Interval exchange maps

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Circle rotations

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(Standard)interval exchange maps

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Circle diffeomorphisms

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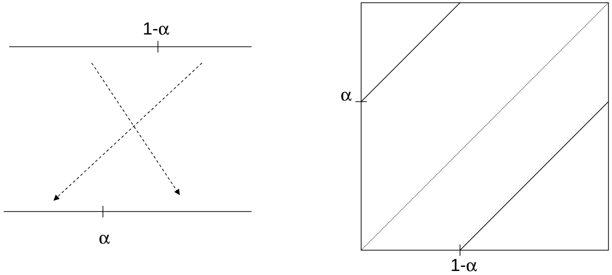
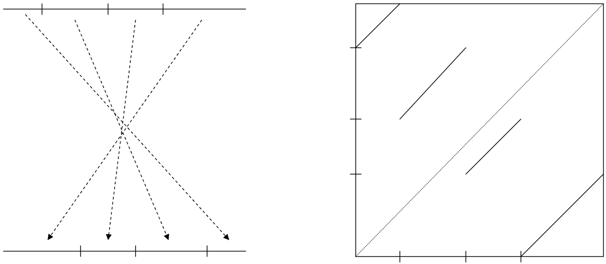
Generalized interval exchange maps

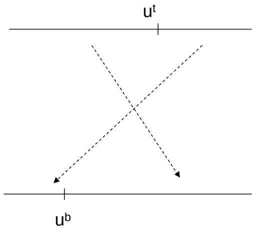
Circle rotations

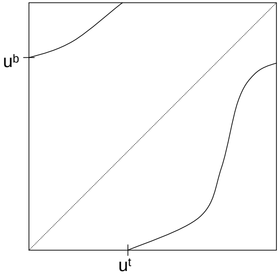


Standard interval exchange maps

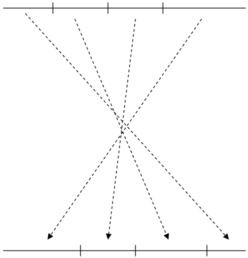


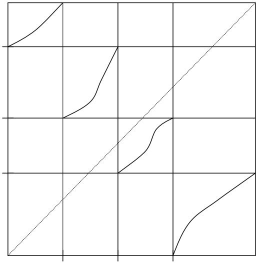
Circle homeomorphisms and diffeomorphisms





Generalized interval exchange maps





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2. If α is irrational, the transformation R_α is *minimal, ergodic and uniquely ergodic*.

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This implies that μ is equal to Lebesgue measure.

Interval exchange maps (i.e.m): combinatorial data

We use a finite alphabet $\mathcal{A}$ (with $\#\mathcal{A} = d$) to label the d open subintervals exchanged by an i.e.m T: we have

$$I = \bigsqcup_{\alpha \in \mathcal{A}} I^t_\alpha = \bigsqcup_{\alpha \in \mathcal{A}} I^b_\alpha,$$

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and the restriction of T to I_α^t is a translation (an homeomorphism if T is a generalized i.e.m) onto I_α^b.
The combinatorial data of T, i.e the order in which the subintervals appear in the two partitions, are given by two bijections π_t, π_b from $\mathcal{A}$ onto $\{1, \ldots, d\}$ and represented by

$$\begin{pmatrix} \pi_t^{-1}(1) & \ldots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \ldots & \pi_b^{-1}(d) \end{pmatrix}.$$

Irreducible combinatorial data

Definition: The combinatorial data $\pi = (\pi_t, \pi_b)$ are *irreducible* if, for all $1 \leqslant k < d$, we have

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Length and Translation vectors

A (standard) i.e.m T is determined by its combinatorial data π and its *length vector* $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}_+^{\mathcal{A}}$:

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The translation vector $\delta \in \mathbb{R}^{\mathcal{A}}$ is determined by

$$I_\alpha^b = I_\alpha^t + \delta_\alpha, \quad \alpha \in \mathcal{A}.$$

Intersection matrix and genus of an i.e.m

Let $\pi = (\pi_t, \pi_b)$ be irreducible combinatorial data. Define the *intersection matrix* by

$$\Omega_{\alpha\,\beta} = \begin{cases} +1 & \text{if } \pi_t(\alpha) < \pi_t(\beta), \pi_b(\alpha) > \pi_b(\beta), \\ -1 & \text{if } \pi_t(\alpha) > \pi_t(\beta), \pi_b(\alpha) < \pi_b(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

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Example: For $d = 2, 3$, the genus is equal to 1. For $\pi = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$, the rank is 4 and the genus is 2.

