A Unified Approach to Periodic Wavelets

Gerlind Plonka and Manfred Tasche

Abstract. We sketch a new approach to p–periodic wavelets for general periodic scaling functions. Our method is based on properties of periodic shift–invariant spaces and related bracket products.
A special way to construct periodic wavelets is the periodization of a known cardinal multiresolution. Using FFT–algorithms, efficient decomposition and reconstruction algorithms are proposed.

§0. Introduction

A theory of periodic wavelets is the basic tool for an investigation of periodic processes in signal processing and numerical analysis. One way to construct periodic wavelets is the periodization of known cardinal wavelets. Y. Meyer [13] was the first to study such periodic multiresolutions (see also I. Daubechies [8], pp. 304 – 307). Further, V. Perrier and C. Basdevant [14] (see [20] for a different approach) investigated orthogonal periodic spline wavelets defined by periodization of Battle–Lemarié wavelets [1,10]. Recently, the authors [15] considered semiorthogonal periodic spline wavelets which can be obtained by periodization of Chui–Wang wavelets [4,6,7].

On the other hand, there are constructions of periodic wavelets which do not use this periodization technique. Various trigonometric wavelets were studied by C. K. Chui and H. N. Mhaskar [5] and by J. Prestin and E. Quak [17,18] (see also [19]) without using knowledge about possibly existing corresponding cardinal multiresolutions.

The aim of this paper is a unified introduction to periodic univariate wavelets and to the corresponding decomposition and reconstruction algorithms based on Fourier technique. It should be stressed that the presented theory does not depend on the cardinal approach, i.e., it is not derived by periodization of a cardinal multiresolution. But of course, special periodic multiresolutions obtained by periodization are included.

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It turns out that similar ideas as used for the construction of cardinal wavelets also succeed in the periodic case. The basic tool of our method is the detailed analysis of p–periodic shift–invariant subspaces of the Hilbert space L_p^2 of all p–periodic square integrable functions. For $j \in \mathbb{N}_0$, we put $h_j := p/d_j$ with $d_j := 2^j d$ $(d \in \mathbb{N})$. We are especially interested in h_j–shift–invariant spaces $S_j(\varphi_j)$ generated by $h_j\mathbb{Z}$–translations of one function $\varphi_j \in L_p^2$. Our research is influenced by [2,3,9]. In that papers, cardinal shift–invariant spaces in $L^2(\mathbb{R}^d)$ have systematically been studied, and the results have been applied to a new approach to wavelets on $\mathbb{R}^d$.
The main idea in studying cardinal multiresolutions and wavelets is to consider the corresponding problems in the Fourier transformed domain. In case of periodic multiresolutions we will use the bijective mapping by the finite Fourier transform instead.

The outline of our paper is as follows. In Section 1 we consider p–periodic shift–invariant subspaces of L_p^2, which we describe by their finite Fourier transforms. The scalar product between functions of p–periodic h_j–shift–invariant spaces can be simplified to a finite sum by means of the so-called bracket product, which is closely related to the p–periodic autocorrelation symbol introduced in Section 2. This bracket product is convenient for the description of stable bases of $S_j(\varphi_j)$ $(j \in \mathbb{N}_0)$ as well as for the characterization of orthogonal shift–invariant spaces.
In Section 2 we define a p–periodic multiresolution of L_p^2 by a nested sequence of h_j–shift–invariant spaces $V_j := S_j(\varphi_j)$ $(j \in \mathbb{N}_0)$. The required conditions of a p–periodic multiresolution of L_p^2 and their consequences for φ_j are analyzed in some detail. In Section 3 we introduce the p–periodic wavelet space W_j $(j \in \mathbb{N}_0)$ as orthogonal complement of V_j in V_{j+1}. Periodic wavelets ψ_j are obtained by finding generators for the p–periodic h_j–shift–invariant space W_j. Using the two–scale symbol of φ_j and the periodic autocorrelation symbols of φ_j and φ_{j+1}, the possible wavelets ψ_j are characterized in Theorem 3.4. Further, the close connection between the Fourier transformed two–scale relations of φ_j and ψ_j $(j \in \mathbb{N}_0)$ and the two–scale (2,2)–matrices is discussed. Assuming the stability for the bases of V_j and W_j $(j \in \mathbb{N}_0)$, all two-scale symbol matrices are well–conditioned.
Section 4 is devoted to new efficient decomposition and reconstruction algorithms based on Fourier technique and two–scale symbol matrices. Our wavelet algorithms are very fast, numerically stable and do not contain truncation errors.

§1. Periodic Shift–Invariant Spaces

Let $p > 0$ and $d \in \mathbb{N}$ be fixed. Put $d_j := 2^j d$, $h_j := p/d_j$ $(j \in \mathbb{N}_0)$. Consider the Hilbert space L_p^2 of all p–periodic, square integrable functions $f : \mathbb{R} \to \mathbb{C}$ with the scalar product

$$\langle f, g \rangle := \frac{1}{p} \int_0^p f(t)\, \overline{g(t)}\, \mathrm{d}t \quad (f, g \in L_p^2)$$

and the related norm $\|\cdot\|$. Introduce the finite Fourier transform of $f \in L_p^2$ by $\mathbf{c}(f) := (c_u(f))_{u=-\infty}^{\infty} \in l^2$ with

$$c_u(f) := \langle f, e^{2\pi i u \cdot /p} \rangle \quad (u \in \mathbb{Z}).$$

For $u \in \mathbb{Z}$ and $f \in L_p^2$, we have

$$c_u(f(\cdot - kh_j)) = \omega_j^{uk}\, c_u(f) \quad (k = 0, \ldots, d_j - 1) \tag{1.1}$$

with $\omega_j := \exp(-2\pi i/d_j)$.

