The Vector Helmholtz Equation Revisited: Inverse Obstacle Scattering

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Abstract

The vector Helmholtz equation, from a mathematical point of view, provides a generalization of the time-harmonic Maxwell equations for the propagation of time-harmonic electromagnetic waves. After reviewing some classic results on the two main exterior boundary value problems for the vector Helmholtz equation, i.e., the so-called electric boundary condition and the magnetic boundary condition, we prove reciprocity results for scattering of plane waves and point sources. Then we make use of them for obtaining uniqueness results for the inverse obstacle scattering problem to determine the shape of the scatterer from knowing the far field pattern of the scattered waves and results on the so-called far field operator for the two scattering problems. These results are generalizations of corresponding results for the Maxwell equations and analogues to results for the Dirichlet and Neumann boundary condition for the scalar Helmholtz equation. We also briefly consider the extension of the so-called DB boundary condition from the Maxwell equations to the vector Helmholtz equation.

1 Introduction

About sixty years ago Werner [12] generalized the exterior boundary value problem for scattering of time-harmonic electromagnetic waves from a perfect conductor by replacing the time-harmonic Maxwell equations by the slightly more general form of the vector Helmholtz equation which is obtained by

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elimination of either the magnetic field or the electric field from the Maxwell equations. For a bounded set D that is the open complement of an unbounded domain of class C^2 Werner introduced the exterior boundary value problem to find a vector field E in $\mathbb{R}^3 \setminus D$ that is a radiating solution to the vector Helmholtz equation

$$\Delta E + k^2 E = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{1.1}$$

with positive wave number k and satisfies the *electric boundary condition*

$$\nu \times E = c \quad \text{and} \quad \operatorname{div} E = \gamma \quad \text{on } \partial D \tag{1.2}$$

where c is a given tangential field and γ a given scalar function on the boundary ∂D and ν denotes the unit normal vector to ∂D directed into the exterior of D. Here, a solution E to the vector Helmholtz equation whose domain of definition contains the exterior of some sphere is called *radiating* if it satisfies the *Silver–Müller radiation condition*

$$\lim_{|x|\to\infty} \operatorname{curl} E(x) \times x + x \operatorname{div} E(x) - ik|x|E(x) = 0 \tag{1.3}$$

where the limit is assumed to hold uniformly in all directions $x/|x|$. We note that the Silver–Müller radiation condition (1.3) for the vector field E is equivalent to the Sommerfeld radiation condition for the Cartesian components of E (see [2, Corollary 4.14]). As a main advantage the above reformulation clarifies the relationship between acoustic and electromagnetic scattering more closely.

Partly based on previous work by Knauff and Kress [5], a complete account on existence and uniqueness of the solution to the boundary value problem (1.1)–(1.3) via boundary integral equations in a classical Hölder space setting was presented some fourty years ago by Colton and Kress in their first monograph [3] together with the corresponding analysis for the related problem with the *magnetic boundary condition*

$$\nu \times \operatorname{curl} E = c \quad \text{and} \quad \nu \cdot E = \gamma \quad \text{on} \quad \partial D. \tag{1.4}$$

Due to the vector identities $\operatorname{div}\operatorname{curl} = 0$ and $\operatorname{curl}\operatorname{curl} = -\Delta + \operatorname{grad}\operatorname{div}$ for each solution E, H to the Maxwell equations

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \tag{1.5}$$

both the electric field E and the magnetic field H are divergence free and satisfy the vector Helmholtz equation. Conversely, for a divergence free solution E of the vector Helmholtz equation the pair $E, ikH := \operatorname{curl} E$ solves the Maxwell equations. The homogeneous form of the boundary condition (1.2) corresponds to the homogeneous boundary condition for the electric field in the case of scattering from a perfect conductor and therefore was named electric boundary condition. Because of $\nu \cdot \operatorname{curl} E = -\operatorname{Div} \nu \times E$ with the surface divergence Div for the corresponding magnetic field H we have $\nu \cdot H = 0$ on ∂D and therefore the boundary condition (1.4) was named the magnetic boundary condition. The choice of the two boundary conditions (1.2) and (1.4) is further motivated through the occurrence of their left hand sides in Green's vector integral theorem

$$\begin{aligned} &\int_D \{E \cdot \Delta F - F \cdot \Delta E\}\, dx \\ &= \int_{\partial D} \{\nu \times E \cdot \operatorname{curl} F + \nu \cdot E \operatorname{div} F - \nu \times F \cdot \operatorname{curl} E - \nu \cdot F \operatorname{div} E\}\, ds \end{aligned} \tag{1.6}$$

for vector fields $E, F \in C^2(D) \cap C^1(\bar{D})$ in a bounded domain D of class C^1 with the unit normal vector ν to the boundary ∂D directed into the exterior of D. It follows easily from the Gauß divergence integral theorem applied to $E \times \operatorname{curl} F + E \operatorname{div} F$ and plays an important part in the analysis of the vector Helmholtz equation. It also illustrates that the pair of the two boundary conditions (1.2) and (1.4) can be considered as counterpart to the pair given by the Dirichlet and the Neumann boundary conditions for the scalar Helmholtz equation.

The purpose of this paper is to consider the inverse scattering problems for the two boundary conditions with an emphasis on the question of uniqueness. The plan of the paper is as follows. In Section 2 we shortly review the basic results on the exterior electric and magnetic boundary value problems from [2]. In Section 3 we will introduce the corresponding forward scattering problems with plane waves and point sources as incident fields. In particular we will prove a reciprocity relation for the interchange of the incident and the observation direction for plane wave incidence and a mixed reciprocity relation connecting plane wave and point source incidence. In the following Section 4 we will establish some uniqueness results for the inverse scattering problem that extend corresponding results for the Maxwell equations. In particular, this includes the extension of Karp's theorem [4] as one of the rare explicit analytic solutions rather than approximate solutions for a nonlinear inverse obstacle scattering problem. The final Section 5 is concerned with the extension to the vector Helmholtz equation for the DB boundary condition, that is, the boundary conditions for the normal component of both the electric and the magnetic field in case of the Maxwell equations. Here, it turns out that considering the DB boundary condition for the vector Helmholtz equation the case of scattering with plane wave incidence does not lead to a generalization of the Maxwell case.