Suspension of i.e.m: The case $d = 2$

From a length vector $\lambda = (\lambda_A, \lambda_B)$ and a *suspension vector* $\tau = (\tau_A, \tau_B)$ with $\tau_A > 0 > \tau_B$, one constructs first a parallelogram with sides $\zeta_\alpha = (\lambda_\alpha, \tau_\alpha)$ and then a flat torus by identifying parallel sides.

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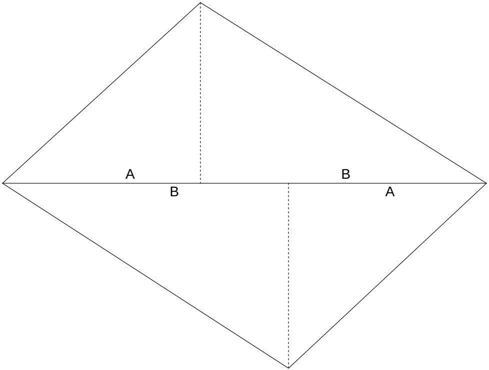

Translation surfaces

To suspend an i.e.m T with combinatorial data π, length vector λ, one also needs a *suspension vector* $\tau \in \mathbb{R}^{\mathcal{A}}$ satisfying the *suspension conditions*

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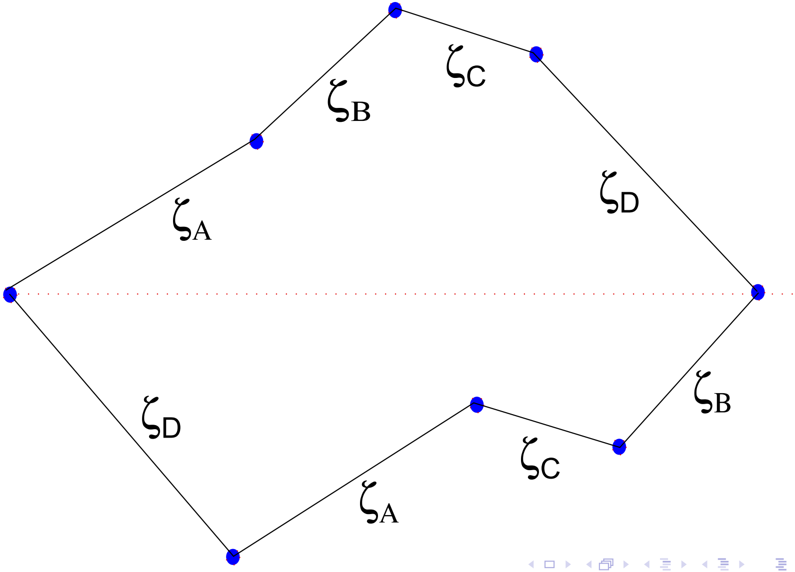
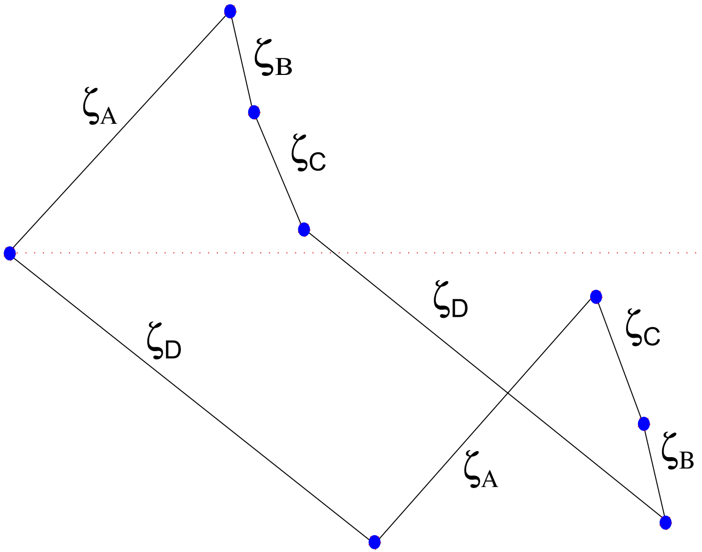
The vertices of the polygon correspond to singular points where the total angle is a multiple of 2π. The genus of M is g. The number s of singular points is related to g and d by

