

# PARABOLIC IMPLOSION A MINI-COURSE

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For the first three sections, the reader may refer to [M], though conventions may be different. For the others, see [D].

### 1. PARABOLIC POINTS

A parabolic point is a periodic point of a one dimensional complex dynamical system with multiplier a root of unity.

For a local study, we may assume  $f$  is defined near the origin in  $\mathbb{C}$  and  $f(z) = \rho z + \dots$  with  $|\rho| = 1$ :

$$\rho = e^{2\pi i p/q}$$

with  $p \wedge q = 1$ . Then

$$f^q(z) = f \circ \dots \circ f(z) = z + \dots$$

**Lemma.** *If  $f$  is a polynomial or a rational map of degree  $> 2$ , or a transcendental entire function, then  $f^q \neq \text{id}$  in any neighborhood of 0.*

*Proof.* For otherwise, by analytic continuation we would have  $f^q = \text{id}$  everywhere, contradicting the fact that the preimage of most points consists in more than one point. □

We now assume that  $f^q \neq \text{id}$ . Let  $Cz^k$  be the next term in the expansion of  $f^q$ :

$$f^q(z) = z + Cz^k + \dots$$

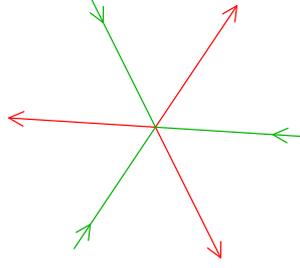
We define attracting axes as the  $k - 1$  half lines whose union is the solution to

$$Cz^k/z \in ]-\infty, 0[$$

and the repelling axes are for

$$Cz^k/z \in ]0, +\infty[.$$

This notion is not invariant by conjugacy of  $f$  by a change of variable, but this problem is solved by considering the axes as living in the tangent space to  $\mathbb{C}$  at the origin.



**Lemma.**  $k = 1 + mq$  for some  $m > 0$ .

*Proof.* Analytic:  $Df_0$  permutes the attracting directions (because  $f$  and  $f^q$  commute). Algebraic: write  $f \circ f^q = f^q \circ f$  whence  $f + \rho Cz^k + h.o.t. = f + C\rho^k z^k + h.o.t.$ .  $\square$

More about the dynamics:

$$f^q(z) = z + s(z), \text{ where } s \text{ may be called the (foot)step.}$$

Here  $s(z) = \mathcal{O}(z^2)$  and there is a heuristic principle that if  $s(z)$  varies slowly enough when  $z$  varies, then  $f$  is comparable to the vector field  $dz/dt = s(z)$ . Of course this has to be made more precise and depends on the situation. Here this serves as a motivation to the following: since  $s(z) \approx Cz^k$ , we will compare  $f$  to the vector field  $dz/dt = Cz^k$ . Let

$$r = k - 1.$$

Then  $r \geq 1$  and the straightening coordinates of the latter v.f. are

$$u = \frac{-1}{rCz^r}.$$

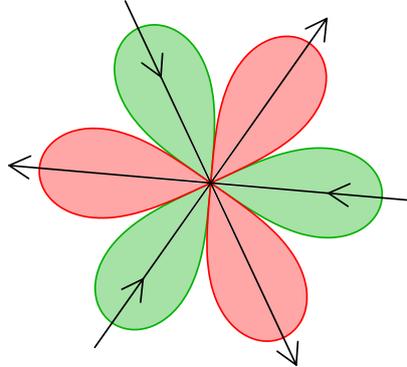
We can call  $u$  the *prepared coordinates* or the *pre-Fatou coordinate*. If

$$z_{n+1} = f^q(z_n)$$

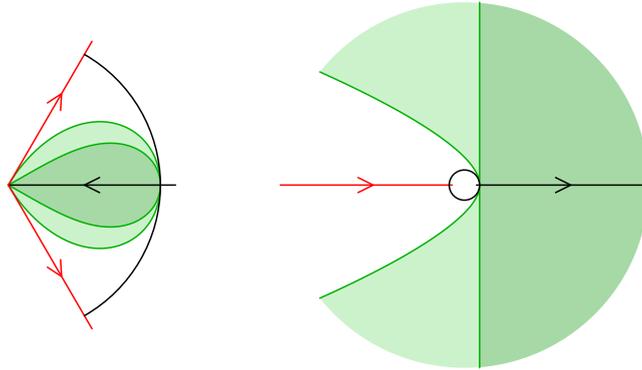
and  $u_n = -1/rCz_n^r$  then

$$u_{n+1} = u_n + 1 + o(|u_n|^{-1/r}).$$

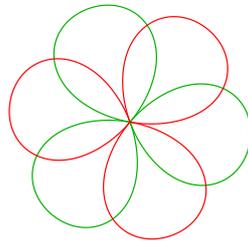
From this it is easy to construct traps: *petals*. (There is no unified definition of petals.) For instance a right half-plane in the  $u$  coordinates, which I call *small petals*. This builds  $r$  attracting petals for  $f$ . Repelling petals are constructed similarly using  $f^{-1}$ .



But one can draw bigger petals. For instance  $\alpha$ -petals, which are sets whose image by the change of variable  $u = -1/rCz^r$  is a sector of angle  $\alpha$ , and whose bisector line is horizontal. There is also what I call big petals, whose image are bounded by a curve with parabolic branches with horizontal asymptotic direction.

