Review of *Calculus Without Derivatives* by Jean-Paul Penot*

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Roz Chast has a wonderful cartoon entitled "Falling off the Math Cliff" [*New Yorker*, March 6, 2006] showing a student at various stages of his mathematical education, visualized as scaling a rugged mountain peak and ending in a precipitous drop just as the student seems confident of his understanding. How true! It is at those moments, just before the fall, that I remember Jamie Sethian saying in one of his undergraduate lectures at Berkeley that mathematicians are junkies for those exhilarating flashes, lasting maybe two-minutes, where they feel like they understand what is going on. For most North American teenagers, calculus stands as a daunting, if not terrifying, barrier to the vast world of science and engineering, dry hills and dark canyons of exotic symbols littered with the corpses of pilgrims who didn't make it. It is known as Introductory Analysis for most other students throughout the world, but these foothills to the greater Analysis Range are pretty much the same everywhere: a highly refined cannon of facts and computational techniques of integration and differentiation through which instructors shepherd their young charges with inspirational peeks at the promised land that lies beyond. It is hard to fully appreciate the amount of midnight oil spent by some of the most celebrated – and many of the uncelebrated – minds of recorded history to characterize, synthesize and simplify the vast array of ideas, arguments and computational tricks tossed about over centuries so that, over the course of one year, a student can leap past hundreds of years of intellectual development and bitter philosophical battles just to get to the beginning of the beginning of what we could call "university mathematics". I'm not sure whether, despite their disagreements, Leibniz and Newton (and Fermat!) would be unanimous in their horror or delight at what has become of their brainchildren, but modern mathematicians would certainly agree that integral and differential calculus today has succumbed to the exigencies of an audience with very broad horizons and strict limitations on time, interest and patience. Not only are functions conflated with their values, but, worse, the study of limits is often crowded out by formal calculation and applications. These formal calculations are so effective that one can easily be lulled into the illusion that smooth functions are the only ones that are relevant, and that the story ends when you hit a kink.

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Ask any beginning student about a tangent to the graph of $x \mapsto |x|$ at $x = 0$ and she will probably tell you that it doesn't exist, or at least, that it is not well-defined. After all, this function is not differentiable at the origin. We equip students to travel through gently rolling hills, so it is quite natural to avoid the rough patches. Until recently, most could ignore this part of the intellectual universe and be no worse off for it. Classical mechanics, is predicated on the continuum and deterministic, one-to-one models. This is the legacy of integral and differential calculus. And then there are the laws of Man, contradictory, asymmetric and political, not logical. Not only Man, but often even Nature resists the tools of differential calculus: quantum phenomena, breaking waves, explosions. The usual calculus of differentiation is useless in these situations and collapses almost completely when you remove the basic assumption that directional derivatives must meet at a point. All of the facts that are more or less taken for granted – the gradient and derivative of a function are one in the same – can no longer be glossed over. An entirely new vocabulary is needed to handle these exceptional cases. For continuous convex functions on open subsets of Banach spaces with separable duals, the set of points where the Fréchet derivative exists is *dense* [1], so we can be comforted by the fact that a smooth handhold is never far away. Yet, often, and particularly in optimization, the solutions to our problems lie *exactly* at the exceptional points where classical differential calculus breaks down, as the problem of minimizing the function $f(x) = |x|$ illustrates.

The title of Jean-Paul Penot's recently published collection of lecture notes in analysis, *Calculus Without Derivatives*, is provocative and evocative of Chast's math cliff. These 478 pages (plus references and index) encapsulate everything that your calculus teacher, following generations of educators, avoided on your first introduction to analysis. It is a bit like learning arithmetic all over again in a university course on real analysis: you see for the first time the bulk of the iceberg beneath the surface. And much of the content is, on mathematical timescales, recent, having its origins in the 20th century. The book collects three different branches of analysis: differential calculus, convex analysis and nonsmooth analysis. The last of these three topics is properly a generalization of the first two, which are evolving to become different species of the same genus called nonsmooth or variational analysis, depending on who you ask. A survey of facts and tools from topology and functional analysis supports these three themes.

