Frame Soft Shrinkage Operators are Proximity Operators

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Abstract

Abstract. In this paper, we show that the commonly used frame shrinkage operator, that maps a given vector $\mathbf{x} \in \mathbb{R}^N$ onto the vector $\mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{x}$, is already a proximity operator, which can therefore be directly used in corresponding splitting algorithms. In our setting, $\mathbf{T} \in \mathbb{R}^{L\times N}$ with $L \geq N$ has full rank N, $\mathbf{T}^\dagger$ denotes the Moore-Penrose inverse of $\mathbf{T}$, and S_γ is the usual soft shrinkage operator with threshold parameter $\gamma > 0$. Our result generalizes the known assertion that $\mathbf{T}^* S_\gamma \mathbf{T}$ is the proximity operator of $\|\mathbf{T}\cdot\|_1$ if $\mathbf{T}$ is an orthogonal (square) matrix. It is well-known that for rectangular frame matrices $\mathbf{T}$ with $L > N$, the proximity operator of $\|\mathbf{T}\cdot\|_1$ is no longer of the above form and can solely be computed iteratively. Showing that the frame soft shrinkage operator is a proximity operator as well, we motivate its application as a replacement of the exact proximity operator of $\|\mathbf{T}\cdot\|_1$. We further give an explanation, why the usage of the frame soft shrinkage operator still provides good results in various applications. We also provide some properties of the subdifferential of the convex functional Φ which leads to the proximity operator $\mathbf{T}^\dagger S_\gamma \mathbf{T}$.

Key words: proximity operator; frame soft shrinkage; maximally cyclically monotone subdifferential; Brouwer's fixed point theorem; splitting algorithms for inverse problems.

1 Introduction

Many reconstruction problems in signal and image processing are ill-posed and commonly solved by means of variational methods. We restrict our considerations to the finite-dimensional case where the ill-posed operator equation $\mathbf{K}\mathbf{x} = \mathbf{f}$ is solved using a regularization approach,

$$\hat{\mathbf{x}} = \operatorname*{arg\,min}_{\mathbf{x}\in\mathbb{R}^N} F(\mathbf{x}) = \operatorname*{arg\,min}_{\mathbf{x}\in\mathbb{R}^N} \left(\frac{1}{2}\|\mathbf{K}\mathbf{x} - \mathbf{f}\|_2^2 + \Phi(\mathbf{x})\right). \tag{1.1}$$

Here, $\mathbf{K}\colon \mathbb{R}^N \to \mathbb{R}^M$ is a known linear or non-linear operator and $\mathbf{f} \in \mathbb{R}^M$ represents the measured (noisy) data. For regularization, a convex, proper and lower semi-continuous functional $\Phi\colon \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ is employed that forces desired properties of $\mathbf{x}$ as regularity or sparsity.

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In many applications, signals tend to be sparse with regard to a suitable encoding, and functionals of the form $\Phi(\mathbf{x}) = \gamma \|\mathbf{x}\|_1$ or $\Phi(\mathbf{x}) = \gamma \|\mathbf{T}\,\mathbf{x}\|_1$ (with $\|\mathbf{x}\|_1 := (\sum_{j=1}^{N} |x_j|)$ and $\gamma > 0$) with a linear transform $\mathbf{T} \in \mathbb{R}^{L\times N}$ are frequently used.

While orthogonal transforms $\mathbf{T}$ have proven to be ideal with regard to numeric stability, they often fail to capture the underlying characteristics of many signal types. Therefore, in numerous applications one uses redundant representations that suit the purpose at hand, as e.g. redundant wavelet transforms, curvelets or shearlets for capturing non-isotropic features in 2D images, cf. [3, 6, 11]. In this case we have rectangular transform matrices $\mathbf{T}$ with $L > N$.

Within the last years, many computational algorithms have been proposed to solve the minimization problem (1.1), which are for example based on operator splitting methods, as forward-backward splitting (FBS), Douglas-Rachford splitting (DRS), or the split Bregman algorithm. These approaches make use of the so-called *proximal mapping* or the *proximity operator* of Φ, which itself is defined as the solver of a minimization problem,

$$\operatorname{prox}_{\Phi}(\mathbf{x}) := \operatorname*{arg\,min}_{\mathbf{y}\in\mathbb{R}^N} \left\{ \tfrac{1}{2}\|\mathbf{y}-\mathbf{x}\|^2 + \Phi(\mathbf{y}) \right\}.$$

Here $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$ denotes the norm corresponding to a fixed scalar product in $\mathbb{R}^N$. If we take the standard scalar product and $\Phi(\mathbf{x}) = \gamma \|\mathbf{x}\|_1$, then the corresponding proximity operator turns out to be the so-called soft shrinkage operator,

$$\operatorname{prox}_{\gamma\|\cdot\|_1}(\mathbf{x}) = S_\gamma(\mathbf{x}) \quad \text{with} \quad [S_\gamma(\mathbf{x})]_j = \begin{cases} x_j - \gamma, & x_j \ge \gamma, \\ x_j + \gamma, & x_j \le -\gamma, \\ 0, & |x_j| < \gamma. \end{cases} \tag{1.2}$$

For $\Phi(\mathbf{x}) = \gamma \|\mathbf{T}\mathbf{x}\|_1$ with $\mathbf{T}$ being an orthonormal matrix, it can be shown that the corresponding proximity operator is

$$\operatorname{prox}_{\gamma\|\mathbf{T}\cdot\|_1}(\mathbf{x}) = \mathbf{T}^*\, S_\gamma(\mathbf{T}\,\mathbf{x}) = \mathbf{T}^{-1}\, S_\gamma(\mathbf{T}\,\mathbf{x}),$$

see Proposition 23.29 in [1]. This, however is no longer true if $\mathbf{T}$ is not orthogonal or if $\mathbf{T}$ not even a basis, i.e., if $\mathbf{T} \in \mathrm{R}^{L\times N}$ for $L > N$, see e.g. [5]. In this case, the proximity operator can no longer be represented in a closed form, and one has to make use of an iteration procedure to compute it.

Yet the question remains what would happen if we replaced the exact proximity operator by the frame soft shrinkage operator. For that investigation denote by $\mathbf{T}^\dagger$ the Moore Penrose inverse of $\mathbf{T}$ and replace $\operatorname{prox}_{\gamma\|\mathbf{T}\cdot\|_1}$ by

$$\mathbf{T}^\dagger\, S_\gamma(\mathbf{T}\cdot)$$

in the iteration procedure. Will the corresponding operator splitting algorithms as FBS and DGS still converge? And, if yes, which regularization functional $\Phi(\mathbf{x})$ will we get as a result as compared to $\|\mathbf{T}\mathbf{x}\|_1$ in this case?

A related question has been posed also by Elad in [5] from a different viewpoint. His main argument to explain the good performance of methods that just employ the frame soft shrinkage operator is based on the connection to the solution of basis pursuit denoising (BPDN) problems [2]. He showed that the application of the frame soft shrinkage operator can be interpreted as the first iteration step of an iterative algorithm to solve the BPDN problem.

We need to keep in mind that we want to enforce sparsity of $\mathbf{Tx}$ using the functional Φ. However, the used ℓ_1-norm only acts as a proxy here – a compromise to the convexity for Φ – which is needed in order to ensure convergence of the iteration algorithms. Instead, we would rather like to have actual sparsity of $\mathbf{Tx}$, i.e., a small number of non-zero components in $\mathbf{Tx}$. Thus, one might wonder whether $\mathbf{T}^\dagger S_\gamma(\mathbf{T}\cdot)$ is doing the job as well as the exact proximity operator of $\|\mathbf{T}\,\mathbf{x}\|_1$.

We would like to present another heuristic argument towards the use of the frame soft shrinkage operator. In practice, e.g. for solving phase retrieval problems, the inverse problem is often formulated as a feasibility problem. In our case we can define the measurement set

$$M \coloneqq \{\mathbf{x} : \mathbf{Kx} = \mathbf{f}\}$$

and a constraints set, as for example

$$C \coloneqq \{\mathbf{x} :\ \|\mathbf{x}\|_2 \leq \|\mathbf{K}^\dagger \mathbf{f}\|_2,\ \mathbf{Tx} \text{ is } M\text{-sparse}\}.$$

We are looking for some

$$\mathbf{x} \in M \cap C. \tag{1.3}$$

It is quite likely that $M \cap C = \emptyset$, particularly for noisy data $\mathbf{f}$. Then the problem is called infeasible. In this case we aim at finding $\mathbf{x} \in C$ with shortest distance to M, or, more generally, $\mathbf{x}$ such that the sum of distances to C and M is the shortest. The most simple iteration algorithms to tackle the feasibility problem (1.3) are alternating projection algorithms. We start with an arbitrary $\mathbf{x}_0$, apply the projection to M, then the projection to C, and then iterate. Another, more sophisticated algorithm is the Relaxed Averaged Alternating Reflections (RAAR) algorithm in [10]. This algorithm can be seen as a combination of alternating projections and Douglas-Rachford iterations and also provides reasonable results in the infeasible case $M \cap C = \emptyset$.

For linear $\mathbf{K}$ the projection onto M is well-defined, since M is an affine subspace of $\mathbb{R}^N$. The set C is bounded but usually no longer convex, and the projection cannot be obtained easily. For given $\mathbf{x} \in C$ we could apply the frame shrinkage operator with a suitable γ such that $S_\gamma \mathbf{T}\,\mathbf{x}$ is M-sparse.

