Projects
Complex Dynamics and Fractals

(a) Study the dynamical properties of the rational function

$$R(z) = z^2 + \frac{1}{z} - 1.$$

Literature: Aaronson, Jon; Denker, Manfred; Urbanski, Mariusz Ergodic theory for Markov fibred systems and parabolic rational maps. Trans. Amer. Math. Soc. 337 (1993), no. 2, 495–548.

(b) Show that the Hausdorff dimension $h(c)$ of the map $z \mapsto z^2+c$ is not continuous at $c = \frac{1}{4}$.

Literature: Zinsmeister, Michel Thermodynamic formalism and holomorphic dynamical systems. Translated from the 1996 French original by C. Greg Anderson. SMF/AMS Texts and Monographs. American Mathematical Society, Providence.

Douady, Adrien; Sentenac, Pierrette; Zinsmeister, Michel Implosion parabolique et dimension de Hausdorff. (French. English, French summary) [Parabolic implosion and Hausdorff dimension] C. R. Acad. Sci. Paris Ser. I Math. 325 (1997), no. 7, 765–772.

(c) For the map $T(z) = z^2 + c$ with real $0 < c < 1/4$ construct points x for which the ergodic averages

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$$

aproximate the values of

$$\int f d\mu$$

where μ is the invariant measure equivalent to the h-dimensional conformal measure (=Gibbs measure for the potential $-h\log|T'|$).

Literature: Denker, Manfred; Duan, Jinqiao; McCourt, Michael Pseudorandom numbers for conformal measures. Dyn. Syst. 24 (2009), no. 4, 439âĂŞ457

(d) Develop the thermodynamic formalism for the quantity

$$P(T,\epsilon,h) = \limsup_{n\to\infty} \log\left(\sum_x \exp\left(\left(\frac{1}{n}\sum_{0\le i,j<n} h(T^i(x),T^j(x))\right)\right)\right).$$

wher ethe first sum extends over a maximal (n,ϵ) separating set. Prove the variational formula in case $h(x,y) = f(x)g(y)$.

Literature: Lecture notes.

(e) Discuss at least one example where the packing dimension is not equal to the Hausdorff dimension. Show that there are metrics on the Sierpinski gasket (with the Euclidean topology) which provide such an example. How about local dimension?

Literature: Koch, Susanne; Denker, Manfred Hausdorff dimension for Martin metrics. Algebraic and topological dynamics, 163–170, Contemp. Math., 385, Amer. Math. Soc., Providence, RI, 2005.

(f) Verify the Poisson limit law for hyperbolic rational maps. Does such a law hold for parabolic rational maps?

Literature: Haydn, Nicolai The distribution of the first return time for rational maps. J. Statist. Phys. 94 (1999), no. 5-6, 1027–1036.

(g) Show that the Julia set $J(R)$ of a hyperbolic rational map which is also a Cantor set, is homeomorphic to the Markov chain which puts equal transition probabilities on preimages of a point $z \notin J(R)$, but close to the Julia set.

Literature: Lecture notes.