

# Counting rational points. Problem sheet 1

Let  $k \in \mathbb{N}$ ,  $k \geq 2$ . Show that there is a positive number  $C_k$  such that

$$\#\{(x,y) \in \mathbb{N}^2 : x^k + y^k \leq N\} = C_k N^{2/k} + O(N^{1/k}).$$

Can you compute or interpret  $C_k$ , for example as an area?

Apply the main result of Lecture 1 to conclude the number  $\Xi(N)$  of solutions to

$$(x^k + y^k)(z^k + w^k) \leq N \quad (1)$$

in natural numbers  $x, y, z, w$  satisfies the asymptotics

$$\Xi(N) = \Gamma N^{2/k} (\log N + E) + O(N^{2/k-\delta}) \quad (2)$$

for suitable real numbers  $\Gamma, E, \delta$  with  $\Gamma > 0, \delta > 0$ .

Express  $\Gamma$  in terms of  $k$  and  $C_k$ .

Now modify the definition of  $\Xi(N)$  to now count the solutions in natural numbers of (1) with the additional coprimality constraints

$$(x; y) = (z; w) = 1.$$

Establish an asymptotic formula similar to (2). Can you say how the new constant  $\Gamma$  is related to the original one?