \in \mathbb{N}$,*

$$\begin{aligned}
&\boldsymbol{G}_{\boldsymbol{y}}(2\omega)\,(D^k\boldsymbol{P})(\omega)\\
&= -\sum_{l=1}^{k}\binom{k}{l}2^l\,(D^l\boldsymbol{G}_{\boldsymbol{y}})(2\omega)\,(D^{k-l}\boldsymbol{P})(\omega) + \left(\frac{1+e^{-i\omega}}{2}\right)\,(D^k\widetilde{\boldsymbol{P}})(\omega)\boldsymbol{G}_{\boldsymbol{y}}(\omega)\\
&+ \frac{1}{2}\sum_{l=1}^{k}\binom{k}{l}(D^{k-l}\widetilde{\boldsymbol{P}})(\omega)\,(-i)^{l-1}\,\left([(2^l-1)e^{-i\omega}+1](D\boldsymbol{G}_{\boldsymbol{y}})(\omega) - ie^{-i\omega}\,\boldsymbol{G}_{\boldsymbol{y}}(\omega)\right).
\end{aligned}$$

In particular,

$$\begin{aligned}
\boldsymbol{G}_{\boldsymbol{y}}(0)\,(D^k\boldsymbol{P})(0) = &-\sum_{l=1}^{k}\binom{k}{l}\,2^l\,(D^l\boldsymbol{G}_{\boldsymbol{y}})(0)\,(D^{k-l}\boldsymbol{P})(0) + (D^k\widetilde{\boldsymbol{P}})(0)\,\boldsymbol{G}_{\boldsymbol{y}}(0)\\
&+\frac{1}{2}\sum_{l=1}^{k}\binom{k}{l}\,(D^{k-l}\widetilde{\boldsymbol{P}})(0)\,(-i)^l\,[\boldsymbol{G}_{\boldsymbol{y}}(0) + 2^l i(D\boldsymbol{G}_{\boldsymbol{y}})(0)]
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{G}_{\boldsymbol{y}}(0)\,(D^k\boldsymbol{P})(\pi) = &-\sum_{l=1}^{k}\binom{k}{l}\,2^l\,(D^l\boldsymbol{G}_{\boldsymbol{y}})(0)\,(D^{k-l}\boldsymbol{P})(\pi)\\
&+\frac{1}{2}\sum_{l=1}^{k}\binom{k}{l}\,(D^{k-l}\widetilde{\boldsymbol{P}})(\pi)\,(-i)^l\,[-\boldsymbol{G}_{\boldsymbol{y}}(\pi) - (2^l-2)i(D\boldsymbol{G}_{\boldsymbol{y}})(\pi)].
\end{aligned}$$

Proof. From (4.1) it follows by differentiation

$$\begin{aligned}
(4.2)\qquad & \sum_{l=0}^{k} \binom{k}{l} 2^l (\mathrm{D}^l \boldsymbol{G}\boldsymbol{y})(2\omega)\, (\mathrm{D}^{k-l} \boldsymbol{P})(\omega) \\
&= \sum_{l=0}^{k} \binom{k}{l} (\mathrm{D}^{k-l} \widetilde{\boldsymbol{P}})(\omega)\, \mathrm{D}^l \left(\left(\frac{1+e^{-i\cdot}}{2} \right) \boldsymbol{G}\boldsymbol{y} \right)(\omega).
\end{aligned}$$

Observing that

$$\mathrm{D}^s \left(\frac{1+e^{-i\cdot}}{2} \right)(\omega) = \begin{cases} \frac{1+e^{-i\omega}}{2} & \text{for } s=0, \\ \frac{(-i)^s}{2}\, e^{-i\omega} & \text{for } s \geq 1 \end{cases}$$

and

$$(4.3)\qquad (\mathrm{D}^s \boldsymbol{G}\boldsymbol{y})(\omega) = \begin{cases} \boldsymbol{G}\boldsymbol{y}(\omega) & \text{for } s=0, \\ (-i)^{s-1}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) & \text{for } s \geq 1, \end{cases}$$