2 Exterior boundary value problems

We begin our analysis by summarizing some basic facts on exterior boundary value problems for the vector Helmholtz equation. Green's vector integral theorem (1.6) together with the Silver–Müller radiation condition (1.3) is the main tool for proving the following representation theorem. In terms of the fundamental solution to the scalar Helmholtz equation

$$\Phi(x,y) := \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y,$$

and under the assumptions on D as introduced in connection with (1.1)–(1.4) for radiating solutions $E \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ to the vector Helnmholtz equation we have the *Stratton–Chu formula*

$$\begin{aligned} E(x) \;=\; & \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\ & - \operatorname{grad} \int_{\partial D} \nu(y) \cdot E(y)\, \Phi(x,y)\, ds(y) \\ & - \int_{\partial D} [\operatorname{curl} E(y) \times \nu(y) + \nu(y) \operatorname{div} E(y)]\, \Phi(x,y)\, ds(y) \end{aligned} \tag{2.1}$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$. For a proof we refer to [5] and [2, Theorem 4.13]. With the aid of the asymptotic of the fundamental solution Φ from the representation (2.1) we obtain that each radiating solution E to the vector Helmholtz equation has the asymptotic form

$$E(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{2.2}$$

uniformly in all directions $\hat{x} = x/|x|$ where the vector field E_∞ defined on the unit sphere $\mathbb{S}^2$ is known as the *far field pattern* of E and given by

$$\begin{aligned} E_\infty(\hat{x}) = \frac{ik}{4\pi} \int_{\partial D} [\hat{x} \times \{\nu(y) \times E(y)\} \,-\, \hat{x}\, \nu(y) \cdot E(y)\,]\, e^{-ik\,\hat{x}\cdot y} ds(y) \\ -\frac{1}{4\pi} \int_{\partial D} [\operatorname{curl} E(y) \times \nu(y) + \nu(y) \operatorname{div} E(y)]\; e^{-ik\,\hat{x}\cdot y} ds(y) \end{aligned} \tag{2.3}$$

for $\hat{x} \in \mathbb{S}^2$. Since the Cartesian components of E_∞ must coincide with the scalar far field patterns of the Cartesian components of E_∞ Rellich's lemma (see [3, Theorem 2.14]) for the scalar Helmholtz equations tells us that $E_\infty = 0$ on $\mathbb{S}^2$ implies that $E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. In the sequel we will refer to this one-to-one correspondence of far field patterns and radiating solutions of the vector Helmholtz equation just as Rellich's lemma.

From the Introduction we recall that both the electric and the magnetic boundary value problem introduced by (1.1)–(1.4) are uniquely solvable under the assumption that the right hand sides c and γ are Hölder continuous. Although it is not stated explicitly in [2], from the existence analysis via the Riesz theory for compact operators it follows that both problems are well posed in the sense that the solutions depend continuously on the boundary data in the Hölder space setting.

3 The scattering problems: reciprocity

Given a solution E^i to the vector Helmholtz equation that is defined in all of $\mathbb{R}^3$ except possible singular points in $\mathbb{R}^3 \setminus \bar{D}$ the direct scattering problem is to find a solution $E = E^i + E^s$ to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ such that the scattered field E^s satisfies the Silver–Müller radiation condition and the total field E satisfies the homogeneous electric boundary condition

$$\nu \times E = 0 \quad \text{and} \quad \operatorname{div} E = 0 \quad \text{on} \quad \partial D \tag{3.1}$$

or the homogeneous magnetic boundary condition

$$\nu \times \operatorname{curl} E = 0 \quad \text{and} \quad \nu \cdot E = 0 \quad \text{on} \quad \partial D. \tag{3.2}$$

As particular incident fields we will consider plane waves and point sources. The *inverse scattering problem* that we are considering then consists of determining the boundary of the scatterer D from a knowledge of the far field pattern for the scattering of both plane waves or point sources.

Plane waves with incident direction $d \in \mathbb{S}^2$ and polarization vector $p \in \mathbb{R}^3$ are described by the matrix $E^i(x,d)$ defined by its multiplication with the polarization vector as

$$E^i(x,d)p := e^{ik\,x\cdot d}p. \tag{3.3}$$

Since the direct scattering problem is linear with respect to the incident field, we can express the scattered wave by a matrix $E^s(x,d)$, the total wave by a matrix $E(x,d)$, and the far field pattern by a matrix $E_\infty(\hat{x},d)$. They map the polarization vector p onto the scattered wave $E^s(x,d)p$, the total wave $E(x,d)p$, and the far field pattern $E_\infty(\hat{x},d)p$, respectively.

Theorem 3.1 *The far field pattern for the scattering of plane waves by a scatterer with electric or magnetic boundary condition satisfies the* reciprocity relation

$$E_\infty(\hat{x},d) = [E_\infty(-d,-\hat{x})]^\top, \quad \hat{x}, d \in \mathbb{S}^2, \tag{3.4}$$

for the interchange of the incident and the observation direction.

Proof. The idea of the prove is the same as for the reciprocity relations in acoustic scattering for the scalar Helmholtz equation and in electromagnetic scattering for the Maxwell equation as provided in [3] and is based totally on Green's vector integral theorem. From the latter together with the vector Helmholtz equation for the incident and the scattered fields and the radiation condition for the scattered field we have

$$\int_{\partial D} \big\{\nu \times E^i(\cdot\,,d)p \,\cdot\, \operatorname{curl} E^i(\cdot\,,-\hat{x})q + \nu\cdot E^i(\cdot\,,d)p \,\operatorname{div} E^i(\cdot\,,-\hat{x})q$$
$$-\nu \times E^i(\cdot\,,-\hat{x})q \,\cdot\, \operatorname{curl} E^i(\cdot\,,d)p - \nu\cdot E^i(\cdot\,,-\hat{x})q \,\operatorname{div} E^i(\cdot\,,d)p\big\}\,ds = 0$$

and

$$\int_{\partial D} \big\{\nu \times E^s(\cdot\,,d)p \,\cdot\, \operatorname{curl} E^s(\cdot\,,-\hat{x})q + \nu\cdot E^s(\cdot\,,d)p \,\operatorname{div} E^s(\cdot\,,-\hat{x})q$$
$$-\nu \times E^s(\cdot\,,-\hat{x})q \,\cdot\, \operatorname{curl} E^s(\cdot\,,d)p - \nu\cdot E^s(\cdot\,,-\hat{x})q \,\operatorname{div} E^s(\cdot\,,d)p\big\}\,ds = 0$$

for all $p, q \in \mathbb{R}^3$. For the second integral we used the fact that for two radiating solutions E and F of the vector Helmholtz equation for a sphere

S_R of radius R centered at the origin we have

$$\int_{S_R} \{\nu \times E \cdot \operatorname{curl} F + \nu \cdot E \operatorname{div} F - \nu \times F \cdot \operatorname{curl} E - \nu \cdot F \operatorname{div} E\}\, ds$$

$$= \frac{1}{R} \int_{|x|=R} \{E(x) \cdot [\operatorname{curl} F(x) \times x + x \operatorname{div} F(x) - ik|x|F(x)]$$

$$-F(x) \cdot [\operatorname{curl} E(x) \times x + x \operatorname{div} E(x) - ik|x|E(x)]\}\, ds(x) \to 0,\ R \to \infty.$$