A linear subspace S of L_p^2 is called *h_j–shift–invariant*, if for each $f \in S$ all h_j–shifts $f(\cdot - kh_j)$ $(k = 0, \ldots, d_j - 1)$ are contained in S. The *h_j–shift–invariant subspace generated by* $\varphi \in L_p^2$ is defined by

$$S_j(\varphi) := \text{span}\,\{\varphi(\cdot - kh_j) :\ k = 0, \ldots, d_j - 1\}.$$

A useful characterization of the functions of $S_j(\varphi)$ can be given by their finite Fourier transforms:

Lemma 1.1 (see [16]) Let $\varphi \in L_p^2$ and $j \in \mathbb{N}_0$ be given.
(i) We have $f \in S_j(\varphi)$ if and only if

$$c_u(f) = \hat{a}_{j,u}(f)\, c_u(\varphi) \quad (u \in \mathbb{Z})$$

with

$$\hat{a}_{j,u}(f) \in \mathbb{C}, \quad \hat{a}_{j,u}(f) = \hat{a}_{j,u+d_j}(f) \quad (u \in \mathbb{Z}).$$

(ii) Let $f \in S_j(\varphi)$. Then $S_j(f) = S_j(\varphi)$ if and only if

$$\text{supp}\ \mathbf{c}(f) = \text{supp}\ \mathbf{c}(\varphi)$$

with the support $\text{supp}\ \mathbf{c}(f) := \{u \in \mathbb{Z} :\ c_u(f) \neq 0\}$ of $\mathbf{c}(f)$.

For a detailed analysis of periodic shift–invariant spaces we introduce the following notion. Let the *bracket product of level j* $(j \in \mathbb{N}_0)$ be defined for $\mathbf{a} := (a_u)_{u=-\infty}^{\infty}$, $\mathbf{b} := (b_u)_{u=-\infty}^{\infty} \in l^2$ by $[\mathbf{a}, \mathbf{b}]_j := ([\mathbf{a}, \mathbf{b}]_{j,k})_{k=0}^{d_j-1}$, where

$$[\mathbf{a}, \mathbf{b}]_{j,k} := \sum_{u=-\infty}^{\infty} a_{k+ud_j}\, \overline{b_{k+ud_j}} \quad (k = 0, \ldots, d_j - 1).$$

Then, $[\mathbf{a}, \mathbf{a}]_{j,k} \geq 0$ $(k = 0, \ldots, d_j - 1)$ for $\mathbf{a} \in l^2$. Further, $[\mathbf{a}, \mathbf{a}]_{j,k} = 0$ $(k = 0, \ldots, d_j - 1)$ if and only if $\mathbf{a} = (0)_{u=-\infty}^{\infty}$. By Cauchy–Schwarz inequality, we have for $k = 0, \ldots, d_j - 1$

$$|[\mathbf{a}, \mathbf{b}]_{j,k}|^2 \leq [\mathbf{a}, \mathbf{a}]_{j,k}\, [\mathbf{b}, \mathbf{b}]_{j,k} \leq \|\mathbf{a}\|_{l^2}^2\, \|\mathbf{b}\|_{l^2}^2 < \infty.$$

The bracket product will be an important tool for the description of shift–invariant spaces as well as for the characterization of their bases.

Lemma 1.2 *Let $\varphi,\ \psi \in L^2_p$ and $j \in \mathbb{N}_0$. Further let $f \in S_j(\varphi)$, $g \in S_j(\psi)$ with*

$$c_u(f) = \hat{a}_{j,u}(f)\, c_u(\varphi), \quad c_u(g) = \hat{b}_{j,u}(g)\, c_u(\psi) \quad (u \in \mathbb{Z})$$

be given, where $\hat{a}_{j,u}(f)$, $\hat{b}_{j,u} \in \mathbb{C}$ possess the properties

$$\hat{a}_{j,u}(f) = \hat{a}_{j,u+d_j}(f), \quad \hat{b}_{j,u}(g) = \hat{b}_{j,u+d_j}(g) \quad (u \in \mathbb{Z}).$$

Then we have

$$\langle f, g \rangle = \sum_{k=0}^{d_j-1} \hat{a}_{j,k}(f)\, \overline{\hat{b}_{j,k}(g)}\, [\mathbf{c}(\varphi), \mathbf{c}(\psi)]_{j,k}.$$

Proof: From the Parseval identity, it follows that

$$\begin{aligned}
\langle f, g \rangle &= \sum_{u=-\infty}^{\infty} c_u(f)\, \overline{c_u(g)} \\
&= \sum_{k=0}^{d_j-1} \sum_{v=-\infty}^{\infty} c_{k+vd_j}(f)\, \overline{c_{k+vd_j}(g)} \\
&= \sum_{k=0}^{d_j-1} a_{j,k}(f)\, \overline{b_{j,k}(g)} \sum_{v=-\infty}^{\infty} c_{k+vd_j}(\varphi)\, \overline{c_{k+vd_j}(\psi)},
\end{aligned}$$

and hence the assertion. ∎

As a consequence of Lemma 1.2 we obtain:

Corollary 1.3 *Let $\varphi,\ \psi \in L^2_p$ and $j \in \mathbb{N}_0$ be given. Then we have*
(i)

$$\langle \varphi(\cdot - lh_j), \psi \rangle = \sum_{k=0}^{d_j-1} \omega_j^{kl}\, [\mathbf{c}(\varphi), \mathbf{c}(\psi)]_{j,k} \quad (l = 0, \ldots, d_j - 1). \tag{1.2}$$

(ii) *$S_j(\varphi) \perp S_j(\psi)$ if and only if*

$$[\mathbf{c}(\varphi), \mathbf{c}(\psi)]_{j,k} = 0 \quad (k = 0, \ldots, d_j - 1).$$

