Sparse Encoding Techniques in X-Ray Imaging

Gerlind Plonka

Universität Göttingen

joint work with:
Robert Beinert, Jakob Geppert, Lina Liu, Stefan Loock, Anne Pein, Tim Salditt, Katrin Wannenwetsch

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- The Phase Retrieval Problem
- The Fresnel Transform
- Sparsity in Wavelet and Shearlet Frames
- Projection Algorithms
- Frame Shrinkage Operators are Proximity Operators
- Numerical Results
- Further Research Results in Project C11

The phase retrieval problem

Far field:

"Given $|\mathcal{F}f|$, recover f."

Near field:

"Given $|\mathcal{R}_\tau f|$, recover f."

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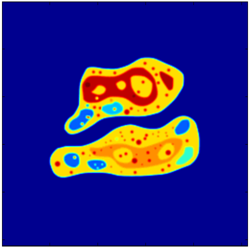
where:

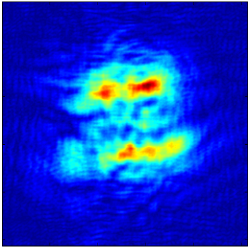
$$\mathcal{F}f(\boldsymbol{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\mathbf{x}\boldsymbol{\xi}}\, f(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

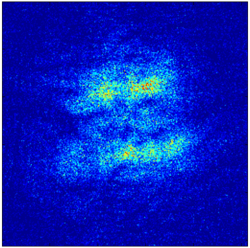
$$\mathcal{R}_\tau f(\boldsymbol{\xi}) = \frac{1}{\tau^2}\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\pi\frac{\|\mathbf{x}-\boldsymbol{\xi}\|^2}{\tau^2}}\, f(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \tau = \sqrt{\lambda d}$$

The Fresnel transform

$$\mathcal{R}_\tau f(\boldsymbol{\xi}) = \frac{1}{\tau^2} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\pi \frac{\|\mathbf{x}-\boldsymbol{\xi}\|^2}{\tau^2}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}$$


Original data


Fresnel transform


Poisson distributed data with

10^5 counted photons

Cell image by Giewekemeyer et al.

The phase retrieval problem in the Fresnel regime

"Given $|\mathcal{R}_\tau f|$, recover f."

a priori information:

- **Usually**:
  object f real, with compact support, positive (non-negative)
- We use **structural information**:
  f is sparse in a suitable function frame enforcing f to be in a certain function space ("smooth").

Shearlet frame: generalization of wavelet bases

We use a representation

$$f(\mathbf{x}) = \sum_{j,k,\mathbf{n}} c_{j,k,\mathbf{n}} \, \psi_{j,k,\mathbf{n}}(\mathbf{x})$$

where

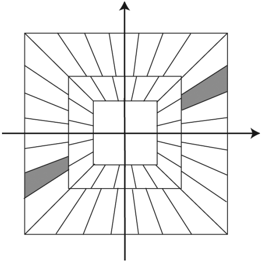
$$\psi_{j,k,\mathbf{n}}(x) = \left(2^{j\lfloor j/2 \rfloor}\right)^{1/2} \psi_0(B_0^k A_0^j \mathbf{x} - \mathbf{n})$$

with

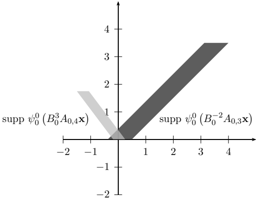
$$A_0^j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{\lfloor j/2 \rfloor} \end{pmatrix}, \qquad B_0^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Here ψ_0 is the mother shearlet. See [Lim2010, Kittipoom2010].