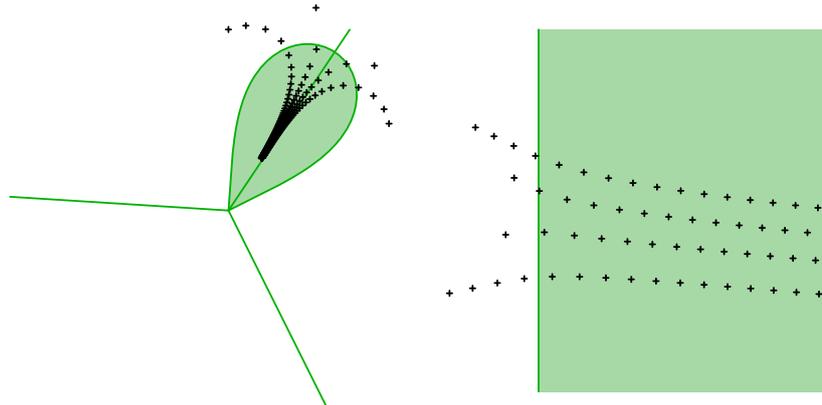


The next picture, illustrates how big petals may overlap.



These definitions of petals are not invariant by conjugacy. An more canonical definition of petals could fix their shape in Fatou coordinates (see below) but requires Fatou coordinates to be already defined, which is not the choice made in this mini-course.

In a petal,  $u_n \sim n$  thus  $|z_n| \sim \frac{L}{n^{1/r}}$  with  $L = \left| \frac{1}{rC} \right|^{1/r}$  and  $z_n \rightarrow 0$  tangentially to the corresponding attracting axis, as on the following picture showing four different orbits under  $f^q$  for  $f(z) = e^{2\pi i/3}z + z^2$  attracted by the same petal and their image in the prepared coordinate  $u$ .



Note also that:

**Lemma.** *Any orbit tending to a parabolic point is captured by an attracting petal.*

## 2. FATOU COORDINATES

There exists holomorphic injective maps  $\phi_-$  and  $\phi_+$  defined on each petal that “conjugate”  $f^q$  to the translation  $T : z \mapsto z + 1$ . One has to be a little careful

because writing  $\phi \circ f^q = T \circ \phi$  would not be correct, as both sides have different sets of definition, and thus are different as functions. What is true is that  $\phi \circ f^q(z) = T \circ \phi(z)$  holds for all  $z$  for which both sides are defined, or equivalently that  $\phi$  conjugates the bi-restriction  $f^q|_P$  to the restriction of  $T$  to some open set.

Their asymptotics can be computed: in particular

$$\phi(u) \sim u.$$

**Remark:** In fact an asymptotic series expansion exists, of the form

$$\phi(z) = \frac{-1}{rCz^r} + \frac{a_{-r+1}}{z^{r-1}} + \cdots + \frac{a_{-1}}{z} + a_0 \log z + c + a_1 z + a_2 z^2 + \cdots,$$

which is asymptotically correct at all finite order (but divergent), and it is the same for all petals (be it attracting or repelling).

A corollary of  $\phi(u) \sim u$  is the form of the image of the petal: it will not be too small. For instance the image of a small petal contains sectors of angle  $\alpha$  arbitrarily close to  $\pi$ .

Another is that the quotient of the petal by identification of  $z$  and  $f^q(z)$  is a bi-infinite cylinder, i.e. isomorphic to  $\mathbb{C}/\mathbb{Z}$  as a Riemann surface. From this one can prove the uniqueness of Fatou coordinates up to addition of a constant. (More precisely: call admissible all forward invariant open set whose points all have  $f^q$ -orbits captured by the petal and such that all points in the petal have an orbit captured by the set. Then the intersection of two admissible sets is admissible, a petal is admissible, and any two Fatou coordinates on admissible sets differ by a constant.) The choice of one is called a *normalization*.

Let us extend  $\phi_-$  to the whole basin of the petal under  $f^q$ , such that

$$\phi_- \circ f^q = T \circ \phi_-$$

here the domains of definition are the same. There is a unique map doing this. It is not as easy to extend  $\phi_+$  so

- (1) one solution is to restrict to a neighborhood of 0 where  $f^q$  is bijective and to work with  $f^{-1}$
- (2) the other solution is to work with the reciprocal of  $\phi_+$

Let the map  $\psi_+ = \phi_+^{-1}$  and extend it the same way as  $\phi_-$ , i.e. so that

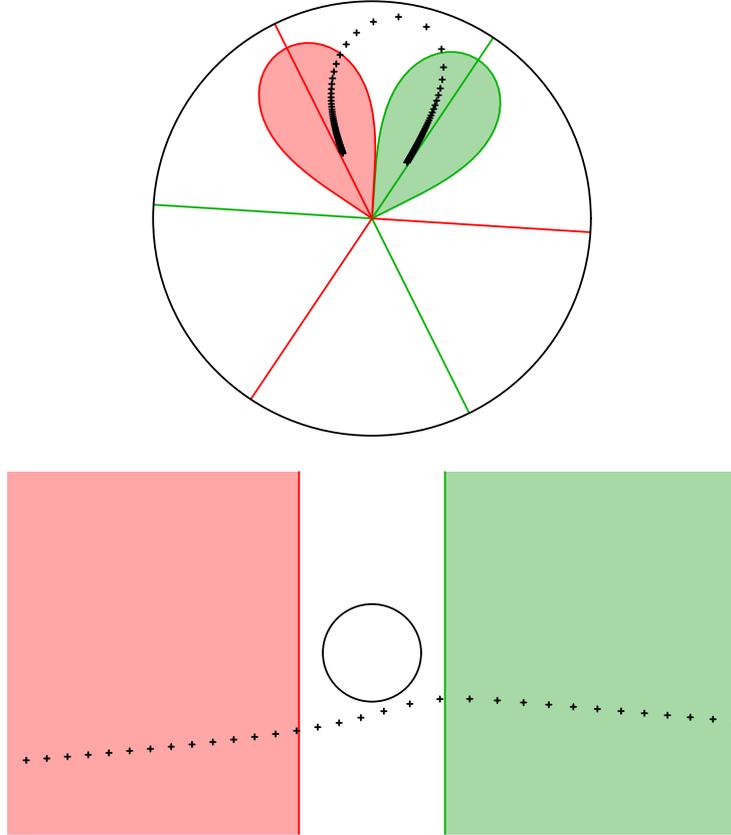
$$f^q \circ \psi_+ = \psi_+ \circ T$$

where the domains of definition are required to be the same. Again the solution is unique. If  $f$  is entire or rational, then  $\text{dom}(\psi_+) = \mathbb{C}$ .