For $\varphi \in L^2_p$, we consider the system $\mathcal{B}_j(\varphi) := \{\varphi(\cdot - lh_j) :\ l = 0, \ldots, d_j - 1\}$. By (1.2), the corresponding Gramian matrix is circulant and reads as follows

$$(\langle \varphi(\cdot - lh_j), \varphi(\cdot - nh_j) \rangle)_{l,n=0}^{d_j-1} = \mathbf{F}_j\, (\mathrm{diag}\ [\mathbf{c}(\varphi), \mathbf{c}(\varphi)]_j)\, \overline{\mathbf{F}}_j \tag{1.3}$$

with the d_j–th Fourier matrix

$$\mathbf{F}_j := (\omega_j^{kl})_{k,l=0}^{d_j-1}.$$

Thus we find:

Lemma 1.4 *Let $\varphi \in L^2_p$ and $j \in \mathbb{N}_0$ be given.*
(i) *$\mathcal{B}_j(\varphi)$ is a basis of $S_j(\varphi)$ if and only if*

$$[\mathbf{c}(\varphi), \mathbf{c}(\varphi)]_{j,k} > 0 \quad (k = 0, \ldots, d_j - 1). \tag{1.4}$$

(ii) *$\mathcal{B}_j(\varphi)$ is an orthonormal basis of $S_j(\varphi)$ if and only if*

$$d_j\, [\mathbf{c}(\varphi), \mathbf{c}(\varphi)]_{j,k} = 1 \quad (k = 0, \ldots, d_j - 1).$$

(iii) *If φ satisfies (1.4) and if $\varphi^\star \in L^2_p$ is defined by*

$$c_u(\varphi^\star) := d_j^{-1/2}\, [\mathbf{c}(\varphi), \mathbf{c}(\varphi)]_{j,u_j}^{-1/2}\, c_u(\varphi) \quad (u \in \mathbb{Z};\; u_j := u \bmod d_j), \tag{1.5}$$

then $\mathcal{B}_j(\varphi^\star)$ is an orthonormal basis of $S_j(\varphi)$.

Proof: 1. By (1.3), the Gramian matrix related to $\mathcal{B}_j(\varphi)$ is regular if and only if diag $[\mathbf{c}(\varphi), \mathbf{c}(\varphi)]_j$ is regular, i.e., if (1.4) is satisfied.
2. Note that

$$\mathbf{F}_j\, \overline{\mathbf{F}}_j = d_j \mathbf{I}_j$$

with the d_j-th identity matrix $\mathbf{I}_j$. Now, $\mathcal{B}_j(\varphi)$ is an orthonormal basis of $S_j(\varphi)$ if and only if the Gramian matrix of $\mathcal{B}_j(\varphi)$ is equal to $\mathbf{I}_j$. This is true if and only if (ii) holds.
3. Since

$$[\mathbf{c}(\varphi^\star), \mathbf{c}(\varphi^\star)]_{j,k} = d_j^{-1} \quad (k = 0, \ldots, d_j - 1),$$

the Gramian matrix of $\mathcal{B}_j(\varphi^\star)$ is equal to $\mathbf{I}_j$. Hence, $\mathcal{B}_j(\varphi^\star)$ is an orthonormal basis of $S_j(\varphi^\star)$. By Lemma 1.1 (ii) and by (1.5), we have $S_j(\varphi^\star) = S_j(\varphi)$. ■

§2. Periodic Multiresolution

For each $j \in \mathbb{N}_0$, we form h_j–shift–invariant subspaces $V_j := S_j(\varphi_j)$ generated by $\varphi_j \in L^2_p$. Put $\varphi_{j,k} := \varphi_j(\cdot - kh_j)$ $(k = 0, \ldots, d_j - 1)$. We say that $\{V_j\}_{j=0}^{\infty}$ forms a *p–periodic multiresolution*, if the following three conditions are satisfied (compare [12,11,3]):

(M1) $\quad V_j \subset V_{j+1} \quad (j \in \mathbb{N}_0)$.

(M2) $\quad \operatorname{clos}\, (\bigcup_{j=0}^{\infty} V_j) = L^2_p$.

(M3) There exist positive constants α, β such that for all $j \in \mathbb{N}_0$ and for any $(a_{j,n})_{n=0}^{d_j-1} \in \mathbb{C}^{d_j}$,

$$\alpha \sum_{n=0}^{d_j-1} |a_{j,n}|^2 \le \| \sum_{n=0}^{d_j-1} a_{j,n}\, d_j^{1/2}\, \varphi_{j,n} \|^2 \le \beta \sum_{n=0}^{d_j-1} |a_{j,n}|^2.$$

By (M3), $\mathcal{B}_j(d_j^{1/2}\varphi_j)$ is a basis of V_j. Furthermore, if the condition (M3) is satisfied, then the system $\{\mathcal{B}_j(d_j^{1/2}\varphi_j) : j \in \mathbb{N}_0\}$ is called L_p^2*–stable*. The h_j–shift–invariant subspace V_j is called *sample space of level* j. A generating function $d_j^{1/2}\,\varphi_j$ of V_j is the *scaling function* or *generator* of V_j. If all systems $\mathcal{B}_j(d_j^{1/2}\,\varphi_j)$ are orthonormal bases of V_j ($j \in \mathbb{N}_0$), then we say that $d_j^{1/2}\,\varphi_j$ is an *orthonormal scaling function of level* j. In this case the constants in condition (M3) read $\alpha = \beta = 1$. Note that dim $V_j = d_j$. Concerning (M2) we observe the following