Exploiting the fact that the triple vector product is unchanged under a circular shift of its three factors we note that

$$\nu(y) \times E^s(y,d)p \cdot \operatorname{curl} E^i(y,-\hat{x})q = ikq \cdot \{\hat{x} \times [\nu(y) \times E^s(y,d)p]\}\, e^{-ik\,\hat{x}\cdot y}$$

$$\nu(y) \cdot E^s(y,d)p \operatorname{div} E^i(y,-\hat{x})q = -ikq \cdot \{\nu(y) \cdot E^s(y,d)p\}\, e^{-ik\,\hat{x}\cdot y}$$

$$\nu(y) \times E^i(y,-\hat{x})q \cdot \operatorname{curl} E^s(y,d)p = q \cdot \{\operatorname{curl} E^s(y,d)p \times \nu(y)\}\, e^{-ik\,\hat{x}\cdot y}$$

$$\nu(y) \cdot E^i(y,-\hat{x})q \operatorname{div} E^s(y,d)p = q \cdot \nu(y) \operatorname{div} E^s(y,d)p\, e^{-ik\,\hat{x}\cdot y}.$$

Using these relations in the far field representation (2.3) we find

$$4\pi\, q \cdot E_\infty(\hat{x},d)p$$

$$= \int_{\partial D} \big\{\nu \times E^s(\cdot\,,d)p \,\cdot\, \operatorname{curl} E^i(\cdot\,,-\hat{x})q + \nu \cdot E^s(\cdot\,,d)p \operatorname{div} E^i(\cdot\,,-\hat{x})q \qquad (3.5)$$

$$-\nu \times E^i(\cdot\,,-\hat{x})q \,\cdot\, \operatorname{curl} E^s(\cdot\,,d)p - \nu \cdot E^i(\cdot\,,-\hat{x})q \operatorname{div} E^s(\cdot\,,d)p\big\}\, ds.$$

From this by interchanging the roles of d and $-\hat{x}$ and of p and q, respectively, we obtain

$$4\pi p \,\cdot E_\infty(-d,-\hat{x})q$$

$$= \int_{\partial D} \big\{\nu \times E^s(\cdot\,,-\hat{x})q \,\cdot\, \operatorname{curl} E^i(\cdot\,,d)p + \nu \cdot E^s(\cdot\,,-\hat{x})q \operatorname{div} E^i(\cdot\,,d)p$$

$$-\nu \times E^i(\cdot\,,d)p \,\cdot\, \operatorname{curl} E^s(\cdot\,,-\hat{x})q - \nu \cdot E^i(\cdot\,,d)p \operatorname{div} E^s(\cdot\,,-\hat{x})q\big\}\, ds.$$

We now subtract the last integral from the sum of the three preceding integrals to obtain

$$4\pi\big\{q \cdot E_\infty(\hat{x}, d)p - p \cdot E_\infty(-d, -\hat{x})q\big\}$$

$$= \int_{\partial D} \big\{\nu \times E(\cdot\,, -\hat{x})q \,\cdot\, \operatorname{curl} E(\cdot\,, d)p + \nu \cdot E(\cdot\,, -\hat{x})q \,\operatorname{div} E(\cdot\,, d)p \tag{3.6}$$

$$-\nu \times E(\cdot\,, d)p \,\cdot\, \operatorname{curl} E(\cdot\,, -\hat{x})q - \nu \cdot E(\cdot\,, d)p \,\operatorname{div} E(\cdot\,, -\hat{x})q\big\}\, ds.$$

From this the reciprocity relation (3.4) is obtained by

$$\nu \times E(\cdot\,, d)p = \nu \times E(\cdot\,, -\hat{x})q = 0$$

and

$$\operatorname{div} E(\cdot\,, d)p = \operatorname{div} E(\cdot\,, -\hat{x})q = 0$$

on ∂D in the case of the electric boundary condition and by

$$\nu \times \operatorname{curl} E(\cdot\,, d)p = \nu \times \operatorname{curl} E(\cdot\,, -\hat{x})q = 0$$

and

$$\nu \cdot E(\cdot\,, d)p = \nu \cdot E(\cdot\,, -\hat{x})q = 0$$

on ∂D in the case of the magnetic boundary condition. □

For the scattering of a point source

$$U^i(x, z)p := \Phi(x, z)p \tag{3.7}$$

with a polarization vector p and a location $z \in \mathbb{R}^3 \setminus \bar{D}$ we denote the scattered wave by $U^s(x, z)$, the total wave by $U(x, z)$, and the far field pattern of the scattered wave by $U^s_\infty(\hat{x}, z)$. Note that as above for plane wave incidence all three quantities are matrices.

Theorem 3.2 *The scattering of point sources and of plane waves are related by the* mixed reciprocity relation

$$4\pi U^s_\infty(-d, z) = [E^s(z, d)]^\top, \quad z \in \mathbb{R}^3 \setminus \bar{D},\ d \in \mathbb{S}^2, \tag{3.8}$$

for both the electric and the magnetic boundary condition.

Proof. As in the proof of Theorem 3.1 from the Green's vector integral theorem, the vector Helmholtz equation and the radiation condition we find

$$\int_{\partial D} \big\{\nu \times U^i(\cdot\,,z)p \,\cdot\, \operatorname{curl} E^i(\cdot\,,d)q + \nu \cdot U^i(\cdot\,,z)p \,\operatorname{div} E^i(\cdot\,,d)q$$

$$-\nu \times E^i(\cdot\,,d)q \,\cdot\, \operatorname{curl} U^i(\cdot\,,z)p - \nu \cdot E^i(\cdot\,,d)q \,\operatorname{div} U^i(\cdot\,,z)p\big\}\, ds = 0$$

and

$$\int_{\partial D} \big\{\nu \times U^s(\cdot\,,z)p \,\cdot\, \operatorname{curl} E^s(\cdot\,,d)q + \nu \cdot U^s(\cdot\,,z)p \,\operatorname{div} E^s(\cdot\,,d)q$$

$$-\nu \times E^s(\cdot\,,d)q \,\cdot\, \operatorname{curl} U^s(\cdot\,,z)p - \nu \cdot E^s(\cdot\,,d)q \,\operatorname{div} U^s(\cdot\,,z)p\big\}\, ds = 0$$