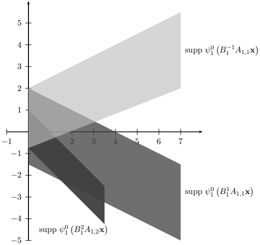
Fourier space tiling & support



Fourier space tiling



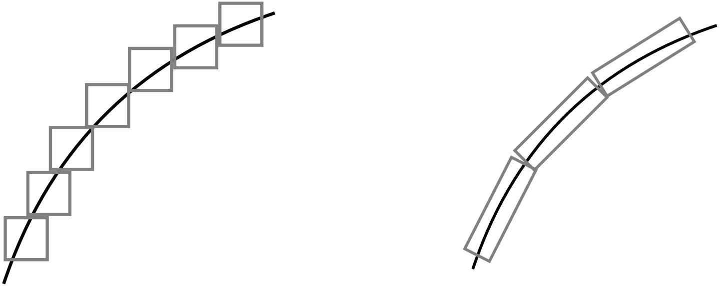
shearlets with compact support



shearlets with compact support

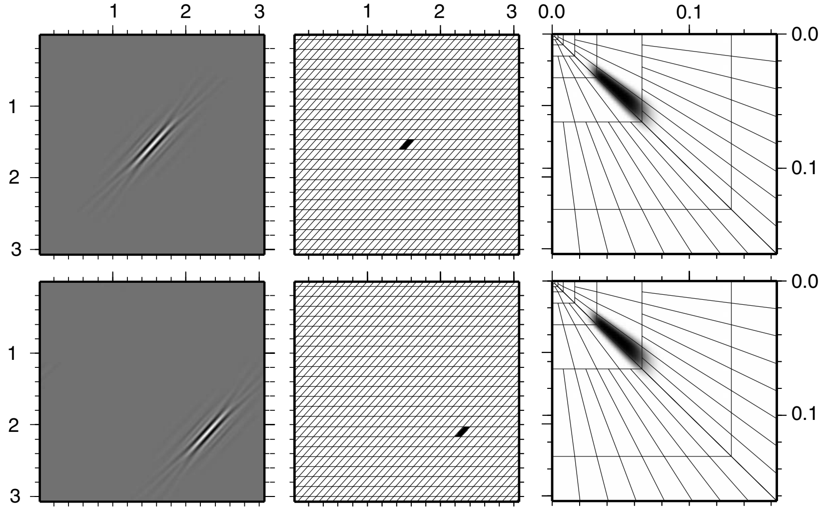
- sparse representation of smooth regions
- sparse representation of singularities along smooth curves
- fast and numerically stable transforms available [Lim2010]

Sparsity of functions in shearlet frames

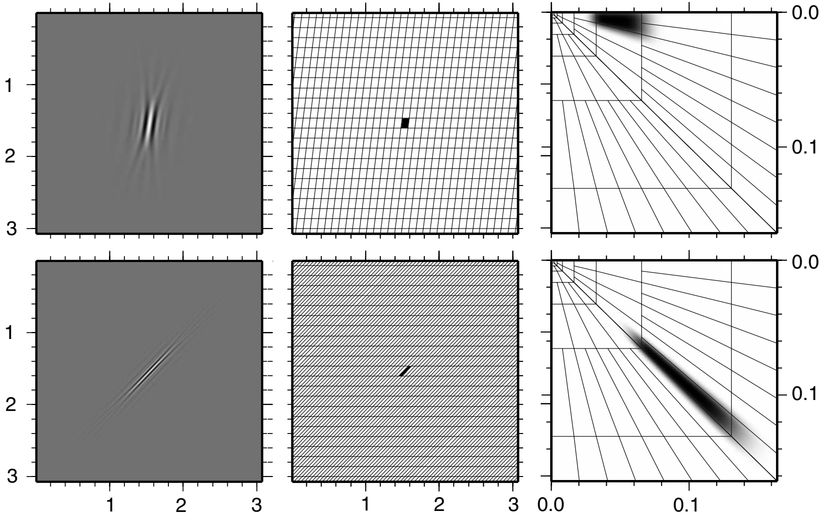


- unlike wavelets, shearlets provide directional information
- curve-like singularities are sparsely represented by shearlets
- shearlets provide sparse representations for images

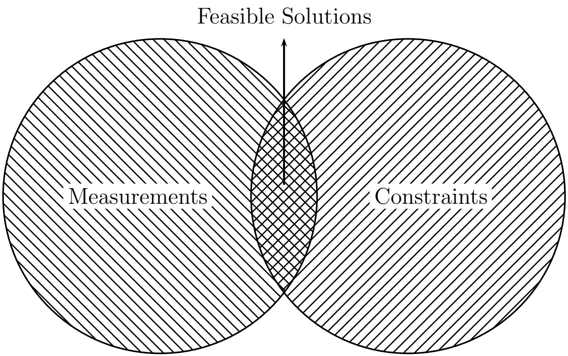
Examples of frame functions in the shearlet frame