However, the frame soft shrinkage operator is not the orthogonal projector onto C, it is not even idempotent. But it obviously provides a vector which is somewhat close to C. In [9, 8], this approach has been used in connection with the RAAR algorithm for phase retrieval problems with very good numerical results.

This paper is structured as follows. In Section 2, we shortly recall the idea of operator splitting methods to solve variational models of the form (1.1). In particular, we recall that these methods involve the proximity operator of the regularization functional Φ. In Section 3, we introduce a set-valued mapping H which is based on the frame shrinkage operator $\mathbf{T}^\dagger S_\gamma(\mathbf{T}\cdot)$. We show that H is well-defined. Further, we derive some structural properties of this mapping. In particular, we prove that H is usually not equal to the subdifferential of $\|\mathbf{T}\cdot\|_1$ but possesses similar properties. In Section 4, we show that the set-valued mapping H is maximally cyclically monotone and therefore the subdifferential of a proper, lower semi-continuous and convex functional Φ. The key idea is to apply a new scalar product in $\mathbb{R}^N$ which is aligned with the linear operator $\mathbf{T}$. We will be able to conclude that the frame soft shrinkage operator is indeed the proximity operator of the functional Φ with $H = \partial\Phi$ and particularly is non-expansive (with respect to the aligned scalar product. Therefore, it can replace the proximity operator of $\|\mathbf{T}\cdot\|_1$ in the operator splitting algorithms in Section 2.

Throughout the paper, we will use the notation $\langle\cdot,\cdot\rangle_2$ and $\|\cdot\|_2$ for the standard scalar product in $\mathbb{R}^N$ and the corresponding norm, i.e. $\langle\mathbf{x},\mathbf{y}\rangle_2 := \mathbf{x}^T\mathbf{y}$. The general notion $\langle\cdot,\cdot\rangle$ denotes a fixed scalar product in $\mathbb{R}^N$ that can be different from the standard scalar product, and $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$ is the corresponding norm.

2 Operator splitting methods and the proximity operator

Let us start with summarizing the idea of operator splitting algorithms to solve minimization problems of the form (1.1). In particular, we will focus on the forward-backward splitting and the Douglas-Rachford algorithm.

For simplicity, we assume that $\mathbf{K}\colon \mathbb{R}^M \to \mathbb{R}^N$ is a linear operator. Further we denote the set of proper, lower semi-continuous and convex functionals $\Phi\colon \mathbb{R}^N \to \mathbb{R}$ by Γ_0.

To solve (1.1), we observe that the first term $\|\mathbf{K}\mathbf{x}-\mathbf{f}\|_2^2$ of the functional F is differentiable. This may no longer hold for the convex functional Φ. Therefore we apply the subdifferential $\partial\Phi$, which is here defined as the set-valued operator

$$\partial\Phi(\mathbf{x}) := \{\mathbf{y}\in\mathbb{R}^N : \langle\mathbf{y},\tilde{\mathbf{x}}-\mathbf{x}\rangle \le \Phi(\tilde{\mathbf{x}}) - \Phi(\mathbf{x}),\, \forall\,\tilde{\mathbf{x}}\in\mathbb{R}^N\}, \tag{2.1}$$

where $\langle\mathbf{x},\,\mathbf{y}\rangle$ denotes a fixed scalar product in $\mathbb{R}^N$. Note that the subdifferential directly generalizes the usual notion of the derivative, and the solution $\hat{\mathbf{x}}$ of (1.1) necessarily satisfies

$$0 \in \partial\left(\frac{1}{2}\|\mathbf{K}\mathbf{x}-\mathbf{f}\|_2^2 + \Phi(\mathbf{x})\right) = \mathbf{K}^*(\mathbf{K}\mathbf{x}-\mathbf{f}) + \partial\Phi(\mathbf{x}).$$

Multiplication with a constant $\lambda > 0$ and addition of $\hat{\mathbf{x}}$ yields the equivalent statements

$$\begin{aligned}\hat{\mathbf{x}} - \lambda\,\mathbf{K}^*(\mathbf{K}\hat{\mathbf{x}}-\mathbf{f}) &\in \hat{\mathbf{x}} + \lambda\,\partial\Phi(\hat{\mathbf{x}})\\ (\mathbf{I}_N - \lambda\,\mathbf{K}^*\mathbf{K})\hat{\mathbf{x}} + \lambda\,\mathbf{K}^*\mathbf{f} &\in (\mathbf{I}_N + \lambda\,\partial\Phi)^{-1}(\hat{\mathbf{x}}),\end{aligned}$$

with the $N\times N$ identity matrix $\mathbf{I}_N$. Thus formally

$$\hat{\mathbf{x}} = (\mathbf{I}_N + \lambda\partial\Phi)^{-1}\left[(\mathbf{I}_N - \lambda\,\mathbf{K}^*\mathbf{K})\hat{\mathbf{x}} + \lambda\,\mathbf{K}^*\mathbf{f}\right]. \tag{2.2}$$

The operator $(\mathbf{I}_N + \lambda\partial\Phi)^{-1}$ is also called the resolvent of $\lambda\,\partial\Phi$, and we have the following, see e.g. [4].

Lemma 2.1 *The proximity operator* $\operatorname{prox}_{\lambda\Phi}$ *of a functional* $\lambda\Phi\in\Gamma_0$*,*

$$\operatorname{prox}_{\lambda\Phi}(\mathbf{x}) := \operatorname*{arg\,min}_{\mathbf{y}\in\mathbb{R}^N}\left\{\tfrac{1}{2}\|\mathbf{y}-\mathbf{x}\|^2 + \lambda\Phi(\mathbf{y})\right\}$$

exists, is uniquely defined, single-valued, and we have

$$\operatorname{prox}_{\lambda\Phi}(\mathbf{x}) = (\mathbf{I}_N + \lambda\partial\Phi)^{-1}(\mathbf{x}).$$

Indeed, we observe that $\frac{1}{2}\|\mathbf{y}-\mathbf{x}\|^2 + \lambda\Phi(\mathbf{y})$ is strictly convex and

$$\begin{aligned}\mathbf{y} = \operatorname{prox}_{\lambda\Phi}(\mathbf{x}) \quad &\Longleftrightarrow \quad \mathbf{0} \in (\mathbf{y}-\mathbf{x}) + \lambda\,\partial\Phi(\mathbf{y}) = -\mathbf{x} + (\mathbf{I}_N + \lambda\,\partial\Phi)(\mathbf{y}) \\ &\Longleftrightarrow \quad \mathbf{y} = (\mathbf{I}_N + \lambda\,\partial\Phi)^{-1}(\mathbf{x}).\end{aligned}$$

In particular,

$$\mathbf{x}-\mathbf{y} \in \lambda\partial\Phi(\mathbf{y}) \quad \Longleftrightarrow \quad \mathbf{y} = \operatorname{prox}_{\lambda\Phi}(\mathbf{x}),$$

or equivalently,

$$\mathbf{t} \in \lambda\,\partial\Phi(\mathbf{y}) \quad \Longleftrightarrow \quad \mathbf{y} = \operatorname{prox}_{\lambda\Phi}(\mathbf{y}+\mathbf{t}), \tag{2.3}$$

see Proposition 16.34 in [1]. Now, (2.2) provides already an iterative algorithm, the so-called forward-backward splitting iteration, see e.g. [7, 1, 14]:

Algorithm 2.2 (***Forward-backward splitting***)
For an arbitrary starting vector $\mathbf{x}^{(0)} \in \mathbb{R}^N$, *iterate*

1. $\mathbf{y}^{(j)} := (\mathbf{I}_N - \lambda\mathbf{K}^*\mathbf{K})\,\mathbf{x}^{(j)} + \lambda\,\mathbf{K}^*\mathbf{f}$,
2. $\mathbf{x}^{(j+1)} := \operatorname{prox}_{\lambda\Phi}(\mathbf{y}^{(j)})$.

This iteration converges for $\lambda \in (0, 2/\|\mathbf{K}^*\|^2)$, see e.g. [14], since $\frac{1}{\|\mathbf{K}^*\|^2}(\mathbf{K}^*\mathbf{K}\mathbf{x} + \mathbf{K}^*\mathbf{f})$ is firmly nonexpansive, i.e., we have

$$\langle \mathbf{x}-\mathbf{y}, \mathbf{K}^*\mathbf{K}(\mathbf{x}-\mathbf{y})\rangle = \|\mathbf{K}(\mathbf{x}-\mathbf{y})\|^2 \geq \frac{1}{\|\mathbf{K}^*\|^2}\|\mathbf{K}^*\mathbf{K}(\mathbf{x}-\mathbf{y})\|^2.$$

Let us consider a second operator splitting method. Observe that the fixed point relation (2.2) can be rewritten as

$$\hat{\mathbf{x}} + \lambda\mathbf{K}^*\mathbf{K}\hat{\mathbf{x}} \in \operatorname{prox}_{\lambda\Phi}\left[(\mathbf{I}_N - \lambda\mathbf{K}^*\mathbf{K})\hat{\mathbf{x}} + \lambda\mathbf{K}^*\mathbf{f}\right] + \lambda\mathbf{K}^*\mathbf{K}\hat{\mathbf{x}}.$$

With $\hat{\mathbf{t}} := (\mathbf{I}_N + \lambda\mathbf{K}^*\mathbf{K})\hat{\mathbf{x}}$ it follows that

$$\hat{\mathbf{t}} \in \operatorname{prox}_{\lambda\Phi}(2\hat{\mathbf{x}} - \hat{\mathbf{t}} + \lambda\mathbf{K}^*\mathbf{f}) + \hat{\mathbf{t}} - \hat{\mathbf{x}}.$$

This leads to the Douglas-Rachford iteration, which converges in a large parameter range, see [1, 14].