for all $p, q \in \mathbb{R}^3$ and $z \in \mathbb{R}^3 \setminus \bar{D}$. From the far field representation (2.3), again using the properties of the triple product, we find

$$4\pi\, q \cdot U_\infty(-d, z)p$$

$$= \int_{\partial D} \big\{\nu \times U^s(\cdot\,,z)p \,\cdot\, \operatorname{curl} E^i(\cdot\,,d)q + \nu \cdot U^s(\cdot\,,z)p \,\operatorname{div} E^i(\cdot\,,d)q$$

$$-\nu \times E^i(\cdot\,,d)q \,\cdot\, \operatorname{curl} U^s(\cdot\,,z)p - \nu \cdot E^i(\cdot\,,d)q \,\operatorname{div} U^s(\cdot\,,z)p\big\}\, ds.$$

and from the Stratton-Chu formula (2.1) we have

$$p \cdot E^s(z, d)q$$

$$= \int_{\partial D} \big\{\nu \times E^s(\cdot\,,d)q \,\cdot\, \operatorname{curl} U^i(\cdot\,,z)p + \nu \cdot E^s(\cdot\,,d)q \,\operatorname{div} U^i(\cdot\,,z)p$$

$$-\nu \times U^i(\cdot\,,z)p \,\cdot\, \operatorname{curl} E^s(\cdot\,,d)q - \nu \cdot U^i(\cdot\,,z)p \,\operatorname{div} E^s(\cdot\,,d)q\big\}\, ds.$$

Now the proof can be completed as for the previous theorem. $\square$

These reciprocity principles can be used for the analysis of the inverse scattering problem for the vector Helmholtz equation as in the cases of the scalar Helmholtz equation or the Maxwell equations. We demonstrate this by some analysis on the far field operator which is defined as an operator on the space of L^2 vector fields on $\mathbb{S}^2$. Analogous to the notion of Herglotz wave functions and Herglotz pairs in [3] we call fields of the form

$$E(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{3.9}$$

with some $g \in L^2(\mathbb{S}^2)$ a Herglotz field with kernel g. Clearly, a field is a Herglotz wave field if and only if its Cartesian components are Herglotz wave functions. For the basic properties of Herglotz wave functions we refer to [3, Section 3.4].

Theorem 3.3 *For both the electric or magnetic boundary condition the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by*

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{3.10}$$

is injective and has dense range if and only if there does not exist a solution E of the vector Helmholtz equation in D which satisfies the corresponding homogeneous boundary condition on ∂D which is a Herglotz wave field.

Proof. We begin by noting the following consequence of the linearity and well-posedness of the scattering problem for both boundary conditions (see [3, Lemma 3.28]). For $g \in L^2(\mathbb{S}^2)$ the solution to the scattering problem for the incident wave

$$\tilde{E}^i(x) := \int_{\mathbb{S}^2} E^i(x, d) g(d)\, ds(d), \quad x \in \mathbb{R}^3,$$

is given by

$$\tilde{E}^s(x) = \int_{\mathbb{S}^2} E^s(x, d) g(d)\, ds(d)$$

for $x \in \mathbb{R}^3 \setminus D$ and has the far field pattern

$$\tilde{E}_\infty(\hat{x}) = \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d) = (Fg)(\hat{x})$$

for $\hat{x} \in \mathbb{S}^2$.

From the reciprocity relation (3.4) it follows that the L^2 adjoint $F^* : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ of F is given by

$$F^* g = \overline{RFR\bar{g}}, \tag{3.11}$$

where $R : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by $(Rg)(d) := g(-d)$. Now the proof is analogous to that of Theorem 3.30 in [3]. $\square$

For an example of a domain for which the far field operator is not injective, let S_R be a sphere of radius R centered at the origin and consider the spherical wave function

$$u_n(x) := j_n(k|x|)\, Y_n\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^3,$$

where j_n is a spherical Bessel function and Y_n a spherical harmonic of order n. By the Funk–Hecke formula (see [3, p. 36]) u_n is a Herglotz wave function. Since derivatives of Herglotz wave functions again are Herglotz wave functions, the field

$$E_n := \operatorname{grad} u_n$$

is a Herglotz wave field. It satisfies the homogeneous electric boundary condition $\nu \times E_n = 0$ and $\operatorname{div} E_n = 0$ on S_R if kR is equal to a zero of j_n.

The spherical vector wave functions

$$M_n(x) := \operatorname{curl} \{x u_n(x)\}, \quad x \in \mathbb{R}^3,$$

and $N_n := \operatorname{curl} M_n$ both solve the vector Helmholtz equation and are Herglotz wave fields (see [3, p. 264]). For $x \in S_R$ we have $x = R\nu(x)$ and consequently

$$\nu(x) \times M_n(x) = \nu(x) \times \{\operatorname{grad} u_n(x) \times x\} = R \operatorname{Grad} u_n(x) \tag{3.12}$$

with the surface gradient Grad. This means that the divergence free Herglotz wave field M_n also satisfies the homogeneous electric boundary condition on S_R if kR is equal to a zero of j_n.

From (3.12) it follows that on S_R we have

$$\nu \cdot N_n = \nu \cdot \operatorname{curl} M_n = -\operatorname{Div}\{\nu \times M_n\} = -R \operatorname{Div} \operatorname{Grad} u_n$$

and, in view of $\operatorname{curl} N_n = \operatorname{curl} \operatorname{curl} M_n = -\Delta M_n = k^2 M_n$, also

$$\nu \times \operatorname{curl} N_n = k^2 R \operatorname{Grad} u_n.$$

Therefore if kR is equal to a zero of j_n the Herglotz wave field N_n satisfies the homogeneous magnetic boundary condition on S_R.

We will conclude our considerations on the far field operator by showing that it is a normal operator. To this end, let E_g^i and E_h^i be the Herglotz wave fields with kernels $g, h \in L^2(\mathbb{S}^2)$ and let $E_g = E_g^i + E_g^s$ and $E_h = E_h^i + E_h^s$ be the corresponding solutions to the scattering problem with E_g^i and E_h^i as incident fields, respectively. By $E_{g,\infty}$ and $E_{h,\infty}$ we denote the far field patterns of E_g^s and E_h^s, respectively. Proceeding as in the proof of the reciprocity relation in Theorem 3.1 from Green's vector integral theorem and the radiation condition for the scattered fields we obtain that

$$\int_{\partial D} \left\{ \nu \times E_g^i \cdot \operatorname{curl} \overline{E_h^i} + \nu \cdot E_g^i \operatorname{div} \overline{E_h^i} - \nu \times \overline{E_h^i} \cdot \operatorname{curl} E_g^i - \nu \cdot \overline{E_h^i} \operatorname{div} E_g^i \right\} ds = 0$$