it follows for $l > 0$ that

$$\begin{aligned}
\mathrm{D}^l & \left(\left(\frac{1+e^{-i\cdot}}{2} \right) \boldsymbol{G}\boldsymbol{y} \right)(\omega) = \sum_{s=0}^{l} \binom{l}{s} \mathrm{D}^s \left(\frac{1+e^{-i\cdot}}{2} \right)(\omega)\, (\mathrm{D}^{l-s} \boldsymbol{G}\boldsymbol{y})(\omega) \\
&= \left(\frac{1+e^{-i\omega}}{2} \right) (-i)^{l-1}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) + \frac{(-i)^l}{2}\, e^{-i\omega}\, \boldsymbol{G}\boldsymbol{y}(\omega) \\
&\quad + \frac{e^{-i\omega}}{2} \sum_{s=1}^{l-1} \binom{l}{s} (-i)^{l-1}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) \\
&= \frac{(-i)^{l-1}}{2} (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega)\, \left((1+e^{-i\omega}) + e^{-i\omega}(2^l - 2) \right) + \frac{(-i)^l}{2}\, e^{-i\omega}\, \boldsymbol{G}\boldsymbol{y}(\omega) \\
&= \frac{(-i)^{l-1}}{2} \left((2^l - 1)e^{-i\omega} + 1 \right) (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) + \frac{(-i)^l}{2}\, e^{-i\omega}\, \boldsymbol{G}\boldsymbol{y}(\omega).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sum_{l=0}^{k} \binom{k}{l} (\mathrm{D}^{k-l} \widetilde{\boldsymbol{P}})(\omega)\, \mathrm{D}^l \left(\left(\frac{1+e^{-i\cdot}}{2} \right) \boldsymbol{G}\boldsymbol{y} \right)(\omega) \\
&= \left(\frac{1+e^{-i\omega}}{2} \right) (\mathrm{D}^k \widetilde{\boldsymbol{P}})(\omega)\, \boldsymbol{G}\boldsymbol{y}(\omega) \\
&+ \frac{1}{2} \sum_{l=1}^{k} \binom{k}{l} (\mathrm{D}^{k-l} \widetilde{\boldsymbol{P}})(\omega)\, (-i)^{l-1} \left([(2^l - 1)e^{-i\omega} + 1]\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) - i e^{-i\omega}\, \boldsymbol{G}\boldsymbol{y}(\omega) \right).
\end{aligned}$$

Together with (4.2) the assertion of Lemma 4.1 follows. ∎

Proof. (*of Theorem* 2.6) By (2.18) for $k=0$ we have $\boldsymbol{y}_0^{\mathrm{T}} = \widetilde{\boldsymbol{y}}_0^{\mathrm{T}}\, \boldsymbol{G}\boldsymbol{y}(0) = \widetilde{\boldsymbol{y}}_0^{\mathrm{T}}\, \boldsymbol{e}\, \boldsymbol{y}^{\mathrm{T}}$, where $\boldsymbol{e}$ corresponds to $\boldsymbol{y}$ via (2.10). Further, note that $\widetilde{\boldsymbol{P}}(0)\, \boldsymbol{e} = \boldsymbol{e}$ implies

$$(4.4)\qquad \widetilde{\boldsymbol{P}}(0)\, \boldsymbol{G}\boldsymbol{y}(0) = \boldsymbol{G}\boldsymbol{y}(0).$$

By assumption, $\widetilde{\boldsymbol{P}}$ satisfies the conditions (2.1)–(2.2) for $n = 0, \ldots, m-1$ with $\widetilde{\boldsymbol{y}}_0, \ldots, \widetilde{\boldsymbol{y}}_{m-1}$. Hence, we get

$$\begin{aligned}
&\frac{2^k}{2^k-1} \sum_{l=0}^{k-1} \binom{k}{l} (2i)^{l-k}\, \widetilde{\boldsymbol{y}}_l^{\mathrm{T}}\, (\mathrm{D}^{k-l}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{G}\boldsymbol{y}(0) \\
&= \frac{2^k}{2^k-1} \left(\frac{1}{2^k}\, \widetilde{\boldsymbol{y}}_k^{\mathrm{T}}\, \boldsymbol{E}(0) - \widetilde{\boldsymbol{y}}_k^{\mathrm{T}}\, \widetilde{\boldsymbol{P}}(0)\, \boldsymbol{E}(0) \right) = -\widetilde{\boldsymbol{y}}_k^{\mathrm{T}}\, \boldsymbol{E}(0),
\end{aligned}$$

such that $\boldsymbol{y}_k^{\mathrm{T}}$ defined in (2.18)–(2.19) can be represented for $k = 0, \ldots, m$ in the form

$$\begin{aligned}
\boldsymbol{y}_k^{\mathrm{T}} = {} & (-ik)\widetilde{\boldsymbol{y}}_{k-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) + \sum_{l=0}^{k-1} \binom{k}{l} \widetilde{B}_{k-l}\, \widetilde{\boldsymbol{y}}_l^{\mathrm{T}}\, \boldsymbol{G}\boldsymbol{y}(0) \\
& - \frac{2^k}{2^k-1} \sum_{l=0}^{k-1} \binom{k}{l} (2i)^{l-k}\, \widetilde{\boldsymbol{y}}_l^{\mathrm{T}}\, (\mathrm{D}^{k-l}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{E}(0).
\end{aligned} \tag{4.5}$$