Algorithm 2.3 (***Douglas-Rachford splitting***)
For any starting vectors $\mathbf{x}^{(0)}, \mathbf{t}^{(0)} \in \mathbb{R}^N$, *iterate*

1. $\mathbf{t}^{(j+1)} := \operatorname{prox}_{\lambda\Phi}(2\mathbf{x}^{(j)} - \mathbf{t}^{(j)} + \lambda\mathbf{K}^*\mathbf{f}) + \mathbf{t}^{(j)} - \mathbf{x}^{(j)}$,
2. $\mathbf{x}^{(j+1)} := (\mathbf{I}_N + \lambda\mathbf{K}^*\mathbf{K})^{-1}\mathbf{t}^{(j+1)}$.

For the two considered algorithms we need to evaluate the proximity operator of the regularization functional $\lambda\,\Phi$.

3 A closer look at the frame soft shrinkage operator

Our goal is to show that for any frame matrix $\mathbf{T} \in \mathbb{R}^{L\times N}$ with $L \geq N$ and full rank N the operator

$$\mathbf{T}^\dagger\, S_\gamma \mathbf{T},$$

with S_γ the soft threshold operator in (1.2), is the proximity operator of a convex, proper, lower semi-continuous functional, i.e., of some $\Phi \in \Gamma_0$. In a first step we define the set-valued mapping $H\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ by

$$\mathbf{y} \in H(\mathbf{x}) \quad :\Longleftrightarrow \quad \mathbf{x} = \mathbf{T}^\dagger S_\gamma \mathbf{T}\,(\mathbf{x}+\mathbf{y}). \tag{3.1}$$

Then, according to (2.3) we have to show that H is the subdifferential of a functional $\Phi \in \Gamma_0$.

In this section, we will show that H in (3.1) is well-defined, and we will study some properties of H. In Section 4, we will finally show that indeed $H = \partial\Phi$ for some $\Phi \in \Gamma_0$.

In order to get a first idea of what happens here, we start off with a toy example.

Lemma 3.1 *For* $\mathbf{T} = \begin{pmatrix} 1 \\ c \end{pmatrix}$ *with* $c \geq 1$ *and* $\gamma > 0$ *we find for* H *in* (3.1) *for* $x \geq 0$

$$H(x) = \begin{cases} \gamma[-\frac{1}{c}, \frac{1}{c}] & x = 0, \\ \frac{\gamma}{c} + \frac{x}{c^2} & x \in (0, \frac{\gamma(c-1)c}{c^2+1}], \\ \gamma\left(\frac{1+c}{1+c^2}\right) & x > \frac{\gamma(c-1)c}{c^2+1}. \end{cases}$$

For $x < 0$ *we have* $H(x) = -H(-x)$. *Then* H *is the subdifferential of the even function*

$$\Phi(x) = \begin{cases} \frac{\gamma x}{c} + \frac{x^2}{2c^2} & x \in [0, \frac{\gamma(c-1)c}{c^2+1}], \\ \gamma\left(\frac{1+c}{1+c^2}\right)x - \frac{\gamma^2(c-1)^2}{2(c^2+1)^2} & x > \frac{\gamma(c-1)c}{c^2+1}, \\ \Phi(-x) & x < 0. \end{cases}$$

Proof: For $x = 0$, it follows from (3.1) with $\mathbf{T}^\dagger = \frac{1}{1+c^2}(1, c)$ that

$$y \in H(0) \quad \Longleftrightarrow \quad 0 = \frac{1}{1+c^2}(1,c)\, S_\gamma \begin{pmatrix} 1 \\ c \end{pmatrix} y.$$

This is only true if $S_\gamma\,(c\,y) = 0$ i.e., $y \in [-\frac{\gamma}{c}, \frac{\gamma}{c}]$.

For $x > 0$ we find

$$x = \frac{1}{1+c^2}(1,c)\, S_\gamma \begin{pmatrix} 1 \\ c \end{pmatrix} (y+x).$$

Since $x > 0$, we need that $S_\gamma \mathbf{T}\,(x+y) > \mathbf{0}$. Thus $c(x+y) > \gamma$ as well as $x+y > 0$. We consider two cases.

1. If $x + y \leq \gamma$ and $c(x+y) > \gamma$, then

$$x = \mathbf{T}^\dagger S_\gamma \mathbf{T}(x+y) = \frac{1}{1+c^2}(1,c) \begin{pmatrix} 0 \\ c(x+y)-\gamma \end{pmatrix}$$

implies

$$y = \frac{x}{c^2} + \frac{\gamma}{c}.$$

Further, the condition $x + y = x + (\frac{x}{c^2} + \frac{\gamma}{c}) \le \gamma$ yields $x \le \frac{c(c-1)\gamma}{1+c^2}$.

2. Let now $x + y > \gamma$ and $c(x + y) > \gamma$, then

$$x = \mathbf{T}^\dagger S_\gamma \mathbf{T}(x + y) = \frac{1}{1 + c^2}(1, c) \begin{pmatrix} x + y - \gamma \\ c(x + y) - \gamma \end{pmatrix} = x + y - \frac{\gamma(c + 1)}{c^2 + 1}.$$

Thus, we find $y = \frac{\gamma(c+1)}{c^2+1}$, and $x+y > \gamma$ is true for $x > \frac{\gamma c(c-1)}{c^2+1}$. Similar considerations for $x < 0$ yield $H(-x) = -H(x)$.

Integration gives $\Phi(x)$ as asserted. ■

Example 3.2 *If we employ Lemma* 3.1 *for $c = 2$ and $\gamma = \frac{5}{3}$, we find*

$$H(x) = \begin{cases} [-\frac{5}{6}, \frac{5}{6}] & x = 0, \\ \frac{5}{6} + \frac{x}{4} & x \in (0, \frac{2}{3}], \\ 1 & x > \frac{2}{3}, \\ -H(-x) & x < 0, \end{cases} \quad \textit{and} \quad \Phi(x) = \begin{cases} \frac{5x}{6} + \frac{x^2}{8} & x \in [0, \frac{2}{3}] \\ x - \frac{1}{18} & x > \frac{2}{3} \\ \Phi(-x) & x < 0. \end{cases}$$

Thus $\Phi(x)$ approximates $|x|$, see Figure 1.

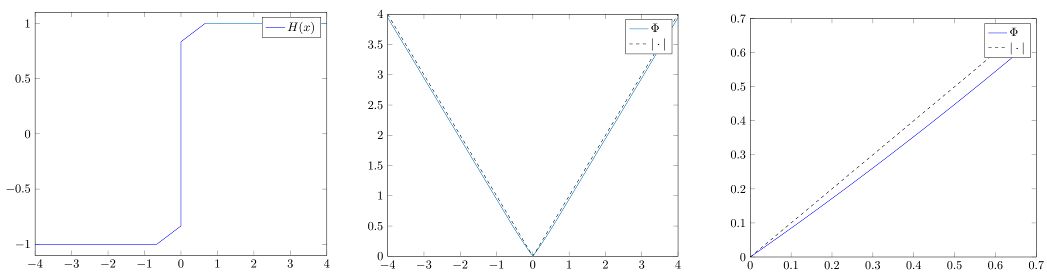

Figure 1: Visualization of $H(x)$ and $\Phi(x)$ in Example 3.2.

Now we inspect the function H in (3.1) and show first that for any fixed $\gamma > 0$ and each $\mathbf{x} \in \mathbb{R}^N$ the set $H(\mathbf{x})$ is not empty.