and

$$\int_{\partial D} \left\{ \nu \times E_g^s \cdot \operatorname{curl} \overline{E_h^s} + \nu \cdot E_g^s \operatorname{div} \overline{E_h^s} - \nu \times \overline{E_h^s} \cdot \operatorname{curl} E_g^s - \nu \cdot \overline{E_h^s} \operatorname{div} E_g^s \right\} ds$$

$$= -2ik \int_{\mathbb{S}^2} E_{g,\infty} \overline{E_{h,\infty}} \, ds = -2ik \, (Fg, Fh)$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\mathbb{S}^2)$. Using (3.5) and interchanging the order of integration we find that

$$4\pi(Fg, h) = 4\pi \int_{\mathbb{S}^2} \overline{h(\hat{x})} \cdot \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d) \, ds(d) \, ds(\hat{x})$$

$$= \int_{\partial D} \left\{ \nu \times E_g^s \cdot \operatorname{curl} \overline{E_h^i} + \nu \cdot E_g^s \operatorname{div} \overline{E_h^i} - \nu \times \overline{E_h^i} \cdot \operatorname{curl} E_g^s - \nu \cdot \overline{E_h^i} \operatorname{div} E_g^s \right\} ds.$$

Interchanging the roles of g and h in the last equation and taking complex conjugates we obtain

$$4\pi(g, Fh)$$

$$= \int_{\partial D} \left\{ \nu \times \overline{E_h^s} \cdot \operatorname{curl} E_g^i + \nu \cdot \overline{E_h^s} \operatorname{div} E_g^i - \nu \times E_g^i \cdot \operatorname{curl} \overline{E_h^s} - \nu \cdot E_g^i \operatorname{div} \overline{E_h^s} \right\} ds.$$

Subtracting the last equation from the sum of the three preceding equations we finally arrive at

$$-2ik \, (Fg, Fh) + 4\pi(Fg, h) - 4\pi(g, Fh)$$

$$= \int_{\partial D} \left\{ \nu \times E_g \cdot \operatorname{curl} \overline{E_h} + \nu \cdot E_g \operatorname{div} \overline{E_h} - \nu \times \overline{E_h} \cdot \operatorname{curl} E_g - \nu \cdot \overline{E_h} \operatorname{div} E_g \right\} ds$$

and from this for both boundary conditions we obtain that

$$-2ik\,(Fg, Fh) + 4\pi(Fg, h) - 4\pi(g, Fh) = 0.$$

Using this together with (3.11) the proof of the following theorem is completely analogous to that for the corresponding result in acoustic scattering, that is, for Theorem 3.32 in [3].

Theorem 3.4 *The far field operator F is compact and normal, that is, $FF^* = F^*F$ and has an infinite number of eigenvalues.*

4 Inverse scattering: uniqueness

Theorem 4.1 *Assume that D_1 and D_2 are two scatterers with either electric or magnetic boundary condition such that for a fixed wave number k for all plane waves of the form* (3.3) *the far field patterns for both scatterers coincide for all incident directions $d \in \mathbb{S}^2$ and all polarizations $p \in \mathbb{R}^3$ (that means for three linearly independent polarizations $p \in \mathbb{R}^3$). Then $D_1 = D_2$.*

Proof. Our proof follows the ideas of the proof of Theorem 5.6 in [3] for the corresponding uniqueness result in inverse acoustic scattering. We denote the scattered waves for D_1 and D_2 for point source incidence by U_1^s and U_2^s, respectively. In a first step by two applications of Rellichs's lemma together with the mixed reciprocity principle of Theorem 3.2 from the coincidence of the far field patterns for all incident directions d we conclude that the scattered waves for point source incidence coincide in the unbounded component G of the complement of $\bar{D}_1 \cup \bar{D}_2$, that is,

$$U_1^s(x, z) = U_2^s(x, z) \tag{4.1}$$

for all $x, z \in G$.

Then in the second step assuming that $D_1 \neq D_2$ (and without loss of generality that $D_1 \setminus (\bar{D}_1 \cap \bar{D}_2)$ is nonempty), we will arrive at a contradiction by letting the source location z tend to a boundary point of ∂D_1 which does not belong to $\bar{D}_2$. Without loss of generality we assume there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. Then we can choose $h > 0$ such that the sequence

$$x_m := x^* + \frac{h}{m}\,\nu(x^*), \quad m = 1, 2, \ldots, \tag{4.2}$$

is contained in G. Consider the fields in (3.7) with z replaced by x_m and polarization p such that $p \times \nu(x^*) \neq 0$. Since $x^* \notin \bar{D}_2$, for scattering from D_2 we have

$$\begin{aligned} & \|\nu \times [U^i(\cdot, x_m)p] - \nu \times [U^i(\cdot, x^*)p]\|_{C^{0,\alpha}(\partial D_2)} \\ = \ & \|\nu \times p \{\Phi(\cdot, x_m) - \Phi(\cdot, x^*)\}\|_{C^{0,\alpha}(\partial D_2)} \to 0, \quad m \to \infty. \end{aligned}$$

Therefore, by the well-posedness of the direct scattering problem with electric boundary condition we have that

$$\lim_{m\to\infty} \nu(x^*) \times [U_2^s(x^*, x_m)p] = \nu(x^*) \times [U_2^s(x^*, x^*)p]. \tag{4.3}$$

On the other hand, from the boundary condition corresponding to the obstacle D_1 we find that

$$|\nu(x^*) \times [U_1^s(x^*, x_m)p]| = |\Phi(x^*, x_m)\nu(x^*) \times p| = \frac{|\nu(x^*) \times p|}{4\pi|x^* - x_m|} \to \infty$$

as $m \to \infty$. This is a contradiction to (4.1) and (4.3). Therefore $D_1 = D_2$ and the proof is complete for the electric boundary condition. The proof for the magnetic boundary condition is obtained in the same way by considering the boundary condition for the normal component of the scattered field and $p = \nu(x^*)$. □

In addition to this uniqueness result for one wave number it is also possible to prove a uniqueness theorem for fixed incident direction and polarization.

Theorem 4.2 *Assume that D_1 and D_2 are two scatterers with electric or magnetic boundary condition such that for plane waves with one fixed incident direction and polarization the far field patterns of both scatterers coincide for all wave numbers contained in some open interval in $(0, \infty)$. Then $D_1 = D_2$.*

The proof is the same as that for the corresponding result on the perfect conductor boundary condition in [3, Theorem 7.2]. For uniqueness with only a few waves we have the following result.