Theorem 2.1 (see [16]) Let $\{V_j\}_{j=0}^{\infty}$ be a nested sequence of h_j–shift–invariant subspaces $V_j := S_j(\varphi_j)$ with $\varphi_j \in L_p^2$. Then we have

$$\text{clos}\,(\bigcup_{j=0}^{\infty} V_j) = L_p^2$$

if and only if

$$\bigcup_{j=0}^{\infty} \text{supp}\,\mathbf{c}(\varphi_j) = \mathbb{Z}. \tag{2.1}$$

For (M3) the following equivalence is known:

Theorem 2.2 (see [16]) The system $\{\mathcal{B}_j(d_j^{1/2}\varphi_j) : j \in \mathbb{N}_0\}$ is L_p^2–stable with positive constants α, β if and only if for $n = 0, \ldots, d_j - 1$ and for $j \in \mathbb{N}_0$,

$$\alpha \le d_j^2\,[\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,n} \le \beta. \tag{2.2}$$

Further, a basis $\mathcal{B}_j(d_j^{1/2}\varphi_j)$ ($j \in \mathbb{N}_0$) is orthonormal if and only if

$$d_j^2\,[\mathbf{c}(\varphi), \mathbf{c}(\varphi)]_{j,n} = 1 \quad (n = 0, \ldots, d_j - 1).$$

Remark. By dim $V_j < \infty$, we can find positive constants α_j, β_j satisfying (2.2) in each level $j \in \mathbb{N}$ if (1.4) is supposed. But for L_p^2-stability we need that

$$\alpha := \inf\{\alpha_j : j \in \mathbb{N}_0\} > 0, \quad \beta := \sup\{\beta_j : j \in \mathbb{N}_0\} < \infty,$$

i.e., (2.2) sharpens (1.4).

In the following we assume that (2.2) is satisfied. From (M1), it follows $\varphi_j \in V_{j+1}$, i.e., there exist unique coefficients $\alpha_{j+1,k} \in \mathbb{C}$ ($k = 0, \ldots, d_{j+1} - 1$) such that

$$\varphi_j = \sum_{k=0}^{d_{j+1}-1} \alpha_{j+1,k}\,\varphi_{j+1,k} \quad (j \in \mathbb{N}_0).$$

This is the so–called *two–scale relation* or *refinement equation* of φ_j. Using (1.1), we obtain the Fourier transformed two–scale relation of φ_j

$$c_u(\varphi_j) = 2\, A_{j+1}(\omega_{j+1}^u)\, c_u(\varphi_{j+1}) \quad (u \in \mathbb{Z}) \tag{2.3}$$

with the *two–scale symbol* or *refinement mask* of φ_j

$$A_{j+1}(z) := \frac{1}{2} \sum_{k=0}^{d_{j+1}-1} \alpha_{j+1,k}\, z^k \quad (z \in \mathcal{T}_{j+1}),$$

where $\mathcal{T}_j := \{\omega_j^n : \; n = 0, \ldots, d_j - 1\}$ denotes the set of all d_j–th complex roots of unity.

If a scaling function $d_j^{1/2}\, \varphi_j$ $(j \in \mathbb{N}_0)$ satisfying (2.2) is given, then an orthonormal basis $\mathcal{B}_j(d_j^{1/2}\, \varphi_j^\star)$ $(j \in \mathbb{N}_0)$ can easily be obtained by the following orthogonalization trick: Let $\varphi_j^\star$ $(j \in \mathbb{N}_0)$ be defined by their Fourier coefficients

$$c_u(\varphi_j^\star) := \frac{1}{d_j\, ([\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,u_j})^{1/2}}\, c_u(\varphi_j) \quad (u \in \mathbb{Z})$$

with $u_j := u \bmod d_j$. Then from Lemma 1.4 (iii), it follows that $\mathcal{B}_j(d_j^{1/2}\, \varphi_j^\star)$ is an orthonormal basis of $V_j = S_j(\varphi_j)$ and the relation $d_j^2\, [\mathbf{c}(\varphi_j^\star), \mathbf{c}(\varphi_j^\star)]_{j,n} = 1$ $(n = 0, \ldots, d_j - 1)$ is obvious. Furthermore, the two–scale symbol $A_{j+1}^\star$ satisfying

$$c_u(\varphi_j^\star) = 2\, A_{j+1}^\star(\omega_{j+1}^u)\, c_u(\varphi_{j+1}^\star) \quad (u \in \mathbb{Z})$$

is connected with A_{j+1} for $n = 0, \ldots, d_{j+1} - 1$ by

$$A_{j+1}^\star(\omega_{j+1}^n) := 2 \left(\frac{[\mathbf{c}(\varphi_{j+1}), \mathbf{c}(\varphi_{j+1})]_{j+1,n}}{[\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,n}} \right)^{1/2} A_{j+1}(\omega_{j+1}^n).$$

A different approach to the bracket product $[\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_j$ can be described by the so–called *p–periodic autocorrelation symbol* of φ_j defined by

$$\Phi_j(z) := \sum_{l=0}^{d_j-1} \langle \varphi_{j,-l}, \varphi_j \rangle\, z^l \quad (z \in \mathcal{T}_j).$$

We observe a close connection between the bracket product $[\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_j$, the two–scale symbol A_{j+1} and the p–periodic autocorrelation symbols Φ_j and Φ_{j+1}.