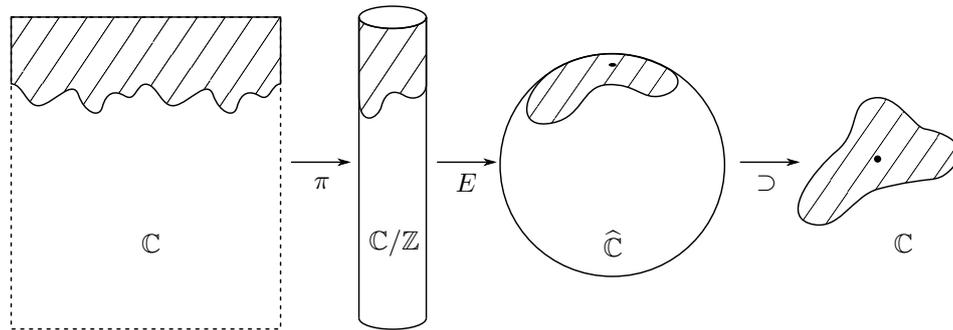
Then consider an adjacent pair of repelling and attracting petals, and define the map  $h = \psi_+ \circ \phi_-$ . This is the *extended horn map* associated to this pair. It commutes with  $T$ , its domain of definition is invariant by  $T$  and contains an upper or a lower half plane (both if  $q = 1$ ) on which it is injective and satisfies  $h(z) \sim z$  as  $\text{Im}(z) \rightarrow \pm\infty$ . If  $q \neq 1$  its domain of definition is disjoint from some lower or upper half plane. All this can be deduced from the fact that  $\phi(u) \sim u$  on big petals.

Points  $w$  where  $h$  is defined correspond to those bi-infinite orbits of  $f^q$  (i.e. sequences  $z_n$  indexed by  $\mathbb{Z}$  and such that  $\forall n \in \mathbb{Z}, z_n \in \text{dom}(f^q)$  and  $z_{n+1} = f^q(z_n)$ ) which tend to 0 in the chosen attracting petal as  $n \rightarrow +\infty$  and tend to 0 in the chosen repelling petal as  $n \rightarrow -\infty$ : given  $w$  let  $z_n = \psi_+(w + n)$ , and conversely given  $z_n$  let  $w = \phi_+(z_n) - n$  for  $n$  close enough to  $-\infty$ . Moreover  $v = h(w)$  implies that for  $n$  close enough to  $-\infty$ ,  $z_n$  is in the repelling petal and its repelling Fatou coordinate  $\phi_+(z_n)$  is equal to  $w$  modulo  $\mathbb{Z}$  and that for  $n$  close enough to  $+\infty$ ,  $z_n$  is in the attracting petal and its (unextended) attracting Fatou coordinate is equal to  $v$  modulo  $\mathbb{Z}$ . This is illustrated on the next picture, where the upper

part represents a finite portion of some bi-orbit and the lower part its image in the prepared coordinates  $u = -1/rCz^r$ . Note also that



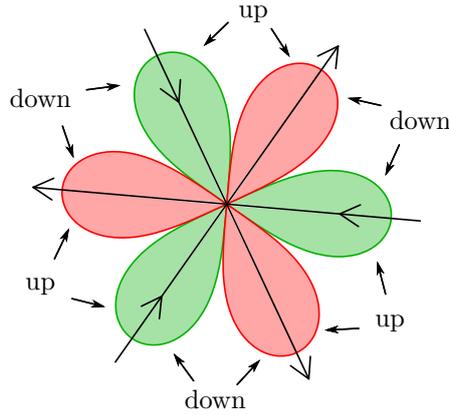
We may identify the cylinder  $\mathbb{C}/\mathbb{Z}$  with  $\mathbb{C}^*$  by means of the map  $E : z \mapsto e^{i2\pi z}$ . This and the inclusion  $\mathbb{C}^* \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  allows to extend the Riemann surface  $\mathbb{C}/\mathbb{Z}$  into a compact Riemann surface: this adds two points to the cylinder, called the upper end and the lower end.



The map  $h$  commutes with  $T$  hence it induces a quotient map  $\tilde{h}$  defined on an open subset of the cylinder and mapping to the cylinder:  $\tilde{h} \circ \pi = \pi \circ h$ . Let us work with the case  $\text{dom}(h)$  contains an upper half plane, the other case being similar. From  $h(z) \sim z$  as  $\text{Im}(z) \rightarrow +\infty$  it follows that  $\tilde{h}$  has a removable singularity at the upper end of the cylinder. In the  $\mathbb{C}^*$  coordinates,  $E \circ \tilde{h} \circ E^{-1}$  has thus a

Laurent series expansion of the form  $\sum_{k \geq k_0} a_k z^k$  with  $k_0 \in \mathbb{Z}$  and  $a_{k_0} \neq 0$ . This implies that for  $|\operatorname{Im} z|$  big enough,  $h(z)$  has the form  $h(z) = k_0 z + g(e^{\pm 2\pi i z})$  where  $g$  is holomorphic in a neighborhood of 0. Whence  $h \circ T = T_{k_0} \circ h$ . Thus  $k_0 = 1$ . Thus the extension of  $\tilde{h}$  fixes the upper end and has a non-vanishing derivative there.

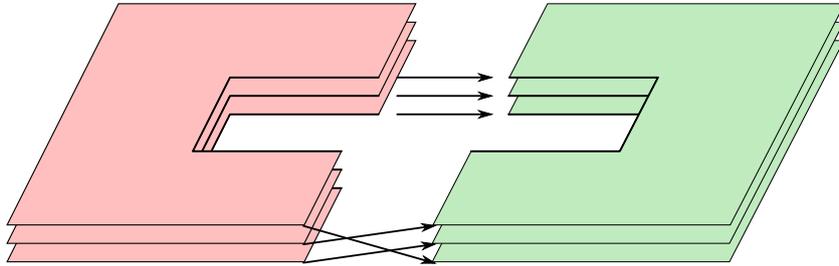
If  $q > 1$  there are  $2r$  horn maps, because there are  $2r$  pairs of adjacent petals, half of them give horn maps with an extension at the upper end, half at the lower end.