1. We have to show that $\boldsymbol{P}(\omega)$ satisfies the equations (2.1) – (2.2) for $n = 0, \ldots, m$ with $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_m$. That means, by (2.18) and (4.5) we have to show that for $n = 0, \ldots, m$

$$\boldsymbol{A}_n(0) + \boldsymbol{B}_n(0) + \boldsymbol{C}_n(0) + \boldsymbol{D}_n(0) = 2^{-n}\, \boldsymbol{y}_n^{\mathrm{T}}$$

and

$$\boldsymbol{A}_n(\pi) + \boldsymbol{B}_n(\pi) + \boldsymbol{C}_n(\pi) + \boldsymbol{D}_n(\pi) = \boldsymbol{0}^{\mathrm{T}}$$

are satisfied with

$$\begin{aligned}
\boldsymbol{A}_n(\omega) &:= \sum_{l=0}^{n} \binom{n}{l} (2i)^{l-n}\, (-il)\, \widetilde{\boldsymbol{y}}_{l-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^{n-l}\boldsymbol{P})(\omega), \\
\boldsymbol{B}_n(\omega) &:= \sum_{l=0}^{n-1} \binom{n}{l} (2i)^{l-n} \sum_{s=0}^{l} \binom{l}{s} \widetilde{B}_{l-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, \boldsymbol{G}\boldsymbol{y}(0)\, (\mathrm{D}^{n-l}\boldsymbol{P})(\omega), \\
\boldsymbol{C}_n(\omega) &:= \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{B}_{n-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, \boldsymbol{G}\boldsymbol{y}(0)\, \boldsymbol{P}(\omega), \\
\boldsymbol{D}_n(\omega) &:= -\frac{2^n}{2^n-1} \sum_{s=0}^{n-1} \binom{n}{s} (2i)^{s-n}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}^{n-s}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{G}\boldsymbol{y}(0)\, \boldsymbol{P}(\omega).
\end{aligned}$$

For $\omega = 0$ and $\omega = \pi$, we replace $\boldsymbol{G}\boldsymbol{y}(0)\, (\mathrm{D}^{n-l}\boldsymbol{P})(\omega)$ in $\boldsymbol{B}_n(\omega)$ by the corresponding expressions given in Lemma 4.1 and obtain

$$\boldsymbol{B}_n(\omega) = \boldsymbol{B}_n^0(\omega) + \boldsymbol{B}_n^1(\omega) + \boldsymbol{B}_n^2(\omega)$$

with

$$\begin{aligned}
\boldsymbol{B}_n^0(\omega) := - & \sum_{l=0}^{n-1} \binom{n}{l} (2i)^{l-n} \sum_{s=0}^{l} \binom{l}{s} \widetilde{B}_{l-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{r=1}^{n-l} \binom{n-l}{r} 2^r\, (\mathrm{D}^r\boldsymbol{G}\boldsymbol{y})(0) \\
& \times (\mathrm{D}^{n-l-r}\boldsymbol{P})(\omega),
\end{aligned}$$

$$\boldsymbol{B}_n^1(\omega) := \left(\frac{1+e^{-i\omega}}{2}\right) \sum_{l=0}^{n-1} \binom{n}{l} (2i)^{l-n} \sum_{s=0}^{l} \binom{l}{s} \widetilde{B}_{l-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}^{n-l}\widetilde{\boldsymbol{P}})(\omega)\, \boldsymbol{G}\boldsymbol{y}(\omega),$$

$$\boldsymbol{B}_n^2(\omega) := \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (2i)^{l-n} \sum_{s=0}^{l} \binom{l}{s} \widetilde{B}_{l-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{r=1}^{n-l} \binom{n-l}{r} (\mathrm{D}^{n-l-r}\widetilde{\boldsymbol{P}})(\omega)\, (-i)^{r-1}$$
$$\times \left([(2^r-1)e^{-i\omega}+1](\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) - ie^{-i\omega}\, \boldsymbol{G}\boldsymbol{y}(\omega)\right).$$

2. First we show that for $\omega = 0$ and $\omega = \pi$,

$$\boldsymbol{A}_n(\omega) + \boldsymbol{B}_n^0(\omega) = \boldsymbol{0}^{\mathrm{T}}.$$