Lemma 2.3 *For $j \in \mathbb{N}_0$ and $k = 0, \ldots, d_j - 1$, we have*

$$\Phi_j(\omega_j^k) = d_j\, [\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,k}, \tag{2.4}$$

$$\Phi_j(z^2) = 2\, |A_{j+1}(z)|^2\, \Phi_{j+1}(z) + 2\, |A_{j+1}(-z)|^2\, \Phi_{j+1}(-z) \quad (z \in \mathcal{T}_{j+1}). \tag{2.5}$$

The condition (M3) *is equivalent to*

$$0 < \alpha \le d_j \, \Phi_j(\omega_j^k) \le \beta < \infty \quad (j \in \mathbb{N}_0, \; k = 0, \ldots, d_j - 1). \tag{2.6}$$

Proof: Let $j \in \mathbb{N}_0$. By (2.3), we obtain for $k = 0, \ldots, d_j - 1$,

$$\begin{aligned} [\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,k} &= \sum_{u=-\infty}^{\infty} |c_{k+ud_{j+1}}(\varphi_j)|^2 + \sum_{u=-\infty}^{\infty} |c_{k+d_j+ud_{j+1}}(\varphi_j)|^2 \\ &= 4\,|A_{j+1}(\omega_{j+1}^k)|^2 \,[\mathbf{c}(\varphi_{j+1}), \mathbf{c}(\varphi_{j+1})]_{j+1,k} \\ &\quad + 4\,|A_{j+1}(-\omega_{j+1}^k)|^2 [\mathbf{c}(\varphi_{j+1}), \mathbf{c}(\varphi_{j+1})]_{j+1,k+d_j}. \end{aligned}$$

By (1.2), we have for $l = 0, \ldots d_j - 1$,

$$\langle \varphi_{j,-l}, \varphi_j \rangle = \sum_{n=0}^{d_j-1} \omega_j^{-nl} \, [\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,n},$$

and hence,

$$\Phi_j(z) = \sum_{l,n=0}^{d_j-1} \omega_j^{-nl} \, [\mathbf{c}(\varphi_j), \mathbf{c}(\varphi_j)]_{j,n} \, z^l \quad (z \in \mathcal{T}_j).$$

For $z = \omega_j^k$ $(k = 0, \ldots, d_j - 1)$, this yields (2.4) by

$$\sum_{l=0}^{d_j-1} \omega_j^{(k-n)l} = d_j \, \delta_{k,n}.$$

From (2.4), it follows immediately (2.5). ■

§3. Periodic Wavelet Spaces

Now we define the *p–periodic wavelet space* W_j *of level* j $(j \in \mathbb{N}_0)$ as the orthogonal complement of V_j in V_{j+1}:

$$W_j := V_{j+1} \ominus V_j \quad (j \in \mathbb{N}_0).$$

Then it follows $\dim W_j = d_{j+1} - d_j = d_j$ and the orthogonal sum representation

$$V_{j+1} = V_j \oplus W_j \quad (j \in \mathbb{N}_0). \tag{3.1}$$

By definition, the wavelet spaces W_j $(j \in \mathbb{N}_0)$ are mutually orthogonal. Note that $f \in V_{j+1}$ implies $f(\cdot - 2h_{j+1}) = f(\cdot - h_j) \in V_{j+1}$. It can be easily observed that W_j is h_j–shift–invariant.

By (M1) – (M2), we obtain the orthogonal sum decomposition

$$L_p^2 = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j.$$

Assume that for each $j \in \mathbb{N}_0$, the h_j–shift–invariant subspace W_j is generated by a function $\psi_j \in W_j$, i.e. $W_j = S_j(\psi_j)$. Further, we suppose that there exist positive constants γ, δ such that for all $j \in \mathbb{N}_0$ and for any $(b_{j,n})_{n=0}^{d_j-1} \in \mathbb{C}^{d_j}$

$$\gamma \sum_{n=0}^{d_j-1} |b_{j,n}|^2 \le \| \sum_{n=0}^{d_j-1} b_{j,n}\, d_j^{1/2}\, \psi_{j,n}\|^2 \le \delta \sum_{n=0}^{d_j-1} |b_{j,n}|^2. \tag{3.2}$$

In other words, $\{\mathcal{B}_j(d_j^{1/2}\, \psi_j) :\ j \in \mathbb{N}_0\}$ is L_p^2–stable. Under these assumptions, $d_j^{1/2}\, \psi_j$ is called *p–periodic semiorthogonal wavelet of level j* or *p–periodic pre-wavelet of level j*. If all $\mathcal{B}_j(d_j^{1/2}\, \psi_j)$ $(j \in \mathbb{N}_0)$ are orthonormal bases, then $d_j^{1/2}\psi_j$ $(j \in \mathbb{N}_0)$ is called *p–periodic orthonormal wavelet of level j*.

For orthonormal wavelets, the property (3.2) is automatically satisfied, since for $(b_{j,n})_{n=0}^{d_j-1} \in \mathbb{C}^{d_j}$,

$$\| \sum_{n=0}^{d_j-1} b_{j,n}\, d_j^{1/2}\, \psi_{j,n}\|^2 = \sum_{n=0}^{d_j-1} |b_{j,n}|^2,$$

i.e. $\gamma = \delta = 1$. From (M1) and (3.1), it follows $\psi_j \in V_{j+1}$. Then there exist unique coefficients $\beta_{j+1,k} \in \mathbb{C}$ $(k = 0, \ldots, d_{j+1} - 1)$ such that

$$\psi_j = \sum_{k=0}^{d_{j+1}-1} \beta_{j+1,k}\, \varphi_{j+1,k}.$$

This is the so–called *two–scale relation* or *refinement equation* of ψ_j. By (1.1), we obtain the Fourier transformed two–scale relation of ψ_j

$$c_u(\psi_j) = 2B_{j+1}(\omega_{j+1}^u)\, c_u(\varphi_{j+1}) \quad (u \in \mathbb{Z}) \tag{3.3}$$

with the *two–scale symbol* or *refinement mask* of ψ_j

$$B_{j+1}(z) := \frac{1}{2} \sum_{k=0}^{d_{j+1}-1} \beta_{j+1,k}\, z^k \quad (z \in \mathcal{T}_{j+1}).$$

Further, we introduce the *p–periodic autocorrelation symbol* of ψ_j by

$$\Psi_j(z) := \sum_{l=0}^{d_j-1} \langle \psi_{j,-l}, \psi_j \rangle\, z^l \quad (z \in \mathcal{T}_j).$$

Then we observe the following connection between the bracket product $[\mathbf{c}(\psi_j), \mathbf{c}(\psi_j)]_j$, the two–scale symbol B_{j+1} and the p–periodic autocorrelation symbols Ψ_j and Ψ_{j+1}.