The effect of a change of normalization of the  $2r$  Fatou coordinates is to post and pre compose these horn maps by  $2r$  translations, in correspondence with the definition of the horn maps. Be careful: changing one Fatou coordinate will change two horn maps. Details are left to the reader.

The germs of these  $2r$  maps  $\tilde{h}$  at their respective end of the cylinder, modulo post and pre composition by translations are invariant by conjugacy of  $f$  under a change of variable. But better can be said: a *complete conjugacy invariant* is given by the data of the  $2r$  “germs of lifts”  $h$ , modulo the action of  $2r$  translations by pre and post conjugacy, in correspondence with the definition of the horn maps. By germ of lift, we mean an equivalence class of map of the form  $z + \sum_{n \geq 0} a_n e^{\pm n 2\pi i z}$ .

It is equivalent to the data of a sequence  $a_n \in \mathbb{C}$  such that  $\sum a_n z^n$  has positive radius of convergence. Note that since  $f$  commutes with  $f^q$ , the horn maps are not independent. The interested reader should work out the details.



Conversely, given such “germs of lifts”, one reconstructs a map with a parabolic fixed point whose horn maps correspond to these germs: as illustrated above, consider  $r$  copies of an open set, call them attracting regions, union of an upper a lower and a right half plane, and  $r$  copies of another open set, call them repelling regions,

union of an upper, a lower and a left half plane. Now glue them together using the germs of to be horn maps, as illustrated. This gives a Riemann surface, and it can be shown to be isomorphic to a bounded open subset  $U$  of  $\mathbb{C}$  with the origin removed, the latter corresponding to infinity in the regions. Consider on them the dynamical system given by the translation by 1: since the horn maps commute with  $T$ , it is well defined and yields on  $U$  a dynamical system with a removable singularity at the origin. One can prove that the origin is parabolic with  $r$  petals and that its horn maps have the germs we started from. This map has rotation number 0. To get a map with a different rotation number, one must modify the construction: permute the copies according to a rotation of angle  $p/q$  and on one repelling region only and one attracting region only, compose this with  $T$ .

### 3. PERTURBATION OF PARABOLIC POINTS

Let  $f$  belong to an analytically varying family of analytic maps  $f_\varepsilon$  with  $\varepsilon \in \mathbb{C}$  close to 0, such that  $f = f_0$ .

We will assume in the rest of the article that

$$m = 1$$

i.e.  $r = q$ . Then it is easier to describe the explosion of the fixed point and the phase portrait of the dynamical system, which is still comparable to a vector field.

This situation is generic. Moreover, using:

**Theorem (Fatou).** *If  $f$  is a rational map, each cycle of petal must attract a critical point.*

*Proof.* Otherwise, prove that  $\phi_-$  is a bijection from the immediate basin of any petal in this cycle, to  $\mathbb{C}$ , which is impossible since the basin is disjoint from the Julia set which is bigger than a single point.  $\square$

we get that for degree 2 polynomials, we necessarily have  $m = 1$ .

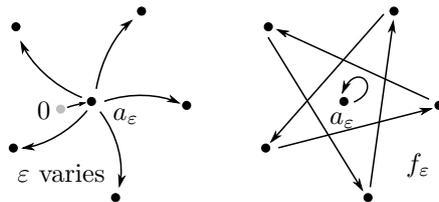
Since  $f^q(z) - z$  has a multiple zero at the origin, with multiplicity  $1 + q$ , this implies that  $f_\varepsilon$  will also have  $1 + q$  zeros counted with multiplicity near the origin by Hurwitz's theorem. We now need to distinguish two cases:

Case  $q > 1$ :

A fixed point of an analytic family of holomorphic maps can be locally holomorphically followed as soon as its multiplier is  $\neq 1$ . This can be proved using the implicit function theorem on the equation  $f_\varepsilon(z) - z = 0$ . One needs to check that the partial derivative  $\partial/\partial z$  does not vanish at the considered point  $(z, \varepsilon)$  solution of the equation, and this quantity is precisely equal there to the multiplier minus one.

This is the case here for  $(z, \varepsilon) = (0, 0)$  because  $q > 1$ . Call  $a_\varepsilon$  the corresponding fixed point of  $f_\varepsilon$  and  $\rho_\varepsilon$  its multiplier.

We now make the supplementary assumption that  $\rho_\varepsilon$  is not constant, so that  $a_\varepsilon$  is a simple root of  $f_\varepsilon^q(z) - z$  for  $\varepsilon \neq 0$  small enough. Since  $f_\varepsilon$  is close to a rotation of angle  $p/q$ , it follows that the other  $q$  roots will form a cycle of period  $q$ , that sits approximately on a regular  $q$ -gon centered on  $a_\varepsilon$ .



The distance of this cycle to  $a_\varepsilon$  is equivalent to  $\left|\frac{1-\rho_\varepsilon^q}{C}\right|^{1/q}$  when  $\varepsilon \rightarrow 0$ .

*Proof.* Compute  $\int \frac{(z - a_\varepsilon)^q}{f_\varepsilon(z) - z} dz$  and  $\int \frac{1}{f_\varepsilon(z) - z} dz$  around a small circle.  $\square$

So generically, this distance is  $\sim \text{cst} |\varepsilon|^{1/q}$ .

Case  $q = 1$ :

If  $q = 1$  then the double fixed point 0 of  $f_\varepsilon$  usually splits into two fixed points. We will make the assumption that this is the case. These points separate at some speed, generically  $\sim \text{cst} |\varepsilon|^{1/2}$ .