By (3.1), we have $\mathbf{y} \in H(\mathbf{x})$ if $\mathbf{x} = \mathbf{T}^\dagger S_\gamma \mathbf{T}(\mathbf{x}+\mathbf{y})$. With the substitution $\mathbf{t} = \mathbf{x}+\mathbf{y}$, we get the equivalent fixed point representation

$$\mathbf{t} - \mathbf{x} \in H(\mathbf{x}) \qquad \Longleftrightarrow \qquad \mathbf{t} = \mathbf{x} + (\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})\, \mathbf{t}.$$

Thus, if the function $f_{\mathbf{x},\mathbf{T}} \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with

$$f_{\mathbf{x},\mathbf{T}}(\mathbf{t}) \coloneqq \mathbf{x} + (\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})\, \mathbf{t} \tag{3.2}$$

possesses a fixed point $\mathbf{t}$, then $\mathbf{y} = \mathbf{t} - \mathbf{x}$ is an element of $H(\mathbf{x})$.
In order to show the existence of fixed points for $f_{\mathbf{x},\mathbf{T}}$ for each $\mathbf{x} \in \mathbb{R}^N$, we recall that the matrix $\mathbf{T} \in \mathbb{R}^{L\times N}$ possesses a singular value decomposition

$$\mathbf{T} = \mathbf{P}\,\mathbf{D}\,\mathbf{Q}^*, \tag{3.3}$$

where $\mathbf{P} \in \mathbb{R}^{L\times N}$ has N orthogonal columns, i.e., $\mathbf{P}^*\mathbf{P} = \mathbf{I}_N$, $\mathbf{D} = \mathrm{diag}(d_1, \dots, d_N) \in \mathbb{R}^{N\times N}$ contains the positive singular values of $\mathbf{T}$, and $\mathbf{Q} \in \mathbb{R}^{N\times N}$ is orthogonal. In particular, we have

$$\mathbf{T}^\dagger = \mathbf{Q}\,\mathbf{D}^{-1}\,\mathbf{P}^*.$$

Using the singular value decomposition of $\mathbf{T}$ we discover that the function $f_{\mathbf{DQ}^*\mathbf{x},\mathbf{P}}$, being defined analogously to $f_{\mathbf{x},\mathbf{T}}$ – with $\mathbf{DQ}^*\mathbf{x}$ instead of $\mathbf{x}$ and with the matrix $\mathbf{P}$ instead of $\mathbf{T}$ – satisfies

$$\begin{aligned}
f_{\mathbf{DQ}^*\mathbf{x},\mathbf{P}}(\mathbf{t}) &= \mathbf{DQ}^*\mathbf{x} + (\mathbf{I}_N - \mathbf{P}^* S_\gamma \mathbf{P})\mathbf{t} \\
&= \mathbf{DQ}^*\mathbf{x} + \mathbf{DQ}^*(\mathbf{QD}^{-1})(\mathbf{I}_N - \mathbf{P}^* S_\gamma \mathbf{P})\mathbf{DQ}^*(\mathbf{QD}^{-1}\mathbf{t}) \\
&= \mathbf{DQ}^*\mathbf{x} + \mathbf{DQ}^*(\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})(\mathbf{QD}^{-1}\mathbf{t}) \\
&= \mathbf{DQ}^*(\mathbf{x} + (\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})(\mathbf{QD}^{-1}\mathbf{t})) \\
&= \mathbf{DQ}^*\, f_{\mathbf{x},\mathbf{T}}(\mathbf{QD}^{-1}\mathbf{t}).
\end{aligned}$$

Reversely,

$$f_{\mathbf{x},\mathbf{T}}(\mathbf{t}) = \mathbf{QD}^{-1} f_{\mathbf{DQ}^*\mathbf{x},\mathbf{P}}(\mathbf{DQ}^*\mathbf{t}). \tag{3.4}$$

Theorem 3.3 *Let $\mathbf{T} \in \mathbb{R}^{L\times N}$ with $L \geq N$ have full rank N and let $\gamma > 0$. Then, for each $\mathbf{x} \in \mathbb{R}^N$, we have $H(\mathbf{x}) \neq \emptyset$. Further, the image of H is bounded, i.e., for each $\mathbf{x} \in \mathbb{R}^N$ we have $H(\mathbf{x}) \subset \{\mathbf{y} \in \mathbb{R}^N : \|\mathbf{y}\|_\infty \leq \gamma \|\mathbf{T}^\dagger\|_\infty\}$.*

Proof: To prove that $H(\mathbf{x}) \neq \emptyset$, we show that for each $\mathbf{x} \in \mathbb{R}^N$ the function $f_{\mathbf{x},\mathbf{T}}$ possesses at least one fixed point.

1. First, we consider $f_{\mathbf{x},\mathbf{P}}(\mathbf{t}) = \mathbf{x} + (\mathbf{I}_N - \mathbf{P}^* S_\gamma \mathbf{P})\mathbf{t}$, where the matrix $\mathbf{P} \in \mathbb{R}^{L\times N}$ satisfies $\mathbf{P}^*\mathbf{P} = \mathbf{I}_N$, i.e., $\mathbf{P}^\dagger = \mathbf{P}^*$. We define the closed ball

$$B(\mathbf{x}, \gamma\|\mathbf{P}^*\|_\infty) := \{\mathbf{t} \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{t}\|_\infty \leq \gamma\|\mathbf{P}^*\|_\infty\},$$

where $\|\mathbf{x}\|_\infty := \max_{k=1,\ldots,N} |x_k|$ for $\mathbf{x} = (x_k)_{k=1}^N \in \mathbb{R}^N$. Then $f_{\mathbf{x},\mathbf{P}}(\mathbf{t}) \in B(\mathbf{x}, \|\mathbf{P}^*\|_\infty)$ for each $\mathbf{t} \in \mathbb{R}^N$, since we have

$$\begin{aligned}
\|\mathbf{x} - f_{\mathbf{x},\mathbf{P}}(\mathbf{t})\|_\infty &= \|(\mathbf{I}_N - \mathbf{P}^* S_\gamma \mathbf{P})\mathbf{t}\|_\infty = \|\mathbf{P}^*(\mathbf{I}_L - S_\gamma)\mathbf{P}\mathbf{t}\|_\infty \\
&\leq \|\mathbf{P}^*\|_\infty \|(\mathbf{I}_L - S_\gamma)\mathbf{P}\mathbf{t}\|_\infty \leq \|\mathbf{P}^*\|_\infty \sup_{\mathbf{s}\in\mathbb{R}^L} \|(\mathbf{I}_L - S_\gamma)\mathbf{s}\|_\infty \leq \gamma\|\mathbf{P}^*\|_\infty .
\end{aligned}$$

Since S_γ is continuous, also $f_{\mathbf{x},\mathbf{P}}$ is continuous, and it follows by Brouwer's fixed point theorem [13] within $B(\mathbf{x}, \gamma\|\mathbf{P}^*\|_\infty)$.

Recall that $\mathbf{T} \in \mathbb{R}^{L\times N}$ with rank N can be written as $\mathbf{T} = \mathbf{PDQ}^*$ as given in (3.3). Since $\mathbf{P}$ satisfies $\mathbf{P}^*\mathbf{P} = \mathbf{I}_N$, we already know that $f_{\mathbf{DQ}^*\mathbf{x},\mathbf{P}}$ possesses a fixed point $\bar{\mathbf{x}}$, i.e., $f_{\mathbf{DQ}^*\mathbf{x},\mathbf{P}}(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$. Now, (3.4) implies

$$f_{\mathbf{x},\mathbf{T}}(\mathbf{QD}^{-1}\bar{\mathbf{x}}) = \mathbf{QD}^{-1} f_{\mathbf{DQ}^*\mathbf{x},\mathbf{P}}(\mathbf{DQ}^*(\mathbf{QD}^{-1}\bar{\mathbf{x}})) = \mathbf{QD}^{-1}\bar{\mathbf{x}},$$

i.e., $f_{\mathbf{x},\mathbf{T}}$ possesses at least the fixed point $\mathbf{QD}^{-1}\bar{\mathbf{x}}$.

2. Moreover, using $\mathbf{T}^\dagger\mathbf{T} = \mathbf{I}_N$, we obtain for each fixed point $\mathbf{t} = f_{\mathbf{x},\mathbf{T}}(\mathbf{t})$,

$$\begin{aligned}
\|\mathbf{x} - \mathbf{t}\|_\infty &= \|\mathbf{x} - f_{\mathbf{x},\mathbf{T}}(\mathbf{t})\|_\infty = \|(\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{t}\|_\infty \\
&= \|\mathbf{T}^\dagger(\mathbf{I}_L - S_\gamma)\mathbf{T}\mathbf{t}\|_\infty \leq \|\mathbf{T}^\dagger\|_\infty \|(\mathbf{I}_L - S_\gamma)\mathbf{T}\mathbf{t}\|_\infty \leq \|\mathbf{T}^\dagger\|_\infty\, \gamma.
\end{aligned}$$

Since $\mathbf{t} - \mathbf{x} \in H(\mathbf{x})$, the boundedness of the image of H follows. ■

Thus, we conclude that the mapping $H\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ in (3.1) is well defined. Next we show that $H(\mathbf{0})$ is indeed not single-valued.

Theorem 3.4 *Let* $\mathbf{T} \in \mathbb{R}^{L\times N}$ *with* $L \geq N$ *with full rank* N*. Further let* $\gamma > 0$ *and* H *as in* (3.1)*. Then* $\mathbf{y} \in H(\mathbf{0})$ *if and only if* $\|\mathbf{T}\,\mathbf{y}\|_\infty \leq \gamma$*, where* $\|\mathbf{T}\,\mathbf{y}\|_\infty := \max_{j\in\{1,\ldots,L\}} |[\mathbf{Ty}]_j|$.

Proof: We recall that $\mathbf{T}^\dagger\mathbf{T} = (\mathbf{T}^*\mathbf{T})^{-1}\mathbf{T}^*\mathbf{T} = \mathbf{I}_N$ as $\mathbf{T}$ has full rank N.
First, let $\|\mathbf{T}\,\mathbf{y}\|_\infty \leq \gamma$. Then the definition of S_γ in (1.2) implies $S_\gamma \mathbf{T}\,\mathbf{y} = \mathbf{0}$ and hence also $\mathbf{T}^\dagger S_\gamma \mathbf{T}\,(\mathbf{y}+\mathbf{0}) = \mathbf{0}$, that is, $\mathbf{y} \in H(\mathbf{0})$.