Theorem 4.3 *A convex polyhedron with electric or magnetic boundary condition is uniquely determined by the electric far field patterns for two incident plane waves of the same wave number with two incident directions d_1, d_2 and two linearly independent polarizations p_1, p_2.*

The proof idea is the same as that for the corresponding result on sound soft acoustic scattering in [3, Theorem 5.5]. The condition of the theorem ensures that for each plane P in $\mathbb{R}^3$ at least one of the two plane waves $E^i(\cdot; d_1)p_1$ and $E^i(\cdot; d_2)p_2$ has nonzero tangential component on P in the case of the electric boundary condition. In the case of the magnetic boundary condition it ensures that for each plane P in $\mathbb{R}^3$ for at least one of the two plane waves $\operatorname{curl} E^i(\cdot, d_1)p_1$ and $\operatorname{curl} E^i(\cdot, d_2)p_2$ has nonzero tangential component on P. □

For scattering of time-harmonic acoustic plane waves from a ball centered at the origin the far field pattern depends only on the angle between the incident and the observation direction. By a result due to Karp [4] from 1962 the converse of this property is also true and provides an example for an explicit analytic solution for a nonlinear inverse obstacle scattering problem.

In the case of the vector Helmholtz equation with the electric or magnetic boundary condition if the scatterer D is a ball centered at the origin, it is obvious from symmetry considerations that the far field pattern for incoming plane waves of the form (3.3) satisfies

$$E_\infty(Q\hat{x}, Qd)Qp = QE_\infty(\hat{x}, d)p \tag{4.4}$$

for all $\hat{x}, d \in \mathbb{S}^2$, all $p \in \mathbb{R}^3$ and all rotations Q, i.e., for all real orthogonal matrices Q with $\det Q = 1$. We will show that the converse of this statement is also true. This result extends the corresponding result by Colton and Kress [1] for the Maxwell equations for scattering from a perfect conductor. The proof partly follows the proof of the latter result in [3] and makes it more concise.

Theorem 4.4 *Assume that the far field pattern for the scattering problem with plane wave incidence for the vector Helmholtz equation with electric or magnetic boundary condition satisfies the symmetry relation* (4.4)*. Then the scatterer is a ball centered at the origin.*

Proof. Choosing a fixed but arbitrary vector $p \in \mathbb{R}^3$ and using the Funk–Hecke formula (see [3, p. 36])

$$\int_{\mathbb{S}^2} e^{ik\,x\cdot d} ds(d) = 4\pi\,\frac{\sin k|x|}{k|x|}$$

we consider the superposition of incident plane waves given by

$$\widetilde{E}^i(x,p) := \int_{\mathbb{S}^2} E^i(x,d)p\,ds(d) = 4\pi\,\frac{\sin k|x|}{k|x|}\,p. \tag{4.5}$$

Then, as in the beginning of the proof of Theorem 3.3, the far field pattern of the corresponding scattered wave $\widetilde{E}^s$ is given by the superposition of the far field patterns

$$\widetilde{E}_\infty(\hat{x},p) = \int_{\mathbb{S}^2} E_\infty(\hat{x},d)p\,ds(d).$$

The symmetry condition (4.4) implies that

$$\widetilde{E}_\infty(Q\hat{x},Qp) = Q\widetilde{E}_\infty(\hat{x},p) \tag{4.6}$$

for all $\hat{x} \in \mathbb{S}^2$, all $p \in \mathbb{R}^3$ and all rotations Q.

The vectors $p, p\times\hat{x}$ and $\hat{x}$ form a basis in $\mathbb{R}^3$ provided $p \times \hat{x} \neq 0$. Hence we can write

$$\widetilde{E}_\infty(\hat{x},p) = \gamma_1(\hat{x},p)\,p + \gamma_2(\hat{x},p)\,p\times\hat{x} + \gamma_3(\hat{x},p)\,\hat{x} \tag{4.7}$$

with well defined scalar functions γ_1, γ_2 and γ_3 depending on $\hat{x}$ and p. Since the solution to the scattering problem and correspondingly the far field pattern $\widetilde{E}_\infty$ depends linearly on the polarization p the functions γ_1 and γ_2 do not depend on p and the function γ_3 must be linear in p, that is, $\gamma_3(\hat{x},p) = \widetilde{\gamma}_3(\hat{x})\cdot p$ with some vector function $\widetilde{\gamma}_3$. Finally, the symmetry condition (4.6) implies that γ_1, γ_2 and $\widetilde{\gamma}_3$ are constants and the representation (4.7) simplifies to

$$\widetilde{E}_\infty(\hat{x},p) = c_1\,p + c_2\,p\times\hat{x} + c_3\cdot p\,\hat{x} \tag{4.8}$$

with constant scalars c_1, c_2 and a constant vector c_3 for all $\hat{x} \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$ with $p \times \hat{x} \neq 0$. Because of the linear dependence of $\widetilde{E}$ on the polarization p the equation (4.8) is actually valid for all $\hat{x} \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$ without restriction.

So far we have only exploited the symmetry relation (4.4) and only now will start to make use of the boundary conditions. Since by Rellich's lemma the far field pattern uniquely determines the scattered field, from (4.8) we observe that

$$\widetilde{E}^s(x,p) = c_1\,\frac{e^{ik|x|}}{|x|}\,p + \frac{ic_2}{k}\,\operatorname{curl} p\,\frac{e^{ik|x|}}{|x|} + \frac{c_3\cdot p}{ik}\,\operatorname{grad}\frac{e^{ik|x|}}{|x|}$$

for all $x \in \mathbb{R}^3 \setminus \bar{D}$ and all $p \in \mathbb{R}^3$. Consequently, in view of (4.5), for the total field $\widetilde{E} = \widetilde{E}^i + \widetilde{E}^s$ we have that

$$\widetilde{E}(x,p) = \frac{1}{k}\,\psi(|x|)\,p + \frac{ic_2}{k}\,\operatorname{curl}\varphi(|x|)p + \frac{c_3\cdot p}{ik}\,\operatorname{grad}\varphi(|x|) \tag{4.9}$$

for all $x \in \mathbb{R}^3 \setminus \bar{D}$ and $p \in \mathbb{R}^3$ where we have set

$$\varphi(t) := \frac{e^{ikt}}{t}\,, \quad t > 0,$$

and

$$\psi(t) := 4\pi\,\operatorname{Im}\varphi(t) + c_1 k\varphi(t), \quad t > 0,$$

with the imaginary part $\operatorname{Im}\varphi$ of φ. From this we conclude that

$$\operatorname{div}\widetilde{E}(x,p) = \frac{1}{k|x|}\,\psi'(|x|)\,p\cdot x + ikc_3\cdot p\,\varphi(|x|) \tag{4.10}$$

and

$$\begin{aligned}\operatorname{curl}\widetilde{E}(x,p) = {} & \frac{\psi'(|x|)}{k|x|}\,[x\times p] + \frac{ic_2}{k}\left[k^2\varphi(|x|) + \frac{\varphi'(|x|)}{|x|}\right]p \\ & + \frac{ic_2}{k}\,\frac{|x|\varphi''(|x|) - \varphi'(|x|)}{|x|^3}\,p\cdot x\,x\end{aligned} \tag{4.11}$$

for all $x \in \mathbb{R}^3 \setminus \bar{D}$ and $p \in \mathbb{R}^3$.