**Note:** the assumptions made in the case  $q > 1$  or in the case  $q = 1$  can often be proved to hold: for instance if one considers a parabolic periodic point in the family  $P_c(z) = z^2 + c$  (for which we recall that we always have  $m = 1$ ), with period  $k$ , and let  $f_\varepsilon(z) = P_{c_0+\varepsilon}^k(z_0 + z) - z_0$  then it can be proved that the multiple fixed point is not persistent if  $q = 1$  and that the multiplier of the holomorphically followable fixed point  $a_\varepsilon$  varies if  $q > 1$ . The proof consists in realizing that when  $c$  is big enough, all periodic points are repelling, and in making use of analytic continuation in the algebraic variety defined by  $P_c^k(z) - z = 0$ . It is also interesting to know that for the family  $P_c(z) = z^2 + c$ , the cycle explodes at the generic speed ( $\sim \text{cst} |\varepsilon|^{1/q}$  if  $q > 1$ ,  $\sim \text{cst} |\varepsilon|^{1/2}$  if  $q = 1$ ). However the proof is more complicated. See [O].

In both cases we can reduce to the study of a family of the form

$$f_\varepsilon = \rho(\varepsilon)z + \dots$$

Indeed: If  $q > 1$  it is easy by a translation on  $z$  by  $-a_\lambda$ . If  $q = 1$  it is still possible but we may need to change the parameter:  $\varepsilon = \eta^2$  (this will change the speed of explosion of the double fixed point). Recall that we are assuming  $m = 1$ .

#### 4. PERSISTENCE OF FATOU COORDINATES: PASSING THROUGH THE EGGBEATER

**Remark.** From now on these notes take a more expository form.

Assume that

$$f_\varepsilon = \rho_\varepsilon z + \dots$$

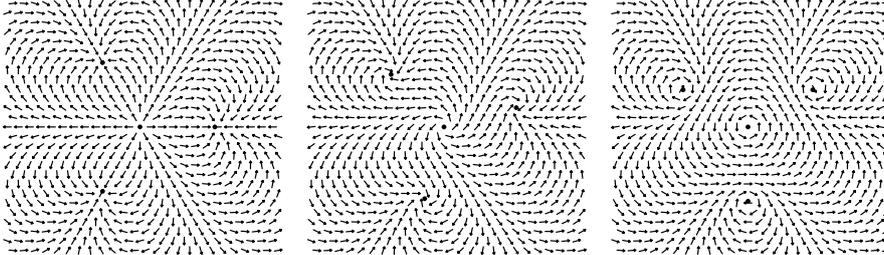
(it is still assumed analytically varying with  $\varepsilon$ ). The dynamics is comparable to the vector field

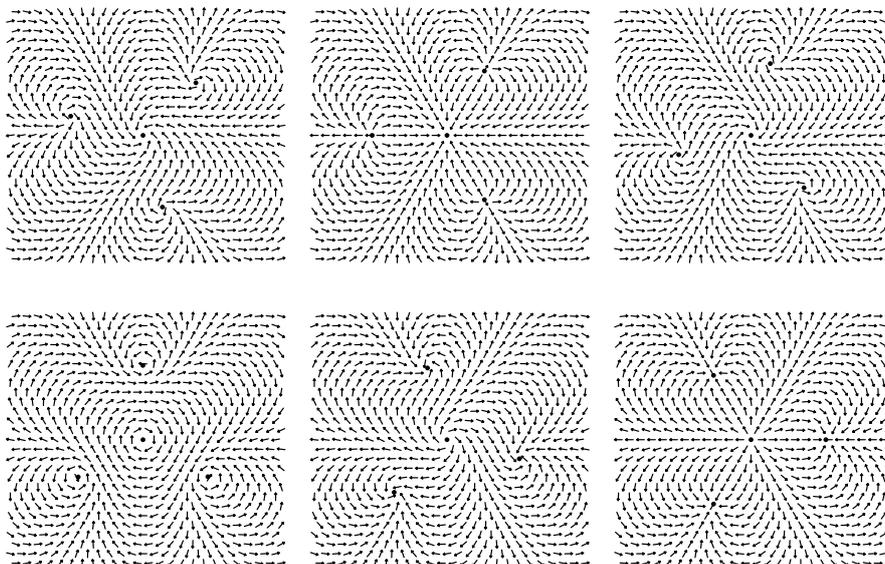
$$dz/dt = f(z) - z$$

which is itself comparable to

$$dz/dt = (\rho^q - 1)z + Cz^{q+1}.$$

On the next pictures, we illustrate the direction field of the latter vector field, for  $\rho$  varying along a small circle centered on  $\rho = 1$ .

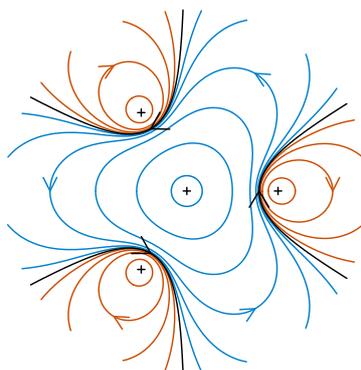




The ideal vector field  $dz/dt = (\rho^q - 1)z + Cz^{q+1}$  is the pull back by  $z^q$  of a simpler vector field: if  $v = z^q$  then  $dv/dt = qv(\rho^q - 1 + Cv)$ , which has only two fixed points: 0 and another one  $v_0 = (1 - \rho^q)/C$ . A further change of coordinates  $w = v/(v - v_0)$  transforms this vector field into the very simple:

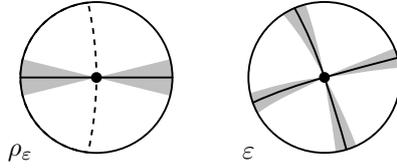
$$dw/dt = \lambda w$$

with  $\lambda = q(\rho^q - 1)$ . From this, one easily understands the dynamics of the ideal vector field. This is left as an exercise. The following picture shows field lines in the case  $\text{Re}(\rho^q - 1) = 0$ . The black curves are those who join infinity in finite time.