Lemma 3.1 *For $j \in \mathbb{N}_0$ and $k = 0, \ldots, d_j - 1$, we have*

$$\Psi_j(\omega_j^k) = d_j \, [\mathbf{c}(\psi_j), \mathbf{c}(\psi_j)]_{j,k},$$

$$\Psi_j(z^2) = 2\,|B_{j+1}(z)|^2 \, \Phi_{j+1}(z) + 2\,|B_{j+1}(-z)|^2 \, \Phi_{j+1}(-z) \quad (z \in \mathcal{T}_{j+1}). \tag{3.4}$$

The condition (3.2) *is equivalent to*

$$0 < \gamma \le d_j \, \Psi_j(\omega_j^k) \le \delta < \infty \quad (j \in \mathbb{N}_0, \; k = 0, \ldots, d_j - 1). \tag{3.5}$$

The proof is similar to that of Lemma 2.3 and is omitted here.

Now the following question is of interest: How can we choose the two–scale symbol B_{j+1} such that $S_j(\psi_j) \perp V_j$ and (3.2) or (3.5) are satisfied? One condition for B_{j+1} follows from the orthogonality of $S_j(\psi_j)$ and V_j.

Lemma 3.2 *For $j \in \mathbb{N}_0$, we have $S_j(\psi_j) \perp V_j$ if and only if for $z \in \mathcal{T}_{j+1}$*

$$A_{j+1}(z)\, \overline{B_{j+1}(z)}\, \Phi_{j+1}(z) + A_{j+1}(-z)\, \overline{B_{j+1}(-z)}\, \Phi_{j+1}(-z) = 0. \tag{3.6}$$

The proof is based on Corollary 1.3 (ii) and the two–scale relations (2.3) and (3.3).

Now let us introduce the *two–scale symbol matrices of the j-th level* ($j \in \mathbb{N}_0$)

$$\mathbf{S}_{j+1}(z) := \begin{pmatrix} A_{j+1}(z) & B_{j+1}(z) \\ A_{j+1}(-z) & B_{j+1}(-z) \end{pmatrix} \quad (z \in \mathcal{T}_{j+1}). \tag{3.7}$$

In the next section, these matrices and their inverses will play the main role for the decomposition and reconstruction algorithms. We investigate the invertibility of $\mathbf{S}_{j+1}(z)$:

Lemma 3.3 *Assume that* (2.6) *and* (3.5) *are true. For $j \in \mathbb{N}_0$, the two–scale symbol matrices $\mathbf{S}_{j+1}(z)$ $(z \in \mathcal{T}_{j+1})$ are regular with*

$$\frac{2\sqrt{\alpha\,\gamma}}{\beta} \le |\det \mathbf{S}_{j+1}(z)| \le \frac{2\sqrt{\beta\,\delta}}{\alpha}. \tag{3.8}$$

Further, we have

$$\mathbf{S}_{j+1}(z)^{-1} = \operatorname{diag}\,(\Phi_j(z^2)^{-1},\, \Psi_j(z^2)^{-1})^{\mathrm{T}}\, \overline{\mathbf{S}_{j+1}(z)}^{\mathrm{T}} \operatorname{diag}\,(\Phi_{j+1}(z),\, \Phi_{j+1}(-z))^{\mathrm{T}}. \tag{3.9}$$

Proof: Using (2.5), (3.4) and (3.6), we obtain for $z \in \mathcal{T}_{j+1}$ that

$$\overline{\mathbf{S}_{j+1}(z)}^{\mathrm{T}} \operatorname{diag}\,(\Phi_{j+1}(z),\, \Phi_{j+1}(-z))^{\mathrm{T}}\, \mathbf{S}_{j+1}(z) = \operatorname{diag}\,(\Phi_j(z^2),\, \Psi_j(z^2))^{\mathrm{T}}. \tag{3.10}$$

By known properties of the determinant, this equation yields

$$\Phi_{j+1}(z)\,\Phi_{j+1}(-z)\,|\det \mathbf{S}_{j+1}(z)|^2 = \Phi_j(z^2)\,\Psi_j(z^2) \quad (z \in \mathcal{T}_{j+1}).$$

Then by (2.6) and (3.5), all matrices $\mathbf{S}_{j+1}(z)$ $(z \in \mathcal{T}_{j+1})$ are regular with (3.8). From (3.10), it follows directly (3.9). ■

Remark. The assertion of Lemma 3.3 emphasizes the importance of the L_p^2–stability of $\{\mathcal{B}_j(d_j^{1/2}\varphi_j) : j \in \mathbb{N}_0\}$ and $\{\mathcal{B}_j(d_j^{1/2}\psi_j) : j \in \mathbb{N}_0\}$. If the systems $\{\mathcal{B}_j(d_j^{1/2}\varphi_j) : j \in \mathbb{N}_0\}$ or $\{\mathcal{B}_j(d_j^{1/2}\psi_j) : j \in \mathbb{N}_0\}$ are not L_p^2–stable, then the two–scale symbol matrices are not well–conditioned such that the existence of numerically stable algorithms for decomposition and reconstruction is not ensured (see Section 4).