From (4.10) and the electric boundary condition $\operatorname{div}\widetilde{E} = 0$ on ∂D we find that

$$\psi'(|x|)\,p\cdot x + ik^2c_3\cdot p\,|x|\varphi(|x|) = 0 \tag{4.12}$$

for all $x \in \partial D$ and $p \in \mathbb{R}^3$. If $c_3 \neq 0$ for a fixed but arbitrary $x \in \partial D$ in (4.12) we may choose p such that $c_3 \cdot p \neq 0$ and $p \cdot x = 0$ to obtain that $\varphi(|x|) = 0$ for all $x \in \partial D$. Since φ is analytic and does not vanish identically it has only discrete zeros and consequently $|x|$ must be constant on ∂D. Hence, D must be a ball centered at the origin. If $c_3 = 0$ then from (4.12) we obtain $\psi'(|x|) = 0$ and the analyticity of ψ again implies that D is a ball centered at the origin.

From (4.11) and the magnetic boundary condition $\nu \times \operatorname{curl} \widetilde{E} = 0$ on ∂D we obtain

$$\begin{gathered}\psi'(|x|)\, \nu(x) \times [x \times p] + ic_2|x| \left[k^2\varphi(|x|) + \frac{\varphi'(|x|)}{|x|}\right] \nu(x) \times p \\ +ic_2\, \frac{|x|\varphi''(|x|) - \varphi'(|x|)}{|x|^2}\, p \cdot x\, \nu(x) \times x = 0\end{gathered} \tag{4.13}$$

for all $x \in \partial D$ and all $p \in \mathbb{R}^3$. For a fixed but arbitrary $x \in \partial D$ we choose $p = \nu(x)$ and take the scalar product of (4.13) with x to find that

$$\left\{|x|^2 - [\nu(x) \cdot x]^2\right\} \psi'(|x|) = 0 \tag{4.14}$$

for all $x \in \partial D$. From (4.9) we observe that the origin $x = 0$ belongs to D. We assume that there are two points x_1 and x_2 in ∂D such that $|x_1| \neq |x_2|$. Then there exists an open subset Ω of ∂D such that for all $x \in \Omega$ we have that x is not perpendicular to the tangent plane at ∂D in x, i.e., x and $\nu(x)$ form an angle different from zero. Therefore, by the Cauchy–Schwarz inequality we have that $|x|^2 - [\nu(x) \cdot x]^2 > 0$ for all $x \in \Omega$. Consequently from (4.14) it follows that $\psi'(|x|) = 0$ for all $x \in \Omega$. Then by the analyticity of ψ' as above we conclude that Ω is a subset of a sphere centered at the origin. This is a contradiction to our assumption on Ω and completes the proof of the theorem for the magnetic boundary condition. □

5 The DB boundary condition

For the Maxwell equations in 1956 Rumsey [11], in addition to the classical boundary value problems for perfect conductors, suggested to consider the problem of finding a solution E, H to (1.5) in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Silver–Müller radiation condition (1.3) and the boundary condition of the form

$$\nu \cdot E = f, \quad \nu \cdot H = g \quad \text{on } \partial D \tag{5.1}$$

with given scalar functions f and g on the boundary ∂D. For simplicity we only consider the case of a simply connected D. (In the case of a multiply connected D with nonzero topological genus for uniqueness of the solution circulations of E and H with respect to a basis of the first homology group of $\mathbb{R}^3 \setminus \bar{D}$ must be prescribed.) We note that by the Maxwell equations and Stokes integral theorem f and g have to satisfy

$$\int_{\partial D} f\, ds = \int_{\partial D} g\, ds = 0 \tag{5.2}$$

as necessary solvability conditions. Uniqueness of a solution was settled in 1970 by Yee [13]. Existence of a solution was established via Hilbert space variational methods by Picard [10] and by boundary integral equation methods in 1986 by the author [6]. For the type (5.1) of boundary conditions more recently for brevity the term DB boundary conditions was introduced by Lindell and Sihvola [8, 9] who also investigated its relations to metamaterials.

Now in the spirit of this paper the exterior DB boundary value problem for the vector Helmholtz equation is to find a radiating solution E in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the boundary conditions

$$\nu \cdot E = f, \quad \nu \cdot \operatorname{curl} E = g, \quad \operatorname{div} E = h \quad \text{on } \partial D \tag{5.3}$$

for given scalar functions f, g and h in $C^{0,\alpha}(\partial D)$. $\operatorname{div} E = 0$ on ∂D implies $\operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by the uniqueness for the exterior Dirichlet problem for the scalar Helmholtz equation. Therefore the pair $E, ikH := \operatorname{curl} E$ solves the Maxwell equation. Hence uniqueness for the DB boundary value problem for the Maxwell equation (see [6, 7, 13]) implies uniqueness for the DB boundary value problem for the vector Helmholtz equation.

Clearly, by Stokes' integral theorem g has to satisfy

$$\int_{\partial D} g\, ds = 0 \tag{5.4}$$

as necessary solvability condition. However, it turns out that as in the Maxwell case the condition

$$\int_{\partial D} f\, ds = 0 \tag{5.5}$$

is also necessary for solvability. We first briefly sketch an existence proof for the case when both (5.4) and (5.5) are satisfied and then show the necessity of (5.5).

For existence we follow [6] and require the knowledge of a tangential vector $c \in C^{1,\alpha}(\partial D)$ such that $\operatorname{Div} c = -g$ which can be constructed in a straightforward manner. Then we consider an auxiliary problem to find a vector field W and a scalar function u both satisfying the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ and the boundary condition

$$\nu \times W = c,$$

$$\nu \cdot W + \frac{\partial u}{\partial \nu} + \int_{\partial D} u \, ds = f,$$

$$\operatorname{div} W - k^2 u = h$$

on ∂D together with the Silver–Müller radiation condition for W and the Sommerfeld radiation condition for u. Uniqueness and existence of a solution to this problem has been established in [6] by an integral equation approach. (In [6] the problem is formulated only for the homogeneous case $h = 0$ but it is obvious that the inhomogeneous case is also solvable by the Riesz theory for compact operators.) Then $E := W + \operatorname{grad} u$ solves the exterior DB boundary value problem.