Note that $\binom{n}{s}\binom{n-s}{l-s} = \binom{n}{l}\binom{l}{s}$. Changing the order of summation over l and s and putting $r' := n-l-r$ it follows

$$\boldsymbol{B}_n^0(\omega) = (-i) \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=s}^{n-1} \binom{n-s}{l-s} (2i)^{l-n}\, \widetilde{B}_{l-s} \sum_{r=1}^{n-l} \binom{n-l}{r} 2^r$$
$$\times (\mathrm{D}^r\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^{n-l-r}\boldsymbol{P})(\omega)$$
$$= (-i) \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=s}^{n-1} \binom{n-s}{l-s} (2i)^{l-n}\, \widetilde{B}_{l-s} \sum_{r'=0}^{n-l-1} \binom{n-l}{r'} (-2i)^{n-l-r'}$$
$$\times (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^{r'}\boldsymbol{P})(\omega),$$

where we have used that $(\mathrm{D}^r\boldsymbol{G}\boldsymbol{y})(0) = (-i)^{r-1}\,(\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) = (-i)^{n-l-r'+1}\,(\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)$ (see (4.3)). Thus,

$$\boldsymbol{B}_n^0(\omega) = (-i) \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=0}^{n-s-1} \binom{n-s}{l} (2i)^{l+s-n}\, \widetilde{B}_l \sum_{r=0}^{n-l-s-1} \binom{n-l-s}{r}$$
$$\times (-2i)^{n-l-s-r}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^r\boldsymbol{P})(\omega)$$
$$= (-i) \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{r=0}^{n-s-1} \binom{n-s}{r} (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^r\boldsymbol{P})(\omega)((2i)^{-r}(-1)^{n-s-r}$$
$$\times \sum_{l=0}^{n-r-s-1} \binom{n-r-s}{l} \widetilde{B}_l\, (-1)^l.$$

Observe that by (2.16),

$$\sum_{l=0}^{k-1} \binom{k}{l} \widetilde{B}_l\, (-1)^l = \begin{cases} 0 & \text{for } k > 1, \\ 1 & \text{for } k = 1. \end{cases} \tag{4.6}$$

Hence, the last term in the last representation of $\boldsymbol{B}_n^0$ vanishes for $n-s-1 \neq r$, and so

$$\boldsymbol{B}_n^0(\omega) = i \sum_{s=0}^{n-1} \binom{n}{s} (n-s)\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^{n-s-1}\boldsymbol{P})(\omega)\, (2i)^{-n+s+1}.$$

Shifting the summation index, we find for $\boldsymbol{A}_n(\omega)$ ($\omega = 0,\ \pi$):

$$\boldsymbol{A}_n(\omega) = (-i) \sum_{l=0}^{n-1} \binom{n}{l+1} (2i)^{l+1-n}\, (l+1)\, \widetilde{\boldsymbol{y}}_l^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0)\, (\mathrm{D}^{n-l+1}\boldsymbol{P})(\omega)$$

$$= (-i) \sum_{l=0}^{n-1} \binom{n}{l} (n-l) \widetilde{\boldsymbol{y}}_l^{\mathrm{T}} \, (\mathrm{D}\boldsymbol{E})(0) \, (\mathrm{D}^{n-l-1}\boldsymbol{P})(\omega) \, (2i)^{-n+l+1}.$$

Hence, $\boldsymbol{B}_n^0(\omega) + \boldsymbol{A}_n(\omega) = \boldsymbol{0}^{\mathrm{T}}$ for $\omega = 0,\ \pi$.

3. Let us consider $\boldsymbol{B}_n^1(\omega)$. We easily observe that $\boldsymbol{B}_n^1(\pi) = \boldsymbol{0}^{\mathrm{T}}$. For $\omega = 0$, we find by changing the order of summations over l and s

$$\begin{aligned}
\boldsymbol{B}_n^1(0) &= \sum_{l=0}^{n-1} \binom{n}{l} (2i)^{l-n} \sum_{s=0}^{l} \binom{l}{s} \widetilde{B}_{l-s} \, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \, (D^{n-l} \widetilde{\boldsymbol{P}})(0) \, \boldsymbol{G}\boldsymbol{y}(0) \\
&= \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=s}^{n-1} \binom{n-s}{l-s} (2i)^{l-n} \, \widetilde{B}_{l-s} \, (\mathrm{D}^{n-l} \widetilde{\boldsymbol{P}})(0) \, \boldsymbol{G}\boldsymbol{y}(0) \\
&= \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=0}^{n-s-1} \binom{n-s}{l} (2i)^{l+s-n} \, \widetilde{B}_l \, (\mathrm{D}^{n-s-l} \widetilde{\boldsymbol{P}})(0) \, \boldsymbol{G}\boldsymbol{y}(0) \\
&= \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_l \sum_{s=0}^{n-1-l} \binom{n-l}{s} (2i)^{-n+l+s} \, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \, (\mathrm{D}^{n-l-s} \boldsymbol{P})(0) \, \boldsymbol{G}\boldsymbol{y}(0).
\end{aligned}$$