Now, with the help of the conditions (3.4) – (3.6) the two–scale symbol B_{j+1} can be described more exactly.

Theorem 3.4 (see [16]) Assume that (2.6) holds. For every $j \in \mathbb{N}_0$, $B_{j+1} : \mathcal{T}_{j+1} \to \mathbb{C}$ is a two–scale symbol of a p–periodic semiorthogonal wavelet $d_j^{1/2}\,\psi_j \in L_p^2$ satisfying the property (3.5) if and only if

$$B_{j+1}(z) = \frac{\Phi_{j+1}(-z)\,\overline{A_{j+1}(-z)}}{z\,\Phi_j(z^2)}\,K_j(z^2) \quad (z \in \mathcal{T}_{j+1}),$$

where $K_j : \mathcal{T}_j \to \mathbb{C}$ satisfies the condition

$$0 < \mu \le |K_j(z)| \le \nu < \infty \quad (z \in \mathcal{T}_j)$$

with positive constants μ, ν.

In the case of orthonormal wavelets the corresponding two–scale symbol $B^\star_{j+1}$ even satisfies the following

Corollary 3.5 *Assume that $\mathcal{B}_j(d_j^{1/2}\,\varphi_j^\star)$ are orthonormal bases of V_j and $A^\star_{j+1}$ $(j \in \mathbb{N}_0)$ the corresponding two–scale symbols of $\varphi_j^\star$. Then for every $j \in \mathbb{N}_0$, $B^\star_{j+1} : \mathcal{T}_{j+1} \to \mathbb{C}$ is a two–scale symbol of a p–periodic orthonormal wavelet $d_j^{1/2}\,\psi_j^\star \in L_p^2$ if and only if $B^\star_{j+1}$ has the form*

$$B^\star_{j+1}(z) = \pm\, z^{2n-1}\overline{A^\star_{j+1}(-z)} \quad (z \in \mathcal{T}_{j+1})$$

with $n \in \{0, \ldots, d_j - 1\}$. For the related two–scale matrices

$$\mathbf{S}^\star_{j+1}(z) := \begin{pmatrix} A^\star_{j+1}(z) & B^\star_{j+1}(z) \\ A^\star_{j+1}(-z) & B^\star_{j+1}(-z) \end{pmatrix} \quad (z \in \mathcal{T}_{j+1}),$$

we have

$$|\det \mathbf{S}^\star_{j+1}(z)| = |A^\star_{j+1}(z)|^2 + |A^\star_{j+1}(-z)|^2 = 1 \quad (z \in \mathcal{T}_{j+1}).$$

The proof follows directly from Theorem 3.4, taking in consideration the relations

$$d_j\, \Phi_j^\star(\omega_j^k) = d_j^2\, [\mathbf{c}(\varphi_j^\star), \mathbf{c}(\varphi_j^\star)]_{j,k} = 1,$$
$$d_j\, \Psi_j^\star(\omega_j^k) = d_j^2\, [\mathbf{c}(\psi_j^\star), \mathbf{c}(\psi_j^\star)]_{j,k} = 1 \quad (k = 0, \ldots, d_j - 1)$$

instead of (2.6) and (3.5).

§4. Decomposition and Reconstruction Algorithms

In this section we shall derive efficient decomposition and reconstruction algorithms based on Fourier technique. In order to decompose a given function $f_{j+1} \in V_{j+1}$ $(j \in \mathbb{N}_0)$ of the form

$$f_{j+1} = \sum_{l=0}^{d_{j+1}-1} a_{j+1,l}\, \varphi_{j+1,l} \quad (a_{j+1,l} \in \mathbb{C}), \tag{4.1}$$

uniquely determined functions $f_j \in V_j$ and $g_j \in W_j$ have to be found such that

$$f_{j+1} = f_j + g_j. \tag{4.2}$$

Assume that the coefficients $a_{j+1,l} \in \mathbb{C}$ $(l = 0, \ldots, d_{j+1} - 1)$ of f_{j+1} or their DFT(d_{j+1}) data

$$\hat{a}_{j+1,k} := \sum_{l=0}^{d_{j+1}-1} a_{j+1,l}\, \omega_{j+1}^{kl} \quad (k = 0, \ldots, d_{j+1} - 1) \tag{4.3}$$

are known. The wanted functions $f_j \in V_j$ and $g_j \in W_j$ can uniquely be represented by

$$f_j = \sum_{n=0}^{d_j-1} a_{j,n}\, \varphi_{j,n}, \quad g_j = \sum_{n=0}^{d_j-1} b_{j,n}\, \psi_{j,n} \tag{4.4}$$

with unknown coefficients $a_{j,n}, b_{j,n} \in \mathbb{C}$.
In order to reconstruct $f_{j+1} \in V_{j+1}$ $(j \in \mathbb{N}_0)$, we have to compute the sum (4.2) with given $f_j \in V_j$ and $g_j \in W_j$. Assume that $a_{j,n}, b_{j,n} \in \mathbb{C}$ $(n = 0, \ldots, d_j - 1)$ in (4.4) or their DFT(d_j) data

$$\hat{a}_{j,k} := \sum_{n=0}^{d_j-1} a_{j,n}\, \omega_j^{kn}, \quad \hat{b}_{j,k} := \sum_{n=0}^{d_j-1} b_{j,n}\, \omega_j^{kn} \quad (k = 0, \ldots, d_j - 1) \tag{4.5}$$

are known. The function $f_{j+1} \in V_{j+1}$ can be uniquely represented in the form (4.1) with unknown coefficients $a_{j+1,l}$ $(l = 0, \ldots, d_{j+1} - 1)$. The decomposition and reconstruction algorithms are based on the following