Examples of frame functions in shearlet frame



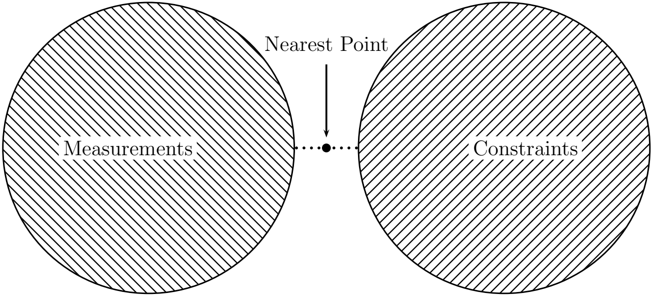
Feasibility problem: "find $g \in M \cap C$"



- Easy, if M, C convex and $M \cap C \neq \emptyset$.
- Much harder, if M or C non-convex.
- Be careful, if $M \cap C = \emptyset$ (e.g. due to noise).

Feasibility problem

Instead: "find g with minimal distance to M and C"



- infeasible case: nearest point $\rightsquigarrow$ local best approx. point

Feasibility problem

Always: valid measurements

$$M = \{f : |\mathcal{R}_\tau f(\boldsymbol{\xi})| = m(\boldsymbol{\xi})\}$$

Usually: compact support constraint

$$C = \{f : f(\mathbf{x}) = 0 \text{ outside } B_r\}$$

Here: sparse representation in suitable frame: $f = \sum_j c_j(f)\psi_j$

$$S = \{f : \|\mathbf{c}(f)\|_0 \leq N\}$$

Projection algorithm with sparsity constraint

Measurements: projection onto magnitude

$$\mathcal{P}_M g(\mathbf{x}) = \left(\mathcal{R}_\tau^{-1} h\right)(\mathbf{x}), \quad h(\xi) = \begin{cases} m(\xi)\frac{\mathcal{R}_\tau g(\xi)}{\left|\mathcal{R}_\tau g(\xi)\right|}, & \text{if } \left|\mathcal{R}_\tau g(\xi)\right| \neq 0 \\ m(\xi), & \text{if } \left|\mathcal{R}_\tau g(\xi)\right| = 0 \end{cases}$$

Sparsity: soft threshold in the shearlet frame

$$\mathcal{P}_{S,\gamma} g(\mathbf{x}) = \left(\mathcal{S}^+ \mathcal{T}_\gamma \mathcal{S} g\right)(\mathbf{x}), \quad \mathcal{T}_\gamma(c) = \begin{cases} c - \gamma, & c > \gamma \\ c + \gamma, & c < -\gamma \\ 0, & \text{otherwise} \end{cases}$$

RAAR algorithm

Relaxed Averaged Alternating Reflections [Luke2005, Luke2008]

$$g_{n+1} = \left[\frac{\beta}{2}\left(R_S R_M + I\right) + (1-\beta)\,\mathcal{P}_M\right] g_n$$

where:

$$R_M := 2\mathcal{P}_M - I, \quad R_S := 2\mathcal{P}_{S,\gamma} - I$$

- convex combination of $\mathcal{P}_M$ and Douglas-Rachford
- parameter $\beta > 0$ relaxes/regularizes the solution
- relaxation parameter can avoid local minima
- finds local best approximation points

Soft frame shrinkage operators are proximity operators!

Theorem (Geppert, Plonka (2019))

Let $\mathcal{S} \in \mathbb{R}^{M \times N}$ with $M \geq N$ be a matrix with full rank N. Then the operator

$$P_{S,\gamma} := \mathcal{S}^{+} \mathcal{T}_{\gamma} \mathcal{S}$$

with

$$\mathcal{T}_{\gamma}(c) = \begin{cases} c - \gamma, & c > \gamma \\ c + \gamma, & c < -\gamma \\ 0, & \textit{otherwise} \end{cases}$$

is the proximity operator of a convex, proper, lower semi-continuous function ψ. This function is usually not equal to $\|\mathcal{S} \cdot \|_1$.

Soft frame shrinkage operators are proximity operators!