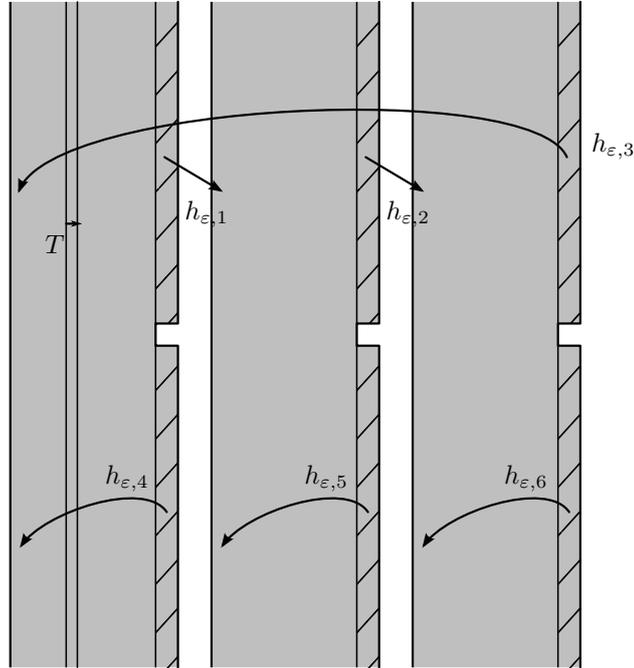
*Recomposition of the dynamics of  $f_\varepsilon$  by translation bands and gluings:*

The description of the local dynamics of small perturbations depends on the argument of  $\rho_\varepsilon^q - 1 \sim q(\rho_\varepsilon - 1)$ . Here we will avoid that  $\rho_\varepsilon \rightarrow 1$  tangentially to the real axis. So we will assume  $\rho_\varepsilon$  close to 1 but  $\arg(\rho_\varepsilon - 1)$  not too close to 0 or  $\pi$ . This means  $\varepsilon$  is close to 0 but avoids a finite number of small sectors. Another case, largely overlapping the first one, is when  $\arg(\rho_\varepsilon - 1)$  is not too close to  $\pi/2$  or  $-\pi/2$ . However we will only treat the first case. The treatment of the second one similar, with everything rotated 90 degrees.



Example where  $\rho_\varepsilon$  and  $\varepsilon$  must avoid the gray regions.  
The dashed line represents  $|\rho| = 1$

Then  $f_\varepsilon$  is conjugated to a model consisting of vertical bands plus tabs, on which  $f$  is a translation and that are glued along the tabs by maps close to the horn maps, as on the following picture:

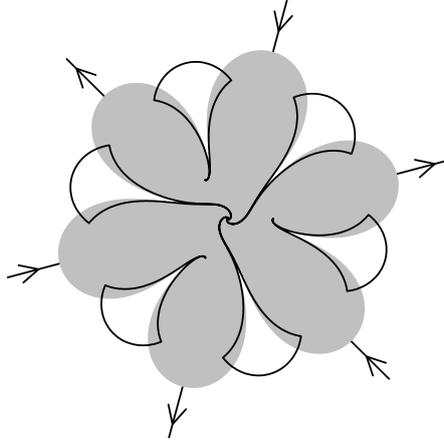


There is an inverse Fatou coord  $\psi_\varepsilon$  on the bands, mapping them to a heart shaped set, tending to a pair of adjacent small petals as  $\varepsilon \rightarrow 0$ , avoiding the bad sectors. See the next picture.

As  $\varepsilon \rightarrow 0$ , avoiding the bad sectors, the width of the bands tends to  $+\infty$  there are complex numbers  $a(\varepsilon)$  and  $b(\varepsilon)$ , which depend which band is concerned, such that

$$\begin{aligned} \psi_\varepsilon(z - a(\varepsilon)) &\longrightarrow \phi_-^{-1}(z) \\ \psi_\varepsilon(z - b(\varepsilon)) &\longrightarrow \phi_+^{-1}(z) \\ b(\varepsilon) - a(\varepsilon) &\underset{\varepsilon \rightarrow 0}{=} \frac{2\pi i}{\rho'_\varepsilon - 1} + \text{cst} + o(1) \end{aligned}$$

This is valid for  $\text{Im}(\rho) > 0$ , otherwise the sign in the last equation should be the opposite. The constant depends on the normalization of the Fatou coordinates.



The gluings are not exactly those on the former picture but almost: they are given by

$$z + a(\varepsilon) \sim h_\varepsilon(z + b(\varepsilon))$$

for  $2r$  different maps  $h_\varepsilon$  and the functions  $a$  and  $b$  of the corresponding bands being glued together (here we omit the indices on  $a$ ,  $b$ , and  $h_\varepsilon$  indicating which one these are). Each map  $h_\varepsilon$  commutes with  $T_1$  and tends uniformly to the corresponding horn map  $h$ .

Now

$$\frac{2\pi i}{\rho^q - 1} + q \frac{2\pi i}{\rho' - 1} \longrightarrow 2i\pi A$$

where  $A = \frac{1}{2\pi i} \int \frac{dz}{f^q(z) - z}$  is called the holomorphic index. Therefore

$$b(\varepsilon) - a(\varepsilon) \underset{\varepsilon \rightarrow 0}{=} -\frac{2\pi i/q}{\rho^q - 1} + \text{cst} + o(1)$$

(use the opposite sign if  $\text{Im}(\rho) < 0$ ).

### 5. LAVAURS THEOREM

Under the same assumptions, choose a sign for  $\text{Im}(\rho)$ , choose an attracting petal and the adjacent repelling petal corresponding to the chosen sign of  $\text{Im}(\rho)$ . Consider the corresponding extended Fatou coordinates  $\phi_-$  and  $\psi_+$ . Let

$$g_\sigma = \psi_+ \circ T_\sigma \circ \phi_-.$$

It is called the *Lavaurs map*.