On the other hand, for $l > 1$, the equations (2.1) for $\widetilde{\boldsymbol{P}}$ and $\widetilde{\boldsymbol{y}}_n$ $(n = 0, \ldots, m-1)$ imply that

$$\begin{aligned}
&\sum_{s=0}^{n-1-l} \binom{n-l}{s} (2i)^{-n+l+s} \, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \, (\mathrm{D}^{n-l-s} \widetilde{\boldsymbol{P}})(0) \, \boldsymbol{G}\boldsymbol{y}(0) \\
&= 2^{-n+l} \, \widetilde{\boldsymbol{y}}_{n-l}^{\mathrm{T}} \, \boldsymbol{G}\boldsymbol{y}(0) - \widetilde{\boldsymbol{y}}_{n-l}^{\mathrm{T}} \, \widetilde{\boldsymbol{P}}(0) \, \boldsymbol{G}\boldsymbol{y}(0) = (2^{-n+l} - 1) \widetilde{\boldsymbol{y}}_{n-l}^{\mathrm{T}} \, \boldsymbol{G}\boldsymbol{y}(0),
\end{aligned}$$

where we have used (4.4). Hence, we can write

$$\begin{aligned}
\boldsymbol{B}_n^1(0) &= \sum_{l=1}^{n-1} \binom{n}{l} \widetilde{B}_l \, (2^{-n+l} - 1) \, \widetilde{\boldsymbol{y}}_{n-l}^{\mathrm{T}} \, \boldsymbol{G}\boldsymbol{y}(0) \\
&\quad + \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \, (2i)^{-n+s} \, (\mathrm{D}^{n-s} \widetilde{\boldsymbol{P}})(0) \, \boldsymbol{G}\boldsymbol{y}(0).
\end{aligned} \tag{4.7}$$

4. Let us concentrate on $\boldsymbol{B}_n^2(\omega)$. Putting

$$\widetilde{\boldsymbol{E}}_r(\omega) := \frac{(-i)^{r-1}}{2} \left([(2^r - 1)e^{-i\omega} + 1](\mathrm{D}\boldsymbol{G}\boldsymbol{y})(\omega) - ie^{-i\omega} \, \boldsymbol{G}\boldsymbol{y}(\omega) \right)$$

we obtain for $\omega = 0,\ \pi$ by changing the order of summations and shifting the summation indices

$$\begin{aligned}
\boldsymbol{B}_n^2(\omega) &= \sum_{l=0}^{n-1} \binom{n}{l} (2i)^{l-n} \sum_{s=0}^{l} \binom{l}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \, \widetilde{B}_{l-s} \sum_{r=1}^{n-l} \binom{n-l}{r} (\mathrm{D}^{n-l-r} \widetilde{\boldsymbol{P}})(\omega) \, \widetilde{\boldsymbol{E}}_r(\omega) \\
&= \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=s}^{n-1} \binom{n-s}{l-s} (2i)^{l-n} \, \widetilde{B}_{l-s} \sum_{r=1}^{n-l} \binom{n-l}{r} (\mathrm{D}^{n-l-r} \widetilde{\boldsymbol{P}})(\omega) \, \widetilde{\boldsymbol{E}}_r(\omega) \\
&= \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}} \sum_{l=0}^{n-s-1} \binom{n-s}{l} (2i)^{l+s-n} \, \widetilde{B}_l \sum_{r=1}^{n-l-s} \binom{n-l-s}{r}
\end{aligned}$$

$$
\begin{aligned}
&\times (\mathrm{D}^{n-l-s-r}\widetilde{\boldsymbol{P}})(\omega)\, \widetilde{\boldsymbol{E}}_r(\omega) \\
&= \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_l \sum_{s=0}^{n-l-1} \binom{n-l}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (2i)^{l+s-n} \sum_{r=1}^{n-l-s} \binom{n-l-s}{r} (\mathrm{D}^{n-l-s-r}\widetilde{\boldsymbol{P}})(\omega)\, \widetilde{\boldsymbol{E}}_r(\omega) \\
&= \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_l \sum_{r=1}^{n-l} \binom{n-l}{r} \left(\sum_{s=0}^{n-l-r} \binom{n-l-r}{s} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (2i)^{-n+l+r+s}\, (\mathrm{D}^{n-l-r-s}\widetilde{\boldsymbol{P}})(\omega) \right) \\
&\times (2i)^{-r}\, \widetilde{\boldsymbol{E}}_r(\omega).
\end{aligned}
$$