Second, let $\mathbf{y} \in H(\mathbf{0})$, i.e.,

$$\mathbf{T}^\dagger S_\gamma \mathbf{T}\,\mathbf{y} = \mathbf{0}. \tag{3.5}$$

We show that then $\|\mathbf{T}\,\mathbf{y}\|_\infty \leq \gamma$. We consider the components $[\mathbf{T}\,\mathbf{y}]_j$, $j = 1,\ldots,L$, and define three index sets I_1, I_2, I_3 that form a partition of $\{1,\ldots,L\}$,

$$\begin{aligned} I_1 &:= \{1 \leq j \leq L : (\mathbf{T}\,\mathbf{y})_j > \gamma\},\\ I_2 &:= \{1 \leq j \leq L : (\mathbf{T}\,\mathbf{y})_j < -\gamma\},\\ I_3 &:= \{1 \leq j \leq L : (\mathbf{T}\,\mathbf{y})_j \in [-\gamma,\gamma]\}. \end{aligned}$$

Suppose that $\|\mathbf{Ty}\|_\infty > \gamma$, which means that $I_1 \cup I_2 \neq \emptyset$. Then

$$S_\gamma \mathbf{T}\,\mathbf{y} = \sum_{j\in I_1} ([\mathbf{T}\,\mathbf{y}]_j - \gamma)\,\mathbf{e}_j + \sum_{j\in I_2} ([\mathbf{T}\,\mathbf{y}]_j + \gamma)\,\mathbf{e}_j, \tag{3.6}$$

where $\mathbf{e}_j$ denotes the j-th unit vector in $\mathbb{R}^L$. Now we combine (3.6) with (3.5) to get

$$\begin{aligned} \mathbf{0} = \mathbf{T}^\dagger S_\gamma \mathbf{T}\,\mathbf{y} &= \sum_{j\in I_1} ([\mathbf{T}\,\mathbf{y}]_j - \gamma)\,\mathbf{T}^\dagger\mathbf{e}_j + \sum_{j\in I_2} ([\mathbf{T}\,\mathbf{y}]_j + \gamma)\,\mathbf{T}^\dagger\mathbf{e}_j\\ &= \sum_{j\in I_1} ([\mathbf{T}\,\mathbf{y}]_j - \gamma)\,\tilde{\mathbf{v}}_j + \sum_{j\in I_2} ([\mathbf{T}\,\mathbf{y}]_j + \gamma)\,\tilde{\mathbf{v}}_j, \end{aligned} \tag{3.7}$$

where $\tilde{\mathbf{v}}_j := \mathbf{T}^\dagger\,\mathbf{e}_j$. In other words, the set $\{\tilde{\mathbf{v}}_j : j \in I_1 \cup I_2\}$ is linearly dependent. At the same time, none of these vectors $\tilde{\mathbf{v}}_j$ vanishes because $\tilde{\mathbf{v}}_j = \mathbf{0}$ for $j \in I_1 \cup I_2$ leads to the following contradiction,

$$\begin{aligned} 0 &= |\langle \tilde{\mathbf{v}}_j, (\mathbf{T}^*\mathbf{T})\,\mathbf{y}\rangle| = \left|\langle \mathbf{T}^\dagger\mathbf{e}_j, (\mathbf{T}^*\mathbf{T})\,\mathbf{y}\rangle\right| = \left|\langle (\mathbf{T}^*\mathbf{T})\,\mathbf{T}^\dagger\mathbf{e}_j,\ \mathbf{y}\rangle\right| = |\langle \mathbf{T}^*\mathbf{e}_j, \mathbf{y}\rangle|\\ &= |[\mathbf{Ty}]_j| > \gamma, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes here the standard scalar product in $\mathbb{R}^N$ and $\mathbb{R}^L$, respectively. Without loss of generality assume that $I_1 \neq \emptyset$ and choose $j_1 \in I_1$. Then $[\mathbf{Ty}]_{j_1} > \gamma$ and (3.7) implies

$$\tilde{\mathbf{v}}_{j_1} = \sum_{\substack{j\in I_1\\ j\neq j_1}} \frac{\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma}\,\tilde{\mathbf{v}}_j \ + \ \sum_{j\in I_2} \frac{-\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma}\,\tilde{\mathbf{v}}_j. \tag{3.8}$$

A closer look at the coefficients shows that

$$\frac{\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma} < 0 \text{ for } j \in I_1 \setminus \{j_1\} \qquad \text{and} \qquad \frac{-\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma} > 0 \text{ for } j \in I_2.$$

Hence, we find with $\mathbf{T}^*\mathbf{T}\tilde{\mathbf{v}}_j = \mathbf{T}^*\mathbf{e}_j$,

$$
\begin{aligned}
[\mathbf{Ty}]_{j_1} &= \langle \mathbf{Ty}, \mathbf{e}_{j_1} \rangle = \langle \mathbf{y}, (\mathbf{T}^*\mathbf{T})(\mathbf{T}^*\mathbf{T})^{-1}\mathbf{T}^*\mathbf{e}_{j_1} \rangle = \langle \mathbf{y}, (\mathbf{T}^*\mathbf{T})\, \tilde{\mathbf{v}}_{j_1} \rangle \\
&= \left\langle \mathbf{y},\ (\mathbf{T}^*\mathbf{T}) \Big(\sum_{\substack{j \in I_1 \\ j \neq j_1}} \frac{\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma}\, \tilde{\mathbf{v}}_j \ + \ \sum_{j \in I_2} \frac{-\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma}\, \tilde{\mathbf{v}}_j \Big) \right\rangle \\
&= \sum_{\substack{j \in I_1 \\ j \neq j_1}} \frac{\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma} \langle \mathbf{y}, (\mathbf{T}^*\mathbf{T})\, \tilde{\mathbf{v}}_j \rangle \ + \ \sum_{j \in I_2} \frac{-\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma} \langle \mathbf{y}, (\mathbf{T}^*\mathbf{T})\, \tilde{\mathbf{v}}_j \rangle \\
&= \sum_{\substack{j \in I_1 \\ j \neq j_1}} \underbrace{\frac{\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma}}_{<0} \underbrace{\langle \mathbf{T}\,\mathbf{y}, \mathbf{e}_j \rangle}_{>\gamma} \ + \ \sum_{j \in I_2} \underbrace{\frac{-\gamma - [\mathbf{Ty}]_j}{[\mathbf{Ty}]_{j_1} - \gamma}}_{>0} \underbrace{\langle \mathbf{T}\,\mathbf{y}, \mathbf{e}_j \rangle}_{<-\gamma} < 0,
\end{aligned}
$$

which contradicts the above assumption that $j_1 \in I_1$. Thus, $I_1 \cup I_2 = \emptyset$, i.e., all indices are located within I_3, which readily shows that $\|\mathbf{Ty}\|_\infty \leq \gamma$. ■

Further, if all components of $\mathbf{Tx}$ have a modulus greater than $\gamma(\|\mathbf{TT}^\dagger\|_\infty + 1)$, we show that $H(\mathbf{x})$ is single-valued.

Theorem 3.5 *Let $\mathbf{T} \in \mathbb{R}^{L \times N}$ with $L \geq N$ with full rank N. Further let $\gamma > 0$ and $\mathbf{x} \in \mathcal{U}_{\mathbf{T},\gamma} := \{\mathbf{v} \in \mathbb{R}^N : |(\mathbf{Tv})_j| > \gamma(\|\mathbf{T}\,\mathbf{T}^\dagger\|_\infty + 1)\ \forall j = 1, \dots, L\}$. Then $H(\mathbf{x})$ with H in (3.1) is single-valued. Moreover, let $\{\mathbf{u}_\ell : \ell = 1, \dots, 2^L\}$ be the set of all possible vectors in $\mathbb{R}^L$ containing only components -1 and 1. Then, on each set $S_\ell := \mathcal{U}_{\mathbf{T},\gamma} \cap \{\mathbf{x} \in \mathbb{R}^N : \mathrm{sign}(\mathbf{Tx}) = \mathbf{u}_\ell\}$ the mapping H evaluates to the constant value $\gamma\mathbf{T}^\dagger\mathbf{u}_\ell$.*

Proof: By Theorem 3.3, $H(\mathbf{x})$ in not empty for each $\mathbf{x} \in \mathbb{R}^N$, and each fixed point of $f_{\mathbf{x},\mathbf{T}}$ in (3.2) provides us an element $\mathbf{y} = \mathbf{t} - \mathbf{x} \in H(\mathbf{x})$. We show that $f_{\mathbf{x},\mathbf{T}}$ possesses only one fixed point, if $\min\limits_{j=1,\dots,L} |(\mathbf{Tx})_j| > \gamma(\|\mathbf{TT}^\dagger\|_\infty + 1)$. Assume that $f_{\mathbf{x},\mathbf{T}}$, with $\mathbf{x} \in \mathcal{U}_{\mathbf{T},\gamma}$, possesses two fixed points $\mathbf{t}_1$ and $\mathbf{t}_2$. Thus

$$
\mathbf{t}_1 = \mathbf{x} + (\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{t}_1, \qquad \mathbf{t}_2 = \mathbf{x} + (\mathbf{I}_N - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{t}_2,
$$

implies

$$
\mathbf{x} = \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{t}_1 = \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{t}_2.
$$

Consequently,

$$
\mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{t}_1 - \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{t}_2 = \mathbf{0}. \tag{3.9}
$$

Now, similarly as in the previous proof, we can show that $\mathbf{T}^\dagger(S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2) = \mathbf{0}$ implies $S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2 = \mathbf{0}$. This time we consider the index sets

$$
\begin{aligned}
I_1 &= \{1 \leq j \leq L : [S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j > 0\}, \\
I_2 &= \{1 \leq j \leq L : [S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j = 0\}, \\
I_3 &= \{1 \leq j \leq L : [S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j < 0\}.
\end{aligned}
$$

We observe that $j \in I_1$ yields $[\mathbf{Tt}_1 - \mathbf{Tt}_2]_j > 0$ and similarly $j \in I_3$ yields $[\mathbf{Tt}_1 - \mathbf{Tt}_2]_j < 0$. We want to show that $I_1 \cup I_3 = \emptyset$.