Theorem 4.1 *Assume that*

$$\operatorname{supp} \mathbf{c}(\varphi_{j+1}) \supseteq \{-d_j, \ldots, d_j - 1\} \quad (j \in \mathbb{N}_0). \tag{4.6}$$

For $j \in \mathbb{N}_0$, let $f_{j+1} \in V_{j+1}$, $f_j \in V_j$ and $g_j \in W_j$ with (4.1) – (4.5) be given. Then we have for $n = 0, \ldots, d_{j+1} - 1$,

$$\hat{a}_{j+1,n} = 2\, \hat{a}_{j,n}\, A_{j+1}(\omega_{j+1}^n) + 2\, \hat{b}_{j,n}\, B_{j+1}(\omega_{j+1}^n), \tag{4.7}$$

i.e., for $k = 0, \ldots, d_j - 1$,

$$\begin{pmatrix} \hat{a}_{j+1,k} \\ \hat{a}_{j+1,k+d_j} \end{pmatrix} = 2\, \boldsymbol{S}_{j+1}(\omega_{j+1}^k) \begin{pmatrix} \hat{a}_{j,k} \\ \hat{b}_{j,k} \end{pmatrix}. \tag{4.8}$$

Proof: From

$$c_u(f_{j+1}) = c_u(f_j) + c_u(g_j) \quad (u \in \mathbb{Z}),$$

it follows by (4.1) and (4.4)

$$\sum_{l=0}^{d_{j+1}-1} a_{j+1,l}\, c_u(\varphi_{j+1,l}) = \sum_{n=0}^{d_j-1} (a_{j,n}\, c_u(\varphi_{j,n}) + b_{j,n}\, c_u(\psi_{j,n})) \quad (u \in \mathbb{Z}),$$

and hence by (1.1), (4.3) and (4.5)

$$\hat{a}_{j+1,u}\, c_u(\varphi_{j+1}) = \hat{a}_{j,u}\, c_u(\varphi_j) + \hat{b}_{j,u}\, c_u(\psi_j) \quad (u \in \mathbb{Z}).$$

Using the Fourier transformed two–scale relations (2.3) and (3.3), we obtain

$$\hat{a}_{j+1,u}\, c_u(\varphi_{j+1}) = (2\, \hat{a}_{j,u}\, A_{j+1}(\omega_{j+1}^u) + 2\, \hat{b}_{j,u}\, B_{j+1}(\omega_{j+1}^u))\, c_u(\varphi_{j+1}) \quad (u \in \mathbb{Z}).$$

Since the coefficients of $c_u(\varphi_{j+1})$ are d_{j+1}-periodic, we conclude from the assumption (4.6) that (4.7) holds. Thus, we have for $k = 0, \ldots, d_j - 1$

$$\begin{aligned} \hat{a}_{j+1,k} &= 2\, \hat{a}_{j,k}\, A_{j+1}(\omega_{j+1}^k) + 2\, \hat{b}_{j,k}\, B_{j+1}(\omega_{j+1}^k), \\ \hat{a}_{j+1,k+d_j} &= 2\, \hat{a}_{j,k}\, A_{j+1}(-\omega_{j+1}^k) + 2\, \hat{b}_{j,k}\, B_{j+1}(-\omega_{j+1}^k), \end{aligned}$$

that means (4.8). ■

Remark. From (4.6) it follows directly (2.1). In all theories on periodic wavelets known up to now (see [5,14,15,17,18]), the condition (4.6) is satisfied.

From Theorem 4.1, we obtain immediately:

Algorithm 4.2 *(Decomposition Algorithm)*
Input:
$j \in \mathbb{N}_0$, $d \in \mathbb{N}$ *(power of* 2*)*, $d_j := 2^j\, d$,
$\hat{a}_{j+1,k} \in \mathbb{C} \quad (k = 0, \ldots, d_{j+1} - 1)$.

1. Precompute $\boldsymbol{S}_{j+1}(\omega_{j+1}^k)^{-1}$ $(k = 0, \ldots, d_j - 1)$ *(given by* (3.9)*) by* FFT.
2. Compute for $k = 0, \ldots, d_j - 1$

$$\begin{pmatrix} \hat{a}_{j,k} \\ \hat{b}_{j,k} \end{pmatrix} := \frac{1}{2}\, \boldsymbol{S}_{j+1}(\omega_{j+1}^k)^{-1} \begin{pmatrix} \hat{a}_{j+1,k} \\ \hat{a}_{j+1,k+d_j} \end{pmatrix}.$$

Output: $\hat{a}_{j,k}$, $\hat{b}_{j,k} \quad (k = 0, \ldots, d_j - 1)$.

Algorithm 4.3 *(Reconstruction Algorithm)*
Input:
$j \in \mathbb{N}_0$, $d \in \mathbb{N}$ *(power of* 2*)*, $d_j := 2^j\, d$,
$\hat{a}_{j,k}$, $\hat{b}_{j,k} \in \mathbb{C} \quad (k = 0, \ldots, d_j - 1)$.

1. Precompute $\boldsymbol{S}_{j+1}(\omega_{j+1}^k)$ $(k = 0, \ldots, d_j - 1)$ *(given by* (3.7)*) by* FFT.
2. Compute (4.8) *for* $k = 0, \ldots, d_j - 1$.
Output: $\hat{a}_{j+1,k} \quad (k = 0, \ldots, d_{j+1} - 1)$.

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Gerlind Plonka
Fachbereich Mathematik, Universität Rostock,
D – 18051 Rostock, Germany
nfa016@nfa.uni-rostock.d400.de

Manfred Tasche
Fachbereich Mathematik, Universität Rostock,
D – 18051 Rostock, Germany
tasche@mathematik.uni-rostock.d400.de