The Factorization Method for Inverse Problems

Andreas Kirsch and Natalia Grinberg

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Ever since Christian Hülsmeyer showed in 1904 that one could use radio waves to detect metallic objects at a distance (the range of the first apparatus was 3000 meters), the race has been on to tease out ever more information from scattered waves. Within months of his first detection demonstration, Hülsmeyer devised a way to determine the distance to the object. At that rate of improvement one might have extrapolated to unimaginable twenty-first-century capabilities. Unfortunately, what has proved to be unimaginable is the difficulty of doing much more than the original device had already accomplished. It would seem that some forms of bionic vision have gone the way of rocket backpacks – that is, until recently.

The newest book by Andreas Kirsch with coauthor Natalia Grinberg, *The Factorization Method for Inverse Problems*, collects over a decade of work by Kirsch and collaborators on a simple method for *shape identification* in inverse scattering. This book belongs to the next generation of monographs on inverse scattering following the now standard works of Colton and Kress [2] (*Inverse Acoustic and Electromagnetic Scattering Theory* (1998)) and Isakov [7] (*Inverse Problems for Partial Differential Equations*).

Kirsch's factorization method arose from experimentation with *noniterative* inverse scattering methods that avoid the computational expense of calculating the solution to the forward problem at each iteration. Noniterative methods attack head-on the inverse problem of determining the scatterer from measured scattered fields by attempting, in principle, simply to invert the scattering operator. In most situations of interest, however, the scattering operator is nonlinear and the inverse problem is ill-posed. Early ideas focused on operator splitting techniques that decompose the scattering operator into a well-posed nonlinear part and an ill-posed linear part, each of which can be inverted stably. Another class of noniterative methods use *indicator functions* to detect the inconsistency or unsolvability of an easily computed auxiliary problem parameterized by points in space. The shape and location of the object is then determined by those points where the auxiliary problem is solvable. This latter generation of techniques, of which the factorization method is one, is both stable *and* computationally fast. What separates the factorization method from most of the other noniterative techniques is that it is mathematically complete: the computable criterion for determining the shape and location of the scatterers is both sufficient *and necessary* while most other techniques rely on only sufficient criteria.

The book consists of seven chapters treating the application of the factorization method for, respectively: simplest cases (namely where the far field operator is normal), refinements for more complicated settings (namely, where the far field operator is not normal), so-called mixed boundary value scattering problems, scattering from inhomogeneous media, Maxwell's equations, impedance tomography, and finally a short survey of alternative and related methods.

The first chapter offers a concise introduction to the factorization method with some new insights. Of particular interest is the characterization of the range of an operator $B : X \to Y$ (X and Y are reflexive Banach spaces) in terms of an infimum of the mapping $h(\psi) : Y^* \to \mathbb{R} \cup \{-\infty, +\infty\} := |\langle \psi, F\psi \rangle|$, where $F := BAB^*$ for $A : X^* \to X$ coercive. The infimum is taken over $\psi \in Y^*$ restricted to $\langle \psi, \phi \rangle = 1$ for a fixed $\phi \in Y$. For those readers familiar with convex analysis it can be shown that their infimal characterization is closely related to the *Fenchel conjugate*, h^*, of h:

$$h^*(\phi) := \sup_{\psi \in Y^*} \{\langle \phi, \psi \rangle - h(\psi)\}.$$

Indeed, the range of B is characterized by those points ϕ where the Fenchel conjugate of h is finite. In convex analysis the theory of Fenchel conjugation is used to gain a deeper understanding of the necessary conditions for the existence of Lagrange multipliers for inequality constrained convex programs. Kirsch and

Grinberg's introduction of the infimal characterization of the range of B opens the door to the possibility of deeper investigations into solvability of a broader class of problems, but this is beyond the scope of the book which is limited to inverse scattering problems.

In the case of scattering, the operator F above is an integral operator whose kernel is made up of the "measured" far field pattern on the sphere at infinity, otherwise known as the *far field operator*. The factor B maps the boundary condition of the governing PDE (the Helmholtz equation) to the far field pattern. Given the right choice of spaces, the mapping B is compact, one-to-one and dense. There are two keys to the factorization method and other sampling techniques (see linear sampling, for instance [1]) for determining the shape and location of scatterers from the far field patterns: first, the construction of the test function ϕ above and, second, the connection of the range of B to that of some operator easily computed from the far field operator F. The secret behind the success of these methods in inverse scattering is, first, that the construction of ϕ is trivial and, second, that there is (usually) a simpler object to work with than the Fenchel conjugate that depends only on the far field operator (usually the only thing that is known). Indeed, the test functions ϕ are simply far field patterns due to point sources: $\phi_z := e^{-ik\widehat{x}\cdot z}$ where $\widehat{x}$ is a point on the unit sphere (the direction of the incident field), k is a nonnegative integer (the frequency of the incident field), and z is some point in space.

The crucial observation of the factorization method is that ϕ_z is in the range of B *if and only if* z is a point *inside* the scatterer. Now, if one does not know where the scatter is, let alone its shape, then one does not know B. Enter the Fenchel conjugate: the Fenchel conjugate depends not on B but on the operator F which is constructed from measured data. However, the Fenchel conjugate, and hence the Kirsch-Grinberg infimal characterization, is very difficult to compute in general. Which leads us to the final piece to the factorization method that makes it fly: depending on the physical setting, there is a functional U of F under which the ranges of $U(F)$ and B coincide. In the case where F is a normal operator, $U(F) = (F^*F)^{1/4}$; for non-normal F the functional U depends more delicately on the physical problem at hand and is only known in a handful of cases. So the algorithm for determining the shape and location of a scatterer amounts to determining those points z where $e^{-ik\widehat{x}\cdot z}$ is in the range of $U(F)$ and where U and F are known and easily computed.

The second chapter deals mainly with technical refinements. In particular, the basic technique detailed in the first chapter is extended to problems for which the far field operator is not normal. This expands the methodology to problems with impedance boundary conditions, limited apertures, near-field measurements from spherical incident fields, and scattering on a half-space. The inf criterion discussed in the first chapter is shown to characterize the support of the scatterer in these more exotic instances, however, as already mentioned, this does not lead to feasible computational strategies. Instead, the authors show that the scatterer can be found by those points z where $e^{-ik\widehat{x}\cdot z}$ is in the range of $U(F) := (|\mathrm{Re}\, F| + |\mathrm{Im}\, F|)^{1/2}$, or, using the authors notation, $F_\sharp^{1/2}$.

The remainder of the special cases for the basic scalar wave scattering problem, mixed boundary value problems and inhomogeneous media are featured in dedicated short chapters. The chapter on inhomogeneous media also contains a brief discussion of the interior transmission eigenvalue problem. This sets the stage for the analogous problem for Maxwell's equations, which are elegantly presented in Chapter 5. Chapter six is another short chapter dedicated to impedance tomography and the factorization of the Neumann-to-Dirichlet map. The final chapter is a brief survey of three alternative methods: the dual space method of Colton and Monk [3, 4], the singular sources method of Potthast [8] and the probe method of Ikehata [5, 6].

This is a nice collection of results on the factorization method for a variety of scattering applications. It provides beginning researchers with a good survey of the basic theoretical approach, and for more experienced researchers working with factorization techniques it is a good reference source. This is, however, a keyhole view into a vast and rapidly evolving field. I await the second edition with great anticipation.

References

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