The relation (3.9) implies

$$\mathbf{0} = \sum_{j \in I_1} [S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j \mathbf{T}^\dagger \mathbf{e}_j + \sum_{j \in I_3} [S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j \mathbf{T}^\dagger \mathbf{e}_j.$$

Now, suppose contrarily that w.l.o.g. $I_1 \neq \emptyset$, then for $j_1 \in I_1$ we find $\tilde{\mathbf{v}}_{j_1} := \mathbf{T}^\dagger \mathbf{e}_{j_1}$ similarly as in (3.8) of the form

$$\tilde{\mathbf{v}}_{j_1} = \sum_{\substack{j \in I_1 \\ j \neq j_1}} \frac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} \tilde{\mathbf{v}}_j + \sum_{j \in I_3} \frac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} \tilde{\mathbf{v}}_j.$$

A closer look at the coefficients shows that

$$\frac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} < 0$$

for $j \in I_1 \setminus \{j_1\}$ and

$$\frac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} > 0$$

for $j \in I_3$.
Hence, we find with $\mathbf{T}^*\mathbf{T}\tilde{\mathbf{v}}_j = \mathbf{T}^*\mathbf{e}_j$,

$$\begin{aligned}
&[\mathbf{T}(\mathbf{t}_1 - \mathbf{t}_2)]_{j_1} \\
&= \langle \mathbf{T}(\mathbf{t}_1 - \mathbf{t}_2), \mathbf{e}_{j_1} \rangle = \langle \mathbf{t}_1 - \mathbf{t}_2, \mathbf{T}^*\mathbf{T}\tilde{\mathbf{v}}_{j_1} \rangle \\
&= \left\langle \mathbf{t}_1 - \mathbf{t}_2, (\mathbf{T}^*\mathbf{T}) \sum_{\substack{j \in I_1 \\ j \neq j_1}} \tfrac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} \tilde{\mathbf{v}}_j + \sum_{j \in I_3} \tfrac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} \tilde{\mathbf{v}}_j \right\rangle \\
&= \sum_{\substack{j \in I_1 \\ j \neq j_1}} \tfrac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} \langle \mathbf{t}_1 - \mathbf{t}_2, (\mathbf{T}^*\mathbf{T})\tilde{\mathbf{v}}_j \rangle + \sum_{j \in I_3} \tfrac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}} \langle \mathbf{t}_1 - \mathbf{t}_2, (\mathbf{T}^*\mathbf{T})\tilde{\mathbf{v}}_j \rangle \\
&= \sum_{\substack{j \in I_1 \\ j \neq j_1}} \underbrace{\tfrac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}}}_{<0} \underbrace{\langle \mathbf{T}(\mathbf{t}_1 - \mathbf{t}_2), \mathbf{e}_j \rangle}_{>0} + \sum_{j \in I_3} \underbrace{\tfrac{-[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_j}{[S_\gamma \mathbf{T}\mathbf{t}_1 - S_\gamma \mathbf{T}\mathbf{t}_2]_{j_1}}}_{>0} \underbrace{\langle \mathbf{T}(\mathbf{t}_1 - \mathbf{t}_2), \mathbf{e}_j \rangle}_{<0} \\
&< 0,
\end{aligned}$$

which contradicts the above assumption that $j_1 \in I_1$. Therefore, $I_1 \cup I_3 = \emptyset$, and for each component it follows either that $[\mathbf{T}\mathbf{t}_1]_j = [\mathbf{T}\mathbf{t}_2]_j$ or $|[\mathbf{T}\mathbf{t}_1]_j| \leq \gamma$ and $|[\mathbf{T}\mathbf{t}_2]_j| \leq \gamma$. Further, observe that

$$\|\mathbf{T}\mathbf{t}_1 - \mathbf{T}\mathbf{x}\|_\infty = \|\mathbf{T}\mathbf{t}_1 - \mathbf{T}\mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{t}_1\|_\infty = \|\mathbf{T}\mathbf{T}^\dagger (\mathbf{I}_L - S_\gamma)\mathbf{T}\mathbf{t}_1\|_\infty \leq \|\mathbf{T}\mathbf{T}^\dagger\|_\infty \gamma,$$

and similarly for the fixed point $\mathbf{t}_2$. Therefore, $|[\mathbf{T}\mathbf{x}]_j| > \gamma(\|\mathbf{T}^\dagger\mathbf{T}\|_\infty + 1)$ for all $j = 1, \ldots, , L$ implies that

$$|[\mathbf{T}\mathbf{t}_1]_j| > |[\mathbf{T}\mathbf{x}]_j| - |[\mathbf{T}\mathbf{t}_1]_j - [\mathbf{T}\mathbf{x}]_j| > \gamma$$

for all $j = 1, \ldots, L$. Thus, all components of $\mathbf{T}\mathbf{t}_1$ and $\mathbf{T}\mathbf{t}_2$ coincide, and we conclude $\mathbf{t}_1 = \mathbf{t}_2$.

2. Since $H(\mathbf{x})$ is single-valued for $\mathbf{x} \in S_\ell$, it follows with $\mathrm{sign}(\mathbf{T}\mathbf{x}) = \mathbf{u}_\ell$ that

$$\mathbf{x} = \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{t}_1 = \mathbf{T}^\dagger(\mathbf{T}\mathbf{t}_1 - \gamma\mathbf{u}_\ell) = \mathbf{t}_1 - \gamma\mathbf{T}^\dagger\mathbf{u}_\ell$$

and thus $H(\mathbf{x}) = \mathbf{t}_1 - \mathbf{x} = \gamma\mathbf{T}^\dagger\mathbf{u}_\ell$. ∎

Remark 3.6 *Our considerations show that* $H(\mathbf{x})$ *does in general not coincide with the subdifferential* $H_{\|\cdot\|_1}(\mathbf{x}) := \partial\|\mathbf{T}\mathbf{x}\|_1 = \mathbf{T}^* \operatorname{sign}(\mathbf{T}\mathbf{x})$*, where* $\operatorname{sign}\mathbf{y} := (\operatorname{sign} y_j)_{j=1}^L$ *for* $\mathbf{y} = (y_j)_{j=1}^L$ *and*

$$\operatorname{sign} y_j := \begin{cases} 1 & y_j > 0, \\ -1 & y_j < 0, \\ [-1,1] & y_j = 0. \end{cases}$$

But similarly to $H(\mathbf{x})$*, we observe that*

$$\mathbf{y} \in H_{\|\cdot\|_1}(\mathbf{0}) \quad \iff \quad \|\mathbf{T}\mathbf{x}\|_\infty \le \gamma,$$

and $H_{\|\cdot\|_1}(\mathbf{x})$ *is single-valued if* $\min_{j=1,\ldots,L} |[\mathbf{T}\mathbf{x}]_j| > \gamma$.

4 The frame soft threshold operator is a proximity operator

Throughout this section, we again assume that $\mathbf{T} \in \mathbb{R}^{L\times N}$ with $L > N$ has full rank N, $\gamma > 0$ and let S_γ the soft shrinkage operator given in (1.2). In this section, we will show that the set-valued function H in (3.1) is the subdifferential of a proper, lower semi-continuous and convex function Φ, i.e., $\Phi \in \Gamma_0$. Let us first recall the following definition.

Definition 4.1 (12.24 in [12]) *Let* $\langle\cdot,\cdot\rangle$ *denote a scalar product in* $\mathbb{R}^N$*. A mapping* $H\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *is called* cyclically monotone *if for any* $m \in \mathbb{N}$*,* $m \ge 2$ *and any choice of points* $\mathbf{x}_1, \ldots, \mathbf{x}_m$ *and elements* $\mathbf{y}_i \in H(\mathbf{x}_i)$ *we have*

$$\langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{y}_1\rangle + \langle \mathbf{x}_3 - \mathbf{x}_2, \mathbf{y}_2\rangle + \cdots \langle \mathbf{x}_1 - \mathbf{x}_m, \mathbf{y}_m\rangle \le 0. \tag{4.1}$$

We call H maximally cyclically monotone *if it is cyclically monotone and its graph cannot be enlarged without destroying this property.*

We will employ the following theorem.