Now assume that for a triple of functions f, g and h with $\int_{\partial D} f \, ds \neq 0$ the DB boundary value problem has a solution E_1. We set

$$f_0 := f - \frac{1}{|\partial D|} \int_{\partial D} f \, ds$$

and observe that f_0 satisfies (5.5). Therefore we also have a solution E_0 to the DB problem with boundary values f_0, g and h. Then the difference $E := E_1 - E_0$ satisfies $\operatorname{div} E = 0$ on ∂D. As in the above uniqueness proof this implies that the pair $E, ikH := \operatorname{curl} E$ solves the Maxwell DB boundary value problem with boundary values

$$\nu \cdot E = \frac{1}{|\partial D|} \int_{\partial D} f \, ds \neq 0$$

which is a contradiction to the necessary solvability condition (5.2) in the case of the Maxwell equations.

Hence the condition (5.5) is necessary also in the case of the vector Helmholtz equation. This now means that for scattering of plane waves as considered in the two previous sections in order to fulfill (5.5) the polarization p and the propagation d have to be orthogonal. This means we are back to the Maxwell case and this ends our consideration of the DB boundary condition for the vector Helmholtz equation. For results on inverse scattering for the Maxwell equation with DB boundary condition analogous to those of Sections 3 and 4 we refer to [7], except Karp's theorem. Therefore we will finish our analysis by briefly discussing the latter since we have most of the necessary tools already available in Section 4.

For the Maxwell equations incident plane waves are divergence free only if the propagation direction and the polarization are orthogonal. Therefore as in [1, 3] we consider plane waves of the form

$$E^i(x,d)p := \frac{i}{k}\,\mathrm{curl}\,\mathrm{curl}\;e^{ik\,x\cdot d}p = ik\,(d\times p)\times d\,e^{ik\,x\cdot d} \tag{5.6}$$

with $d\in\mathbb{S}^2$ and $p\in\mathbb{R}^3$ and the related superposition

$$\widetilde{E}^i(x,p) = \int_{\mathbb{S}^2} E^i(x,d)p\,ds(d) = \frac{4\pi i}{k^2}\,\mathrm{curl}\,\mathrm{curl}\;\frac{\sin k|x|}{|x|}\;p. \tag{5.7}$$

We also need to modify the basis vectors for $\mathbb{R}^3$ from the proof of Theorem 4.4 into the three vectors $\hat{x}$, $p\times\hat{x}$ and $\hat{x}\times(p\times\hat{x})$ such that $p\times\hat{x}\neq 0$. Hence, since the far field pattern for radiating solutions to the Maxwell equation is orthogonal to $\hat{x}$ (see [3, Theorem 6.9]), we can write

$$\widetilde{E}_\infty(\hat{x},p) = c_1(\hat{x},p)\;p\times\hat{x} + c_2(\hat{x},p)\;\hat{x}\times(p\times\hat{x}) \tag{5.8}$$

with well defined scalar functions c_1 and c_2 depending on $\hat{x}$ and p. As in the proof of Theorem 4.4 from linearity in the polarization p and the symmetry relation (4.4) it follows that c_1 and c_2 are constants again and consequently

$$\widetilde{E}_\infty(\hat{x},p) = c_1\,p\times\hat{x} + c_2\hat{x}\times(p\times\hat{x})$$

for all $\hat{x}\in\mathbb{S}^2$ and $p\in\mathbb{R}^3$. This in turn implies

$$\widetilde{E}(x,p) = \mathrm{curl}\,p\psi(|x|) + \mathrm{curl}\,\mathrm{curl}\,p\chi(|x|) \tag{5.9}$$

for all $x\in\mathbb{R}^3\setminus\bar{D}$ and all $p\in\mathbb{R}^3$ where

$$\psi(t) := \frac{ic_1}{k}\,\frac{e^{ikt}}{t} \quad\text{ and }\quad \chi(t) := \frac{4\pi i}{k^2}\,\frac{\sin kt}{t} + \frac{c_2}{k^2}\,\frac{e^{ikt}}{t}\;.$$

From this we find that

$$\nu(x)\cdot\widetilde{E}(x,p) = \frac{\psi'(|x|)}{|x|}\,[x\times p]\cdot\nu(x) + \left[k^2\chi(|x|) + \frac{\chi'(|x|)}{|x|}\right]\,p\cdot\nu(x)$$
$$+\,\frac{|x|\chi''(|x|)-\chi'(|x|)}{|x|^3}\,p\cdot x\,x\cdot\nu(x).$$

for all $x \in \partial D$ and all $p \in \mathbb{R}^3$ and inserting $p = \nu(x)$ yields

$$\nu(x) \cdot \widetilde{E}(x,p) = \frac{1}{k^2} \left[\chi''(|x|) + k^2\chi(|x|)\right] x \cdot \nu(x)$$

for all $x \in \partial D$. This way from the DB boundary condition we obtain that

$$\left[\chi''(|x|) + k^2\chi(|x|)\right] x \cdot \nu(x) = 0$$

for all $x \in \partial D$. By the Helmholtz equation in polar coordinates this implies

$$\chi'(|x|)\, x \cdot \nu(x) = 0 \tag{5.10}$$

for all $x \in \partial D$.

From the Gauß divergence theorem we have

$$\int_{\partial D} x \cdot \nu(x)\, ds(x) = 3 \int_D dx > 0$$

and therefore the open subset

$$\Omega := \{x \in \partial D : x \cdot \nu(x) > 0\}$$

it not empty. Let $\Gamma \subset \Omega$ be a nonempty connected component. Since χ' as a function of one variable is analytic and does not vanish identically it can have only discrete zeros. Consequently in view of (5.10) we have that $|x| = \text{const}$ for all $x \in \Gamma$, i.e., Γ is a subset of a sphere with center at the origin and positive radius $R > 0$. For $x_0 \in \overline{\Gamma}$ we have a sequence (x_n) in Γ with $x_n \to x_0$ as $n \to \infty$, that is, by continuity we have $x_0 \cdot \nu(x_0) = R > 0$. Therefore the subset $\Gamma \subset \partial D$ is also closed whence $\Gamma = \partial D$ and the proof for the following final theorem is complete.

Theorem 5.1 *Assume that the far field pattern for the scattering problem with plane wave incidence for the Maxwell equations with DB boundary condition satisfies the symmetry relation* (4.4)*. Then the scatterer is a ball centered at the origin.*

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