Unlike the usual calculus texts, Penot does not shy away from precision. The book unfolds slowly and deliberately, starting with Zorn's lemma and convergence of nets in topological spaces. The latter is necessary for weak* topologies on dual Banach spaces, which is not only a natural setting for *subdifferentials* (generalized derivatives) and *Fenchel conjugates* of functions on normed spaces, but is necessary for characterizing the right notions of compactness of subsets of, and functions on, normed spaces, which generates new spaces (*weakly compactly generated spaces* in particular) that arise in nonsmooth analysis. Set-valued mappings are treated early and without much ado. In a twist on the usual order, optimization and control theory are briefly mentioned as motivations for set-valued mappings, but without any prior development of variational problems or convex analysis - this comes later. Convex analysis of sets (topological properties and separation) is presented in considerable detail. The Hahn-Banach-Fenchel duality cycle, of which so much is made in [2], for instance, is nearly completed in this subsection, but Penot delays more than a passing mention of Fenchel duality until Chapter 3 where the analytical properties of convex functions (continuity, differentiability, calculus rules) are developed. Variational principles close out the introductory material, which are immediately put to use in a sort of proto-numerical analysis of variational methods including descent methods, error bounds, regularization and penalization methods, metric regularity and the extension of Lipschitz continuity of set-valued mappings. This is not unreasonable, given the focus of the first chapter on topological and metric properties of sets. It is, however, a juxtaposition noticeable in its shift in tone and focus from the previous elements of the chapter. This first quarter of the book concludes with the notion of well-posedness for variational problems.

The famous saying "standing on the shoulders of giants" attributed to Isaac Newton fittingly inaugurates the second chapter, which is dedicated to a much more rigorous and general treatment of differential calculus than you would find in a calculus text. I can't help but wonder if Penot knew of the the snarky context of Newton's lovely words penned in a letter to his rival Robert Hooke: Hooke was not a stout man. The central goal of the chapter is an analysis of the invertability of nonlinear maps culminating in the theorem of Lyusternik-Graves, the inverse mapping theorem and their many applications. Applications to optimization and the calculus of variations, which have long been the driving force behind nonsmooth analysis, are treated in separate subsections. The normal and tangent cones, which figure so prominently in other books on variational/nonsmooth analysis and generalized differentiation, are introduced in the subsection presenting applications to optimization.

The specialization of differential calculus to the class of convex functions is explored in Chapter 3. Here most of the remaining cannon of convex analysis is developed: continuity, (generalized) differentiability, subdifferential calculus, Legendre-Fenchel transformation. The final two subsections of the chapter concern the choice of spaces on which to formulate a given problem. Naturally, vector spaces with strictly convex norm are desirable, as are spaces on which continuous convex functions are differentiable (Fréchet or Hadamard) on dense subsets of open sets.

Subsequent chapters deal with nonsmooth analysis outside of the context of convexity, which requires a closer attention to detail. Here the "four pillars of nonsmooth analysis" are presented in their fullest generality: normal cones, subdifferentials, tangent cones and directional derivatives. (There are even six pillars if you count coderivatives and graphical derivatives.) The convex (Moreau-Rockafellar) subdifferential bifurcates into the Fréchet and the directional (or Dini-Hadamard or Bouligand or contingent) subdifferential, which in finite dimensional normed spaces are equivalent. Calculus rules for sums and compositions of functions are developed, and the relationship with geometrical counterparts, normal cones and tangent cones, are explored. The last section of Chapter 4 contains a broad array of applications - the only direct nod to applications one will encounter in the latter third of the book. Chapter 5 is a survey of one of the most famous generalizations of the derivative due to Clarke where the duality persists between normal and tangent cones and subdifferentials and direction derivatives. The cost of this clean picture is loss of information: the Clarke subdifferential is often too large for an inclusion of the form $0 \in \partial f(x)$ to be meaningful, the consequences this lack of precision leading rather dramatically to a sound and fury signifying nothing. Chapters 6 and 7 are a study of limiting and graded subdifferentials respectively. These offer the most complete and informative theory of generalized differentiation, extending naturally to multi-valued mappings. This is presented at an expert level and is, by the author's own admission, not easy going, but essential for seeing the frontiers of nonsmooth analysis.

The book is not revolutionary in its selection of topics and collection of ideas. Many of the themes appear either directly or indirectly in other books that attempt to encapsulate the foundations and foundational results of nonsmooth analysis. What makes Penot's work stand out is his path through the material and the clean and scholarly presentation. It is well suited for individual study or in a classroom, though classroom use would require a considerable commitment at the outset. It is well worth the investment.

References

[1] Asplund, E.: Fréchet differentiability of convex functions. Acta Math. 121, 31-47 (1968).

[2] Borwein, J. and Vanderwerff, J. *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, Cambridge, 2010]