Idea of proof

Consider the set-valued mapping

$$\mathbf{y} \in H(\mathbf{x}) \;:\Longleftrightarrow\quad \mathbf{y} = \mathcal{S}^{+}\mathcal{T}_{\gamma}\,\mathcal{S}(\mathbf{x}+\mathbf{y})$$

and show that $H = \partial\psi$ for some $\psi \in \Gamma_0(\mathbb{R})$.

Employing the Theorem of Rockafellar, we need to show that $H(\mathbf{x})$ is maximally cyclically monotone.
We can show:

1. $\mathbf{y} \in H(\mathbf{0})$ iff $\|\mathcal{S}\mathbf{y}\|_\infty \le \gamma$
2. For $\min_k |(\mathcal{S}\mathbf{x})_k| > \gamma$ the mapping $H(\mathbf{x})$ is single-valued.

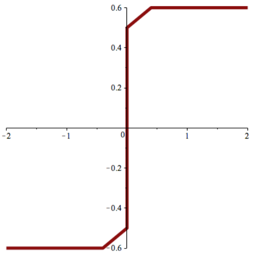
Soft frame shrinkage operators

$$\mathbf{y} \in H(\mathbf{x}) \;:\Longleftrightarrow\quad \mathbf{y} = \mathcal{S}^{+}\mathcal{T}_{\gamma}\,\mathcal{S}(\mathbf{x}+\mathbf{y})$$

Example

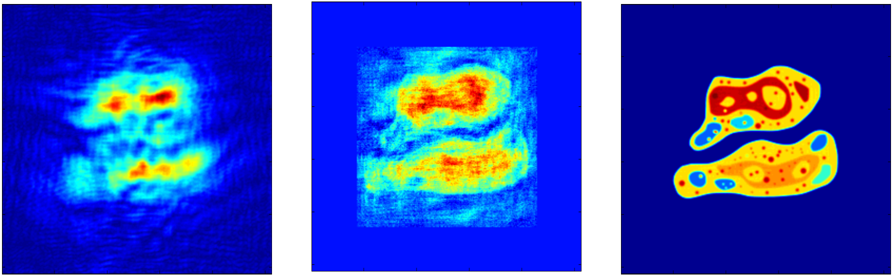
Let $\mathcal{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\gamma > 0$. Then

$$H(x) = \begin{cases} [-\gamma/2, \gamma/2] & x = 0 \\ \gamma/2 + x/4 & x \in (0, \frac{2}{5}\gamma] \\ 3\gamma/5 & x > \frac{2}{5}\gamma. \end{cases}$$



Support vs shearlet constraint

Simulated measurements: $\lambda = 1\text{Å}$, $d = 100\text{mm}$, $dx = 100\text{nm}$

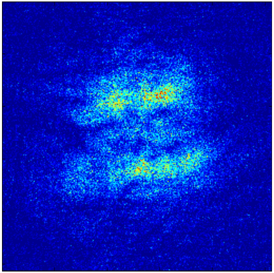


measurements support constraint shearlet constraint

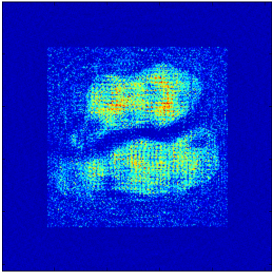
S. Loock, G. Plonka: Phase retrieval for Fresnel measurements using a shearlet sparsity constraint. *Inverse Problems* **30**(5) (2014), 055005.

Support vs shearlet constraint

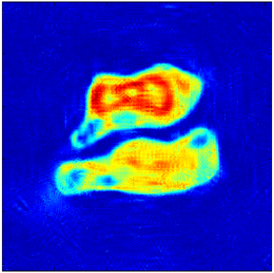
Poisson distributed data with $t = 10^5$



Poisson distributed data with 10^5 photons



Support and positivity constraint

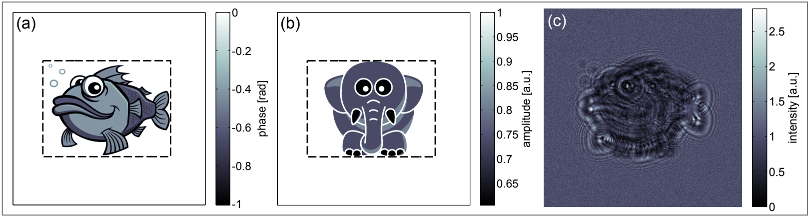


Shearlet and positivity constraint

S. Loock, G. Plonka: Phase retrieval for Fresnel measurements using a shearlet sparsity constraint. *Inverse Problems* **30**(5) (2014), 055005.