Theorem 4.2 ([1]) *A set-valued mapping* $H : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *is the subdifferential of a function* $\Phi \in \Gamma_0$*,i.e.,* $H = \partial\Phi$*, if and only if* H *is maximally cyclically monotone.*

In order to show, that H in (3.1) is indeed maximally cyclically monotone, we need some preliminary lemmas.

Lemma 4.3 *For* $\mathbf{x} \in \mathbb{R}^N$ *and* $\mathbf{y} \in H(\mathbf{x})$ *we have*

$$\mathbf{T}\mathbf{y} + (S_\gamma - \mathbf{I}_L)\mathbf{T}(\mathbf{x} + \mathbf{y}) \in \ker(\mathbf{T}^\dagger) = \ker(\mathbf{T}^*),$$

and

$$\mathbf{y} = \mathbf{T}^\dagger(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x} + \mathbf{y}),$$

where $\mathbf{I}_L$ *denotes the* $L \times L$ *identity matrix.*

Proof: Recall that $\mathbf{T}^\dagger = (\mathbf{T}^*\mathbf{T})^{-1}\mathbf{T}^* \in \mathbb{R}^{N\times L}$ and $\mathbf{T}^\dagger\mathbf{T} = \mathbf{I}_N$. Thus we have

$$\begin{aligned}\mathbf{y} \in H(\mathbf{x}) &\iff \mathbf{x} = \mathbf{T}^\dagger S_\gamma \mathbf{T}(\mathbf{x}+\mathbf{y})\\ &\iff \mathbf{T}^\dagger\mathbf{T}\mathbf{x} = \mathbf{T}^\dagger S_\gamma \mathbf{T}(\mathbf{x}+\mathbf{y})\\ &\iff \exists\, \mathbf{u} \in \ker(\mathbf{T}^\dagger) : \mathbf{u} + \mathbf{T}\mathbf{x} = S_\gamma\mathbf{T}(\mathbf{x}+\mathbf{y})\\ &\iff \exists\, \mathbf{u} \in \ker(\mathbf{T}^\dagger) : \mathbf{u} + \mathbf{T}(\mathbf{x}+\mathbf{y}) - \mathbf{T}\mathbf{y} = S_\gamma\mathbf{T}(\mathbf{x}+\mathbf{y})\\ &\iff \exists\, \mathbf{u} \in \ker(\mathbf{T}^\dagger) : \mathbf{u} + \mathbf{T}(\mathbf{x}+\mathbf{y}) - S_\gamma\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}\mathbf{y}\\ &\iff \exists\, \mathbf{u} \in \ker(\mathbf{T}^\dagger) : \mathbf{u} + (\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}\mathbf{y}. \qquad (4.2)\end{aligned}$$

In other words, $\mathbf{T}\mathbf{y} + (S_\gamma - \mathbf{I}_L)\mathbf{T}(\mathbf{x}+\mathbf{y}) \in \ker(\mathbf{T}^\dagger)$. Further, multiplying (4.2) with $\mathbf{T}^\dagger$, it follows that

$$\mathbf{T}^\dagger\mathbf{T}\mathbf{y} = \mathbf{y} = \mathbf{T}^\dagger(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}+\mathbf{y}),$$

which finishes this proof. ■

Lemma 4.4 *Let* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$ *and* $\mathbf{y}_1 \in H(\mathbf{x}_1)$, $\mathbf{y}_2 \in H(\mathbf{x}_2)$. *Further, let*

$$\mathbf{z}_1 := (\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1), \qquad \mathbf{z}_2 := (\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_2+\mathbf{y}_2).$$

Then

$$\langle S_\gamma\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1), \mathbf{z}_2 - \mathbf{z}_1\rangle_2 \le 0,$$

where $\langle\cdot,\cdot\rangle_2$ *denotes the standard scalar product in* $\mathbb{R}^L$.

Proof: From the definition of S_γ it follows for $x \in \mathbb{R}$

$$(1 - S_\gamma)x = \begin{cases} x - (x-\gamma) = \gamma & x > \gamma,\\ x - (x+\gamma) = -\gamma & x < -\gamma,\\ x & |x| \le \gamma,\end{cases}$$

and therefore for all $\mathbf{x} \in \mathbb{R}^N$,

$$|[(\mathbf{I}_L - S_\gamma)\mathbf{T}\mathbf{x}]_j| \le \gamma, \qquad j = 1,\dots,L. \qquad (4.3)$$

Let us consider the case $[\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j \le -\gamma$. Then $[(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j = -\gamma$ and (4.3) implies $|[(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_2+\mathbf{y}_2)]_j \ge -\gamma$, such that

$$\begin{aligned}[\mathbf{z}_2 - \mathbf{z}_1]_j &= [(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_2+\mathbf{y}_2) - (\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j \qquad (4.4)\\ &\ge -\gamma - [(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j = -\gamma + \gamma = 0.\end{aligned}$$

In the case $[\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j \ge \gamma$ we similarly conclude from $[(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j = \gamma$ and (4.3) that

$$\begin{aligned}[\mathbf{z}_2 - \mathbf{z}_1]_j &= [(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_2+\mathbf{y}_2) - (\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j \qquad (4.5)\\ &\le \gamma - [(\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j = \gamma - \gamma = 0.\end{aligned}$$

Thus, in both cases we have $-\mathrm{sign}[\mathbf{z}_2 - \mathbf{z}_1]_j = \mathrm{sign}[S_\gamma\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j$. Finally, for $|[\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j| < \gamma$ it follows that $[S_\gamma\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j = 0$. We therefore conclude

$$\langle S_\gamma\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1), \mathbf{z}_2 - \mathbf{z}_1\rangle_2 = \sum_{j=1}^{N}[S_\gamma\mathbf{T}(\mathbf{x}_1+\mathbf{y}_1)]_j[\mathbf{z}_2 - \mathbf{z}_1]_j \le 0.$$

and the assertion follows. ■

Now we can show

Theorem 4.5 *The map* $H(\mathbf{x})$ *in* (3.1) *is cyclically monotone.*

Proof: Let $m \in \mathbb{N}$ and $m \geq 2$. Further, for $i \in \{1, \ldots, m\}$ let $\mathbf{y}_i \in H(\mathbf{x}_i)$. Again, we use the singular value decomposition of $\mathbf{T}$, $\mathbf{T} = \mathbf{PDQ}^T$ with $\mathbf{Q} \in \mathbb{R}^{N \times N}$ orthogonal, $\mathbf{D} \in \mathbb{R}^{N \times N}$ being the diagonal matrix with the N positive singular values of $\mathbf{T}$, and $\mathbf{P} \in \mathbb{R}^{L \times N}$ with $\mathbf{P}^*\mathbf{P} = \mathbf{I}_N$. Further, let

$$\begin{aligned} \widetilde{\mathbf{x}}_i &:= \mathbf{DQ}^T\mathbf{x}_i, \qquad \widetilde{\mathbf{y}}_i := \mathbf{DQ}^T\mathbf{y}_i, \\ \mathbf{z}_i &:= (\mathbf{I}_L - S_\gamma)\mathbf{T}(\mathbf{x}_i + \mathbf{y}_i) = (\mathbf{I}_L - S_\gamma)\mathbf{P}(\widetilde{\mathbf{x}}_i + \widetilde{\mathbf{y}}_i), \\ \mathbf{u}_i &:= \mathbf{Ty}_i - \mathbf{z}_i = \mathbf{P}\widetilde{\mathbf{y}}_i - \mathbf{z}_i. \end{aligned} \tag{4.6}$$

For simplicity we use the convention $\mathbf{x}_{m+1} := \mathbf{x}_1$ as well as $\mathbf{y}_{m+1} := \mathbf{y}_1$, and extend that similarly for $\widetilde{\mathbf{x}}_{m+1}$, $\widetilde{\mathbf{y}}_{m+1}$, $\mathbf{z}_{m+1}$, and $\mathbf{u}_{m+1}$. Then, by Lemma 4.3, $\mathbf{u}_i \in \ker \mathbf{T}^\dagger$ and thus also $\mathbf{u}_i \in \ker \mathbf{P}^*$, since $\mathbf{T}^\dagger = \mathbf{QD}^{-1}\mathbf{P}^*$, where $\mathbf{QD}^{-1} \in \mathbb{R}^{N \times N}$ is invertible. Further, with (4.6) we can write

$$\widetilde{\mathbf{y}}_i = \mathbf{P}^*\mathbf{P}\widetilde{\mathbf{y}}_i = \mathbf{P}^*(\mathbf{u}_i + \mathbf{z}_i) = \mathbf{P}^*\mathbf{z}_i.$$

We will show that

$$A := \sum_{i=1}^{m} \langle \widetilde{\mathbf{x}}_{i+1} - \widetilde{\mathbf{x}}_i, \widetilde{\mathbf{y}}_i \rangle_2 \leq 0,$$

where $\langle \cdot, \cdot \rangle_2$ denotes here the standard scalar product in $\mathbb{R}^N$. First, we observe that for all $i = 1, \ldots, m$,

$$\begin{aligned} \mathbf{P}\widetilde{\mathbf{x}}_i + \mathbf{u}_i &= \mathbf{P}\widetilde{\mathbf{x}}_i + \mathbf{P}\widetilde{\mathbf{y}}_i - \mathbf{z}_i \\ &= \mathbf{P}(\widetilde{\mathbf{x}}_i + \widetilde{\mathbf{y}}_i) - (\mathbf{I}_L - S_\gamma)\mathbf{P}(\widetilde{\mathbf{x}}_i + \widetilde{\mathbf{y}}_i) \\ &= S_\gamma \mathbf{P}(\widetilde{\mathbf{x}}_i + \widetilde{\mathbf{y}}_i). \end{aligned} \tag{4.7}$$