Application of (2.1)–(2.2) for $\widetilde{\boldsymbol{P}}$ in the sum over s implies that $\boldsymbol{B}_n^2(\pi) = 0$ and

$$
\begin{aligned}
\boldsymbol{B}_n^2(0) &= \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_l \sum_{r=1}^{n-l} \binom{n-l}{r} 2^{-n+l+r}\, \widetilde{\boldsymbol{y}}_{n-l-r}^{\mathrm{T}}\, (2i)^{-r}\, \widetilde{\boldsymbol{E}}_r(0) \\
&= \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_l \sum_{r=1}^{n-l} \binom{n-l}{r} 2^{-n+l-1}\, \widetilde{\boldsymbol{y}}_{n-l-r}^{\mathrm{T}}\, (-1)^r \left(i2^r (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) + \boldsymbol{G}\boldsymbol{y}(0) \right).
\end{aligned}
$$

Putting $r' := n-l-r$ and changing again the order of summation we get

$$
\begin{aligned}
\boldsymbol{B}_n^2(0) &= \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_l \sum_{r'=0}^{n-l-1} \binom{n-l}{r'} 2^{-n+l-1}\, \widetilde{\boldsymbol{y}}_{r'}^{\mathrm{T}}\, (-1)^{n-l-r'} \\
&\quad \times \left(i2^{n-l-r'} (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) + \boldsymbol{G}\boldsymbol{y}(0) \right) \\
&= i \sum_{r=0}^{n-1} \binom{n}{r} \widetilde{\boldsymbol{y}}_r^{\mathrm{T}} \left(\sum_{l=0}^{n-r-1} \binom{n-r}{l} \widetilde{B}_l\, (-1)^l \right) 2^{-r-1}\, (-1)^{n-r}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) \\
&\quad + \sum_{r=0}^{n-1} \binom{n}{r} \widetilde{\boldsymbol{y}}_r^{\mathrm{T}} \left(\sum_{l=0}^{n-r-1} \binom{n-r}{l} \widetilde{B}_l\, (-2)^l \right) 2^{-n-1}\, (-1)^{n-r}\, \boldsymbol{G}\boldsymbol{y}(0).
\end{aligned}
$$

Using the identities (2.17) and (4.6) for Bernoulli numbers and observing that $(-1)^k \widetilde{B}_k = \widetilde{B}_k$ for $k > 1$, it follows

$$
\begin{aligned}
\boldsymbol{B}_n^2(0) &= \frac{-in}{2^n} \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) - \frac{n}{2^{n+1}}\, \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, \boldsymbol{G}\boldsymbol{y}(0) \\
&\quad + 2^{-n} \sum_{r=0}^{n-2} \binom{n}{r} \widetilde{\boldsymbol{y}}_r^{\mathrm{T}}\, (-2^{n-r} + 1) \widetilde{B}_{n-r}\, (-1)^{n-r} \boldsymbol{G}\boldsymbol{y}(0) \\
&= \frac{-in}{2^n} \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) - \frac{n}{2^{n+1}}\, \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, \boldsymbol{G}\boldsymbol{y}(0) \\
&\quad + \sum_{r=0}^{n-2} \binom{n}{r} \widetilde{\boldsymbol{y}}_r^{\mathrm{T}}\, (2^{-n} - 2^{-r}) \widetilde{B}_{n-r}\, (-1)^{n-r} \boldsymbol{G}\boldsymbol{y}(0) \\
&= \frac{-in}{2^n} \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{G}\boldsymbol{y})(0) + \sum_{r=0}^{n-1} \binom{n}{r} \widetilde{\boldsymbol{y}}_r^{\mathrm{T}}\, (2^{-n} - 2^{-r}) \widetilde{B}_{n-r}\, \boldsymbol{G}\boldsymbol{y}(0).
\end{aligned}
$$

5. Let now $\omega = \pi$. Recall that $\boldsymbol{B}_n^1(\pi) = \boldsymbol{B}_n^2(\pi) = \boldsymbol{A}_n(\pi) + \boldsymbol{B}_n^0(\pi) = \boldsymbol{0}^{\mathrm{T}}$. Further, by $\boldsymbol{G}\boldsymbol{y}(0)\boldsymbol{P}(\pi) = \boldsymbol{0}$ we have $\boldsymbol{C}_n(\pi) = \boldsymbol{D}_n(\pi) = \boldsymbol{0}^{\mathrm{T}}$. Hence,

$$
\boldsymbol{A}_n(\pi) + \boldsymbol{B}_n(\pi) + \boldsymbol{C}_n(\pi) + \boldsymbol{D}_n(\pi) = \boldsymbol{0}^{\mathrm{T}}.
$$

6. Let $\omega = 0$. By $\boldsymbol{Gy}(0)\,\boldsymbol{P}(0) = \widetilde{\boldsymbol{P}}(0)\,\boldsymbol{Gy}(0) = \boldsymbol{Gy}(0)$ we obtain

$$\boldsymbol{C}_n(0) = \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{B}_{n-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, \boldsymbol{Gy}(0),$$

$$\boldsymbol{D}_n(0) = -\frac{2^n}{2^n-1} \sum_{s=0}^{n-1} \binom{n}{s} (2i)^{s-n}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}^{n-s}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{Gy}(0).$$