Separate shearlet constraints for amplitude and phase

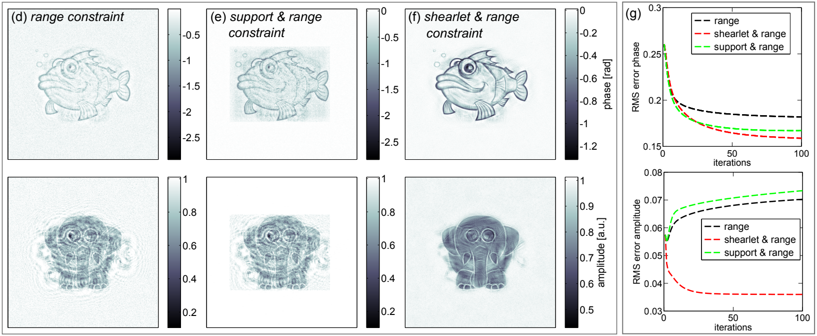
$$\mathcal{P}_S^{\gamma_a,\gamma_p} g = \mathcal{S}^+ \mathcal{T}_{\gamma_a} \mathcal{S}|g| \cdot \exp(\mathrm{i} \cdot \mathcal{S}^+ T_{\gamma_p} \mathcal{S}\phi(g))$$



(a), (b) Phase and amplitude of the complex valued exit wave field in the object plane.

(c) Simulated intensity measurement with Fresnel number $4 \cdot 10^{-3}$ and artificial Poisson noise (50 photons per pixel)

Separate shearlet constraints for amplitude and phase



Exit waves constructed by RAAR with 100 iterations.
Upper row: phase, lower row: amplitude

A. Pein, S. Loock, G. Plonka, T. Salditt: Using sparsity information for iterative phase retrieval in x-ray propagation imaging *Opt. Express* **24**(8) (2016), 8332–8343.

Other results in this project

- with Robert Beinert: Full characterization of ambiguities of the one-dimensional discrete phase retrieval problem
- with Katrin Wannenwetsch: Sublinear algorithms for the discrete sparse FFT
- with Lina Liu: Dictionary learning algorithms for image denoising

S. Loock, G. Plonka: Phase retrieval for Fresnel measurements using a shearlet sparsity constraint. *Inverse Problems* **30**(5) (2014), 055005.

R. Beinert, G. Plonka: Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes. *J. Fourier Anal. Appl.* **21**(6) (2015), 1169–1198.

G. Plonka, K. Wannenwetsch: A deterministic sparse FFT algorithm for vectors with small support. *Numer. Alg.* **71**(4) (2016), 889–905.

A. Pein, S. Loock, G. Plonka, T. Salditt: Using sparsity information for iterative phase retrieval in x-ray propagation imaging *Opt. Express* **24**(8) (2016), 8332–8343.

R. Beinert: One-dimensional phase retrieval with additional interference measurements *Results Math.* **72**(1-2) (2017), 1–24.

R. Beinert: Non-negativity constraints in the one-dimensional discrete-time phase retrieval problem *Inf. Inference* **6** (2017), 213–224.

G. Plonka, K. Wannenwetsch: A sparse Fast Fourier algorithm for real nonnegative vectors. *J. Comput. Appl. Math.* **321** (2017), 532–539.

R. Beinert, G. Plonka: Sparse phase retrieval of one-dimensional signals by Prony's method. *Frontiers Appl. Math. Statist.* **3**(5) (2017), open access, doi: 10.3389/fams.2017.00005.

L. Liu, G. Plonka, J. Ma: Seismic data interpolation and denoising by learning a tensor tight frame. *Inverse Problems* **33**(10) (2017), 105011.

R. Beinert, G. Plonka: Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain. *Appl. Comput. Harmon. Anal.* **45** (2018), 505–525.