Projects "Automorphic forms and applications"

Project 1. Give detailed proofs for Theorem 3 and three other results stated in class without proof.

Project 2. State and prove a selection of analytic properties of the Bessel function $J_k(x)$, in particular bounds, asymptotic formulae (ideally uniformly in k), integral representations etc.

Project 3. We showed in class

$$\sum_{n\leq X} a(n)e(\alpha n) \ll X^{k/2}\log X$$

for Fourier coefficients of a weight k cusp form and any $\alpha\in\mathbb{R}$. Now assume that $\alpha = a/q$ is a *rational* number in lowest terms and f has level 1, i.e., is a cusp form of $SL_2(\mathbb{Z})$. Use the transformation properties of the cusp form f (in particular by the matrix $\begin{pmatrix}\bar a & (1-a\bar a)/q\\ -q & a\end{pmatrix}$ for a fixed representative $\bar a$) to prove stronger estimates (ideally uniformly in q for q not too big in terms of X).

Project 4. [Character sums:] a) While Kloosterman sums to prime modulus cannot be evaluated (in general), a twisted version is easier to handle: Let p be an odd prime, $m,n\in\mathbb{Z}$ with n coprime to p. Define the Salié sum

$$T(n,m;p) := \sum_{\substack{d\,(p)\\ (d,p)=1}} \left(\frac{d}{p}\right) e\left(\frac{m\bar d + nd}{p}\right).$$

These sums play a crucial role in the theory of half-integral weight modular forms. Show

$$T(n,m;p) = \left(\frac{n}{p}\right)\varepsilon_p\sqrt{p}\sum_{y^2\equiv nm\,(p)} e\left(\frac{2y}{p}\right)$$

where as usual $\varepsilon_p = 1$ if $p\equiv 1 \pmod 4$ and $\varepsilon_p = i$ otherwise.
b) Fix $r,n,k\in\mathbb{N}$, $r\geq 2$. Let p be an odd prime, $q=p^r$. As usual let $\tau(\chi)$ denote the Gauß sum associated to a character χ mod q. Show

$$\sum_{\chi\,(q)} \bar\chi(k)\tau(\chi)^n \ll q^{(n+1)/2}.$$

Conclude that

$$\sideset{}{^*}\sum_{\chi\,(q)} \tau(\chi)^n \ll q^{(n+1)/2}$$

where the sum is only over *primitive* characters.

Project 5. Come up with your own interesting question and solve it.