The notation will be the same as in the lectures.

Project 1. Let $\mathbb{I} = (-[1])^{1/(q-1)}$ be the Carlitz imaginary unit and $\tilde{\pi}$ be the Carlitz-π. During the lectures we proved that there is the following decomposition formula

$$\sum_{i=0}^{\infty} \frac{x^{q^i}}{D_i} = x \prod_{\substack{a \in \mathbb{I}\tilde{\pi}A \\ a \neq 0}} \left(1 - \frac{x}{a}\right).$$

Now let Λ be a lattice of rank $d \geq 1$ in $\widehat{C}$. We have associated an exponential function to Λ

$$e_\Lambda(x) = x \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{x}{\lambda}\right)$$

and proved that

$$e_\Lambda(x) = \sum_{i=0}^{\infty} c_i x^{q^i}.$$

The goal of this project is to find some lattices of rank 2 for which the coefficients c_i are given by compact formulas as in the case of the Carlitz exponential. Also, find some conditions under which the coefficients c_i are "rational", i.e., lie in F.

Project 2. Study the endomorphism rings of Drinfeld modules over F of rank 2 and 3. What kind of orders in the extensions of F one obtains? Find explicit examples of Drinfeld modules having the largest possible endomorphism rings; these will be the rings of integers in "imaginary" extensions of F. (A field extension L of F is said to be imaginary if $1/T$ does not split in L; the ring of integers of L is the integral closure of A in L.) In some of the examples determine the image of the Galois representation $\mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{GL}_d(A/\mathfrak{p})$ obtained from the action on $\varphi[\mathfrak{p}]$.

Project 3. Let K be a field and $f(x) \in K[x]$ be a monic irreducible polynomial of degree d. It is a well-known fact from linear algebra that all matrices in $\mathrm{Mat}_d(K)$ with characteristic polynomial f are conjugate in $\mathrm{Mat}_d(K)$.

We have seen that the group $\mathrm{GL}_d(A)$ arises naturally in the problem of classifying isomorphism classes of rank d Drinfeld modules over $\widehat{C}$. In some problems it is important to know whether two matrices in $\mathrm{GL}_d(A)$ with the same irreducible characteristic polynomial are actually conjugate in $\mathrm{GL}_d(A)$, not just $\mathrm{GL}_d(F)$. When A is replaced by $\mathbb{Z}$, this problem was solved by Latimer and MacDuffee in the early 1930's. It turns out that there is a bijection between the $\mathrm{GL}_d(\mathbb{Z})$-conjugacy classes of matrices with characteristic polynomial f and $\mathbb{Z}[\alpha]$-ideal classes in $\mathbb{Q}(\alpha)$, where α is a root of f. (Keith Conrad has a nice expository paper about this problem with lots of examples: "Ideal classes and matrix conjugation over $\mathbb{Z}$". It is available on his webpage.)

The goal of this project is to prove the same theorem for $\mathrm{GL}_d(A)$ and to give some explicit examples.

Project 4. Let $\Gamma := \Gamma_1(\mathfrak{n})$. Let $\alpha \in \Gamma$, and fix an arbitrary point $\omega \in \Omega = \widehat{C} - \widehat{F}$. Consider the function on Ω:

$$u_\alpha(z) = \prod_{\gamma \in \Gamma} \left(\frac{z - \gamma\omega}{z - \gamma\alpha\omega} \right).$$

We have seen that this function comes up in the parametrization problem of elliptic curves by Drinfeld modular curves. It is not too hard to prove that u_α is locally uniformly convergent. This means that for a fixed z_0 the product $u_\alpha(z_0)$ converges in $\widehat{C}$ and does not depend on the order in which one multiplies the terms $\left(\frac{z-\gamma\omega}{z-\gamma\alpha\omega}\right)$. For actual explicit calculations it is important to know how fast the product $u_\alpha(z_0)$ converges. The purpose of this project is to come up with an explicit estimate on the rate of convergence when we multiply $\left(\frac{z-\gamma\omega}{z-\gamma\alpha\omega}\right)$ with respect to some natural ordering of elements of Γ, e.g., according to the maximum of the degrees of entries of the matrix.