Observing that by (4.7)

$$\boldsymbol{D}_n(0) + \boldsymbol{B}_n^1(0) = \sum_{l=1}^{n-1} \binom{n}{l} \widetilde{B}_l\, (2^{-n+l}-1)\, \widetilde{\boldsymbol{y}}_{n-l}^{\mathrm{T}}\, \boldsymbol{Gy}(0)$$
$$-\frac{1}{2^n-1} \sum_{s=0}^{n-1} \binom{n}{s} (2i)^{s-n} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}^{n-s}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{Gy}(0)$$

and using the expression for $\boldsymbol{B}_n^2$ found in part 4, we obtain

$$\begin{aligned}
&\boldsymbol{D}_n(0) + \boldsymbol{B}_n^1(0) + \boldsymbol{B}_n^2(0) + \boldsymbol{C}_n(0)\\
&= -\frac{1}{2^n-1} \sum_{s=0}^{n-1} \binom{n}{s} (2i)^{s-n} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}^{n-s}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{Gy}(0)\\
&\quad + \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_{n-l}\, (2^{-l}-1)\, \widetilde{\boldsymbol{y}}_l^{\mathrm{T}}\, \boldsymbol{Gy}(0)\\
&\quad - \frac{in}{2^n}\, \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{Gy})(0) + \sum_{r=0}^{n-1} \binom{n}{r} \widetilde{\boldsymbol{y}}_r^{\mathrm{T}}\, (2^{-n}-2^{-r})\, \widetilde{B}_{n-r}\, \boldsymbol{Gy}(0)\\
&\quad + \sum_{s=0}^{n-1} \binom{n}{s} \widetilde{B}_{n-s}\, \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, \boldsymbol{Gy}(0)\\
&= \frac{-in}{2^n}\, \widetilde{\boldsymbol{y}}_{n-1}^{\mathrm{T}}\, (\mathrm{D}\boldsymbol{Gy})(0) - \frac{1}{2^n-1} \sum_{s=0}^{n-1} \binom{n}{s} (2i)^{s-n} \widetilde{\boldsymbol{y}}_s^{\mathrm{T}}\, (\mathrm{D}^{n-s}\widetilde{\boldsymbol{P}})(0)\, \boldsymbol{Gy}(0)\\
&\quad + \sum_{l=0}^{n-1} \binom{n}{l} \widetilde{B}_{n-l}\, \widetilde{\boldsymbol{y}}_l^{\mathrm{T}}\, \boldsymbol{Gy}(0)\, \bigl(2^{-l} - 1 + 2^{-n} - 2^{-l} + 1\bigr) = 2^{-n}\, \boldsymbol{y}_n^{\mathrm{T}}.
\end{aligned}$$

Recalling that $\boldsymbol{A}_n(0) + \boldsymbol{B}_n^0(0) = \boldsymbol{0}^{\mathrm{T}}$, the proof is complete. $\square$

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REFERENCES

[AS] M. Abramowitz and J. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[AR] B. K. Alpert and V. Rokhlin, *A fast algorithm for the evaluation of Legendre expansions*, SIAM J. Sci. Statist. Comput., 12 (1991), pp. 158–179.

[CDM] A. Cavaretta, W. Dahmen and C. A. Micchelli, *Stationary Subdivision*, Mem. Amer. Math. Soc., 453 (1991), pp. 1–186.

[CDP] A. Cohen, I. Daubechies, and G. Plonka, *Regularity of refinable function vectors*, J. Fourier Anal. Appl., to appear.

[CL] C. K. Chui and J. A. Lian, *A study of orthonormal multi–wavelets*, Texas A&M University, CAT Report No. 351, 1995.

[DM] W. Dahmen and C. A. Micchelli, *Biorthogonal wavelet expansions*, Const. Approx., to appear.

[D1] I. Daubechies, *Orthonormal bases of wavelets with compact support*, Comm. Pure Appl. Math., 41 (1988), pp. 909–996.

[D2] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[DL1] I. Daubechies and J. Lagarias, *Two–scale difference equations: I. Existence and global regularity of solutions*, SIAM J. Math. Anal., 22 (1991), pp. 1388–1410.

[DL2] I. Daubechies and J. Lagarias, *Two–scale difference equations: II. Local regularity, infinite products of matrices and fractals*, SIAM J. Math. Anal., 23 (1992), pp. 1031–1079.

[DGHM] G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Construction of orthogonal wavelets using fractal interpolation functions*, SIAM J. Math. Anal., to appear.