Consider the following *non-commutative* diagram. Here,  $\mathcal{S}$  is the Riemann surface on which the dynamical system takes place. This could be  $\widehat{\mathbb{C}}$  for a rational map, or  $\mathbb{C}$  for an entire map, or something else.

$$\begin{array}{ccc} & \mathcal{S} & \\ \phi_- \swarrow & & \nwarrow \psi_+ \\ \mathbb{C} & \xrightarrow{T_\sigma} & \mathbb{C} \end{array}$$

The map  $g_\sigma$  consists in turning once around the diagram, starting from the top.

Let  $\varepsilon_n \rightarrow 0$  and denote  $f_n = f_{\varepsilon_n}$ ,  $\rho_n = \rho_{\varepsilon_n}$ , etc... Consider the following quantity, called the *phase*<sup>1</sup>

$$\sigma_n \stackrel{\text{def}}{=} \frac{2i\pi/q}{1 - \rho_n^q}$$

<sup>1</sup>the definition of the phase may differ in other works

(take the opposite complex number if  $\text{Im}(\rho) < 0$ ). Assume that the  $\sigma_n$  has a real part tending to  $+\infty$  but converges modulo  $\mathbb{Z}$ . So

$$\sigma_n \underset{n \rightarrow +\infty}{=} N_n + \sigma + o(1)$$

for some integer sequence  $N_n \rightarrow +\infty$ .

**Theorem** (Lavaurs). *Then*

$$f_n^{qN_n} \rightarrow g_{\sigma+\sigma_0}$$

*uniformly on compact subsets of the basin of attraction of the petal for  $f^q$ , where  $\sigma_0$  depends on the normalizations of the Fatou coordinates.*

To simplify notations, let us now choose the normalization so that  $\sigma_0 = 0$ .

Example of consequence:

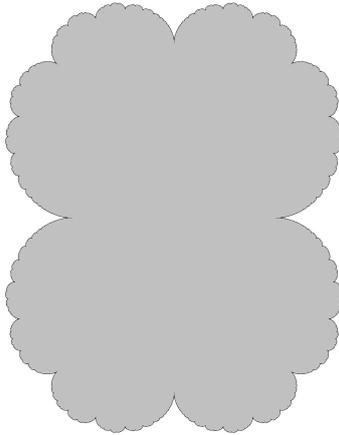
**Theorem.** *If  $f_0$  and the  $f_n$  are polynomials of the same degree then*

$$\begin{aligned} \liminf J(f_n) &\supset J(f_0, g_\sigma) \\ \limsup K(f_n) &\subset K(f_0, g_\sigma) \end{aligned}$$

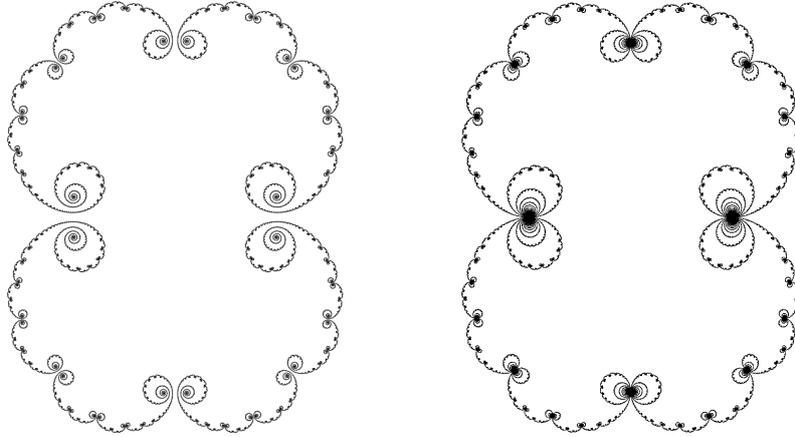
where  $K(f_n)$  is the complement of the basin of infinity  $A(f_n)$ ,  $J(f_n) = \partial K(f_n)$  is the Julia set,  $K(f_0, g_\sigma)$  is the complement of the union over  $n$  of  $g_\sigma^{-n}(A(f_0))$  and  $J(f_0, g_\sigma) = \partial K(f_0, g_\sigma)$ . They are called the *enriched Julia sets*. We have

$$\begin{aligned} J(f_0) &\subsetneq J(f_0, g_\sigma) \\ K(f_0, g_\sigma) &\subsetneq K(f_0) \end{aligned}$$

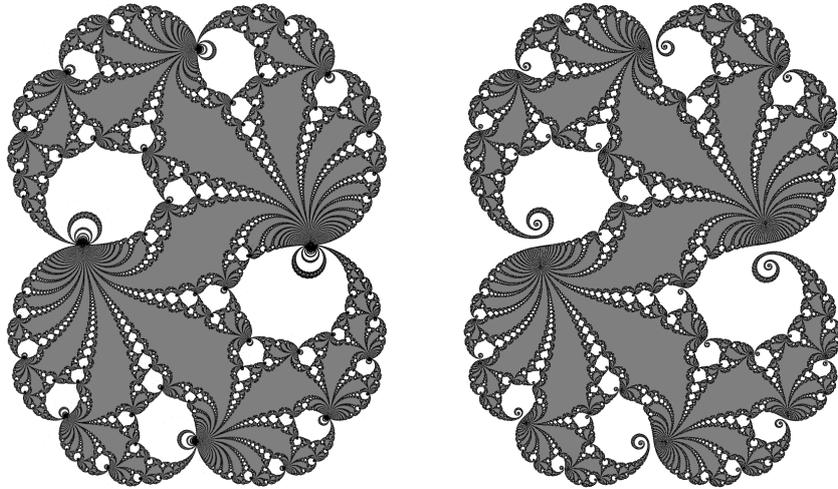
As an illustration, here is the Julia set  $J$  of  $z^2 + 1/4$  in black, with the interior of  $K$  in gray:



And here is what is obtained with  $z^2 + 1/4 + \varepsilon$  with  $\varepsilon$  positive and small and then the enriched Julia set limit for some particular sequence  $(\varepsilon_n > 0) \rightarrow 0$ .