Using (4.7), it follows

$$\begin{aligned} A &= \sum_{i=1}^{m} \langle \widetilde{\mathbf{x}}_{i+1} - \widetilde{\mathbf{x}}_i, \mathbf{P}^*\mathbf{z}_i \rangle_2 = \sum_{i=1}^{m} \langle \mathbf{P}(\widetilde{\mathbf{x}}_{i+1} - \widetilde{\mathbf{x}}_i), \mathbf{z}_i \rangle_2 \\ &= \sum_{i=1}^{m} \langle (\mathbf{P}\widetilde{\mathbf{x}}_{i+1} + \mathbf{u}_{i+1}) - (\mathbf{P}\widetilde{\mathbf{x}}_i + \mathbf{u}_i) - \mathbf{u}_{i+1} + \mathbf{u}_i, \mathbf{z}_i \rangle_2 \\ &= \sum_{i=1}^{m} \langle S_\gamma \mathbf{P}(\widetilde{\mathbf{x}}_{i+1} + \widetilde{\mathbf{y}}_{i+1}), \mathbf{z}_i \rangle_2 - \sum_{i=1}^{m} \langle S_\gamma \mathbf{P}(\widetilde{\mathbf{x}}_i + \widetilde{\mathbf{y}}_i), \mathbf{z}_i \rangle_2 + \sum_{i=1}^{m} \langle -\mathbf{u}_{i+1} + \mathbf{u}_i, \mathbf{z}_i \rangle_2 \\ &= \sum_{i=1}^{m} \langle S_\gamma \mathbf{P}(\widetilde{\mathbf{x}}_{i+1} + \widetilde{\mathbf{y}}_{i+1}), \mathbf{z}_i - \mathbf{z}_{i+1} \rangle_2 + \sum_{i=1}^{m} \langle -\mathbf{u}_{i+1} + \mathbf{u}_i, \mathbf{z}_i \rangle_8. \end{aligned}$$

Bei Lemma 4.4, the first sum is not positive. Thus,

$$\begin{aligned} A &\leq \sum_{i=1}^{m} \langle -\mathbf{u}_{i+1} + \mathbf{u}_i, \mathbf{z}_i \rangle_2 = \sum_{i=1}^{m} \langle -\mathbf{u}_{i+1} + \mathbf{u}_i, \mathbf{P}\widetilde{\mathbf{y}}_i - \mathbf{u}_i \rangle_2 \\ &= \sum_{i=1}^{m} \langle \mathbf{P}^*(-\mathbf{u}_{i+1} + \mathbf{u}_i), \widetilde{\mathbf{y}}_i \rangle_2 - \sum_{i=1}^{m} \langle \mathbf{u}_i, \mathbf{u}_i \rangle_2 + \sum_{i=1}^{m} \langle \mathbf{u}_{i+1}, \mathbf{u}_i \rangle_2. \end{aligned}$$

The first sum vanishes, since $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ are in $\ker \mathbf{P}^*$. Thus

$$A \leq -\sum_{i=1}^{m} \|\mathbf{u}_i\|_2^2 + \sum_{i=1}^{m} \langle \mathbf{u}_{i+1}, \mathbf{u}_i \rangle_2 \leq -\sum_{i=1}^{m} \|\mathbf{u}_i\|_2^2 + \frac{1}{2}\left(\sum_{i=1}^{m} \|\mathbf{u}_i\|_2^2 + \|\mathbf{u}_{i+1}\|_2^2\right) = 0.$$

Now, we define the new scalar product in $\mathbb{R}^N$,

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{DQ}^T} := \mathbf{x}^* \mathbf{Q} \mathbf{D}^T \mathbf{D} \mathbf{Q}^T \mathbf{y}, \tag{4.8}$$

where $\mathbf{D}$ is the diagonal matrix of positive singular values of $\mathbf{T}$ and $\mathbf{Q}$ is the orthogonal matrix in the SVD of $\mathbf{T}$, such that $\mathbf{Q}\mathbf{D}^T\mathbf{D}\mathbf{Q}^T$ is symmetric and positive definite. Then we simply observe that

$$A = \sum_{i=1}^{m} \langle \widetilde{\mathbf{x}}_{i+1} - \widetilde{\mathbf{x}}_i, \widetilde{\mathbf{y}}_i \rangle_2 = \sum_{i=1}^{m} \langle \mathbf{x}_{i+1} - \mathbf{x}_i, \mathbf{y}_i \rangle_{\mathbf{DQ}^T} \leq 0.$$

Thus the assertion of the theorem holds. ■

Now we can conclude the main theorem of this paper.

Theorem 4.6 *Let $\mathbf{T} \in \mathbb{R}^{L \times N}$ with $L \geq N$ and full rank N. Then the operator $\mathbf{T}^\dagger S_\gamma \mathbf{T}$ is a proximity operator of a proper, lower semi-continuous, convex function Φ.*

Proof: First, recall from (2.3) that for any vectors $\mathbf{x}$ and $\mathbf{p}$, we have $\mathbf{x} = \operatorname{prox}_\Phi(\mathbf{p})$ if and only if $\mathbf{x} - \mathbf{p} \in \partial\Phi$. So, by Theorem 4.2, we need to prove that $H(\mathbf{x})$ in (3.1) is maximally cyclically monotone. As shown in the previous theorem, we already have that $H(\mathbf{x})$ is cyclically monotone. Further, by Theorem 3.3, $H(\mathbf{x})$ is bounded, i.e., for all $\mathbf{x} \in \mathbb{R}^N$ we have that $\mathbf{y} \in H(\mathbf{x})$ implies $\|\mathbf{y}\|_\infty \leq \gamma \|\mathbf{T}^*\|_\infty$. Therefore, we observe that the range of the operator $\mathbf{I}_N + H$ is $\mathbb{R}^N$. By Minty's Theorem, see [1], Theorem 21.1, it follows that $H(\mathbf{x})$ is also maximally monotone. The assertion now follows from Theorem 4.2. ■

Corollary 4.7 *Let Let $\mathbf{T} \in \mathbb{R}^{L \times N}$ with $L \geq N$ and full rank N, and let $\mathbf{T} = \mathbf{PDQ}^T$ be its singular value decomposition. Let $\|\cdot\|_{\mathbf{DQ}^T}$ be the norm corresponding to the scalar product in (4.8). Then the operators $\mathbf{T}^\dagger S_\gamma \mathbf{T}$ and $\mathbf{I} - \mathbf{T}^\dagger S_\gamma \mathbf{T}$ are firmly non-expansive, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ we have*

$$\| \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{x} - \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{y}\|_{\mathbf{DQ}^T}^2 + \| (\mathbf{I} - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{x} - (\mathbf{I} - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{y}\|_{\mathbf{DQ}^T}^2 \leq \|\mathbf{x} - \mathbf{y}\|_{\mathbf{DQ}^T}^2.$$

Proof: Observe that with $\mathbf{T}^\dagger = \mathbf{Q}\mathbf{D}^+\mathbf{P}^T$,

$$\| \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{x} - \mathbf{T}^\dagger S_\gamma \mathbf{T}\mathbf{y}\|_{\mathbf{DQ}^T}^2 = \| (\mathbf{DQ}^T)\mathbf{Q}\mathbf{D}^+\mathbf{P}^T (S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y}\|_2^2 = \| S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y}\|_2^2$$

as well as

$$\begin{aligned} & \| (\mathbf{I} - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{x} - (\mathbf{I} - \mathbf{T}^\dagger S_\gamma \mathbf{T})\mathbf{y}\|_{\mathbf{DQ}^T}^2 = \|\mathbf{DQ}^T(\mathbf{x} - \mathbf{y}) - \mathbf{P}^T(S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y})\|_2^2 \\ = \; & \|\mathbf{x} - \mathbf{y}\|_{\mathbf{DQ}^T}^2 + \|S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y}\|_2^2 - 2\langle \mathbf{T}(\mathbf{x} - \mathbf{y}), S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y} \rangle_2 \end{aligned}$$

To prove the assertion of the corollary we hence have to show that

$$\|S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y}\|_2^2 - \langle \mathbf{T}(\mathbf{x} - \mathbf{y}), S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y} \rangle_2 \leq 0.$$

But this assertion is obviously true since for each component $k = 1, \ldots, L$, we have either $(S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y})_k = 0$ or $\operatorname{sign}(S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y})_k) = \operatorname{sign}(\mathbf{T}(\mathbf{x} - \mathbf{y}))_k$ and $|(S_\gamma \mathbf{T}\mathbf{x} - S_\gamma \mathbf{T}\mathbf{y})_k| \leq |(\mathbf{T}(\mathbf{x} - \mathbf{y}))_k|$. ■

Remark 4.8 *Corollary 4.7 particularly implies that* $\mathbf{T}^\dagger S_\gamma \mathbf{T}$ *and* $\mathbf{I} - \mathbf{T}^\dagger S_\gamma \mathbf{T}$ *are non-expansive, see [1].*

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