[GHM] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Fractal functions and wavelet expansions based on several scaling functions*, J. Approx. Theory, 78 (1994), pp. 373–401.

[GL] T. N. T. Goodman and S. L. Lee, *Wavelets of multiplicity r*, Trans. Amer. Math. Soc., 342 (1994), pp. 307–324.

[GLT] T. N. T. Goodman, S. L. Lee and W. S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc., 338 (1993), pp. 639–654.

[HC] C. Heil and D. Colella, *Matrix refinement equations: Existence and uniqueness*, J. Fourier Anal. Appl., 2 (1996), pp. 363-377.

[HSS] C. Heil, G. Strang and V. Strela, *Approximation by translates of refinable functions*, Numer. Math., 73 (1996), pp. 75-94.

[H] L. Hervé, *Multi–Resolution analysis of multiplicity d. Application to dyadic interpolation*, Appl. Comput. Harmonic Anal., 1 (1994), pp. 299–315.

[JL] R. Q. Jia and J. J. Lei, *Approximation by multiinteger translates of functions having global support*, J. Approx. Theory, 72 (1993), pp. 2–23.

[J] Q. Jiang, *On the regularity of matrix refinable functions*, Department of Mathematics, The National University of Singapore, preprint, 1996.

[L] J. A. Lian, *Characterization of the order of polynomial reproduction for multi–scaling functions*, in Approximation Theory VIII, Vol 2: Wavelets and Multilevel Approximation, C. K. Chui and L. L. Schumaker, eds., World Scientific Publishing Co., Singapore, 1995, pp. 251–258.

[MRV] P. R. Massopust, D. K. Ruch and P. J. Van Fleet, *On the support properties of scaling vectors*, Appl. Comput. Harmonic Anal., to appear.

[P1] G. Plonka, *Two-scale symbol and autocorrelation symbol for B-splines with multiple knots*, Adv. Comput. Math., 3 (1995), pp. 1–22.

[P2] G. Plonka, *Generalized spline wavelets*, Constr. Approx., 12 (1996), pp. 127–155.

[P3] G. Plonka, *Approximation order provided by refinable function vectors*, Constr. Approx., to appear.

[P4] G. Plonka, *Factorization of refinement masks of function vectors*, in Approximation Theory VIII, Vol 2: Wavelets and Multilevel Approximation, C. K. Chui and L. L. Schumaker, eds., World Scientific Publishing Co., Singapore, 1995, pp. 317–324.

[Sh] Z. Shen, *Refinable function vectors*, Department of Mathematics, The National University of Singapore, preprint, 1995.

[SB] M. J. T. Smith and T. P. Barnwell, *Exact reconstruction techniques for tree–structured subband coders*, IEEE Trans. Acoust. Speech Signal Process., 34 (1986), pp. 434–441.

[St1] G. Strang, *Eigenvalues of $(\downarrow 2)H$ and convergence of the cascade algorithm*, IEEE Trans. Signal Process., to appear.

[St2] G. Strang, *Introduction to Linear Algebra*, Wellesley-Cambridge Press, Cambridge, 1993.

[SF] G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Wellesley-Cambridge Press, Cambridge, 1973.

[SN] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Cambridge, 1996.

[SS1] G. Strang and V. Strela, *Orthogonal multiwavelets with vanishing moments*, J. Optical Eng. 33 (1994), pp. 2104-2107.

[SS2] G. Strang and V. Strela, *Short wavelets and matrix dilation equations*, IEEE Trans. Signal Process., 43 (1995), pp. 108-115.

[SS3] V. Strela and G. Strang, *Finite element multiwavelets*, in Approximation Theory, Wavelets and Applications, S. P. Singh, ed., Kluwer Academic Publ., Dordrecht, 1995, pp. 485–496.

[SS4] V. Strela and G. Strang, *Biorthogonal multiwavelets and finite elements*, Department of Mathematics, MIT, preprint, 1996.

[S1] V. Strela, *Multiwavelets: regularity, orthogonality and symmetry via two–scale similarity transform*, Stud. Appl. Math., to appear.

[S2] V. Strela, *Multiwavelets: Theory and applications*, Ph. D. thesis, Department of Mathematics, MIT, Cambridge, MA, 1996.

[SHSTH] V. Strela, P. N. Heller, G. Strang, P. Topiwala, and C. Heil, *The application of multiwavelet filter banks to image processing*, IEEE Trans. Signal Process., to appear.

[XGHS] X.-G. Xia, J. S. Geronimo, D. Hardin, and B. Suter, *Design of prefilters for discrete multiwavelet transforms*, IEEE Trans. Signal Process., 44 (1996), pp. 25–35.