Here is another enriched Julia set for the same example  $z^2 + 1/4$ , together with a Julia set tending to it:

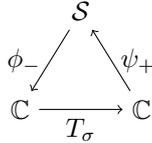


Other consequences include:

- the roots in  $M$  are landing points of external rays (Douady-Hubbard-Sentenac)
- fine study of the size of Siegel disks (Buff-Chéritat)
- $\dim_{\mathbb{H}} \partial M = 2$  (Shishikura)

## 6. PARABOLIC RENORMALIZATION: EMERGENCE OF THE HORN MAP AS A LIMIT DYNAMICAL SYSTEM

Consider the non-commutative diagram again:

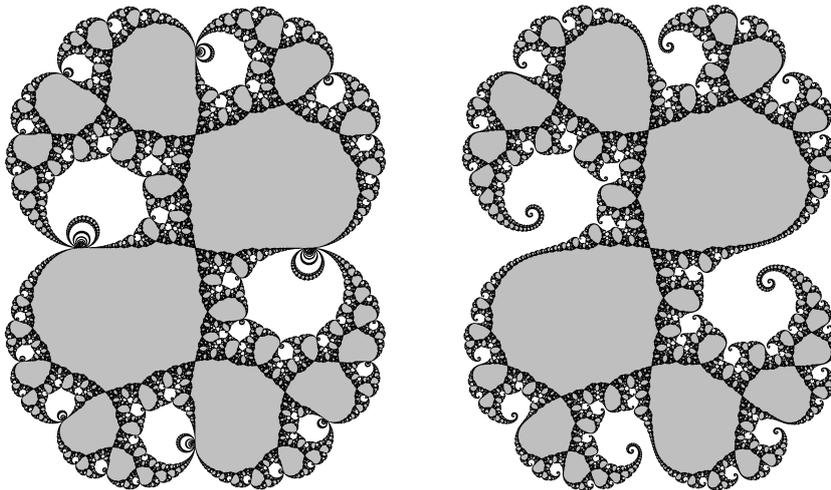


The map  $g_\sigma$  was defined by following the arrows from the upper corner back to it. Let us define the map  $h_\sigma$  in a similar way, but starting from the lower right corner (the repelling Fatou coordinates). Then  $h_\sigma = T_\sigma \circ h$ .

The dynamics of  $h_\sigma$  and  $g_\sigma$  somehow reflect each other: indeed the maps  $h_\sigma$  and  $g_\sigma$  are semi-conjugated in at least two ways as can be seen on the diagram:  $\psi_+$  semi-conjugates  $h_\sigma$  to  $g_\sigma$  and  $T_\sigma \circ \phi_-$  semi-conjugates  $g_\sigma$  to  $h_\sigma$ .

Note that  $h_\sigma = T_\sigma \circ h_0$ . It is maybe better to change variable and map  $\mathbb{C}/\mathbb{Z}$  to  $\mathbb{C}^*$  via  $z \mapsto e^{2\pi iz}$ . Then  $h_\sigma$  gets conjugated to a family of map  $\ell_\sigma = e^{2i\pi\sigma}\ell_0$  that fix the origin and such that  $\ell'_0(z) \neq 0$ . The quantity  $\ell'_\sigma(0)$  is called the *virtual multiplier* of  $g_\sigma$  at the upper end.

For instance on the left part of the next picture, we show an enriched Julia set with a virtual Siegel disk. More precisely  $\sigma$  is chose so that  $\ell_\sigma$  has at the origin an indifferent fixed point of rotation number the Golden mean. In this case the origin is necessarily linearizable for the dynamics of  $\ell_\sigma$ , and its maximal domain on which the map is conjugated to a rotation is called a Siegel disk. The maps  $h_\sigma$  and  $g_\sigma$  do not have a Siegel disk, but instead a domain on which the dynamics of the map and that of  $f$  are conjugated to two horizontal translations on the upper half plane, whose vectors are respectively an irrational equal to the rotation number of the Siegel disk modulo  $\mathbb{Z}$ , and 1. For  $g_\sigma$  this region is that of the two symmetric biggest gray regions that lies on the upper right. On the right part of the picture, we showed the Julia set of  $e^{2\pi i\theta_n}z + z^2$  for  $\theta_n$  being the irrational with continued fraction expansion  $[0; n, 1, 1, 1, \dots]$  and  $n = 20$ . It can be shown that the right figure converges to the left figure as  $n \rightarrow +\infty$ .



**Remark.** A priori the horn map  $h$ , is not a dynamical system but a conjugacy invariant. However, Lavaurs' theorem explains us that the dynamics of  $h_\sigma$  tells us something about the dynamics of perturbations of  $f$ .

There is a unique  $\sigma$  modulo  $\mathbb{Z}$  such that  $\ell'_\sigma(0) = 1$  and we let the *parabolic renormalization* of  $f_0$  be this map  $\ell_\sigma$  or rather its class modulo conjugacy by a linear map:  $\mathcal{R}(f_0)$ . This new map is parabolic at the origin and we may want to look at its parabolic renormalization, and so on... This will enrich further the original dynamical system, and these enriched objects are limits of well chosen perturbations of the original map. It has recently proved very useful to understand what happens when parabolic renormalization is iterated, starting from a polynomial of degree 2, and to control deviations from this situation when *near parabolic renormalization* is done (but this goes far beyond the scope of this survey) see [IS].

Remark: Lavaurs extended theorems of Fatou regarding the link between non repelling cycles and critical points, to horn maps  $\tilde{h}$  of polynomials. Adam Epstein extended them to the much wider class of maps called *finite type maps*, of which these renormalizations are a particular case, and to towers of these objects.

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