$$d = 2g + s - 1.$$

Suspension of an i.e.m with $\sum_\alpha \tau_\alpha = 0$



Suspension of an i.e.m in the general case: a problem . . .



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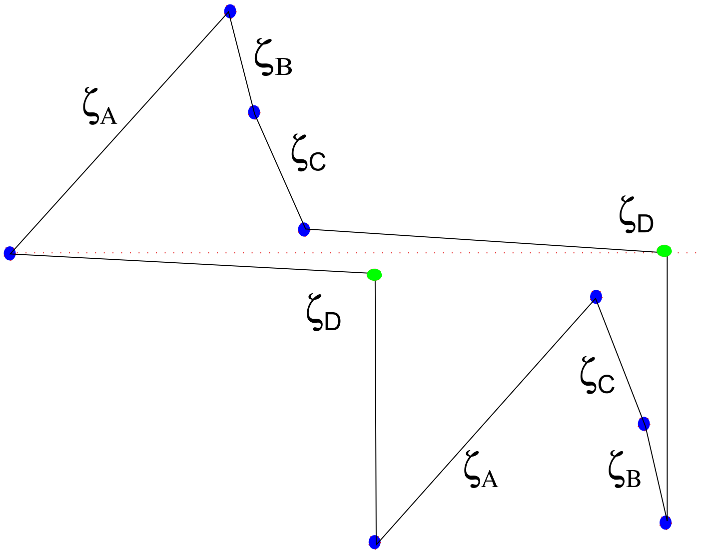

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- A *connection* is a relation $T^m(u_i^b) = u_j^t$ with $1 \leqslant i, j < d$ and $m \geqslant 0$.

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Remark: For $d = 2$, T has no connection iff it is minimal iff its length data are rationally independent. This is no longer true as soon as $d \geqslant 3$: There are minimal i.e.m T with connections, and i.e.m T with rationally dependent length data but no connection.

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Definition: A **standard** i.e.m T is *irrational* if it has no connection.

Invariant measures for an irrational i.e.m

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Proposition: $\mathcal{M}(T_0)$ and $\Lambda(T_0)$ are closed simplices of the same dimension $\in [0, g-1]$ in natural affine one-to-one correspondence.

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Proposition: $\mathcal{M}(T_0)$ and $\Lambda(T_0)$ are closed simplices of the same dimension $\in [0, g-1]$ in natural affine one-to-one correspondence.

Remark: T_0 is uniquely ergodic iff the dimension is 0, i.e $\Lambda(T_0) = \{\lambda^0\}$.

Invariant measures for an irrational i.e.m

Let T_0 be a standard **irrational** i.e.m with combinatorial data π and length data λ^0, normalized by $\sum_{\alpha\in\mathcal{A}} \lambda^0_\alpha = 1$.
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Elements of proof

- For $\lambda \in \Lambda(T_0)$, let h_λ be the homeomorphism of I s.t. $h_\lambda \circ T_{\pi,\lambda} = T_0 \circ h_\lambda$.

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- As every probability in $\mathcal{M}(T_0)$ is a convex combination of *ergodic* T_0-invariant probability measures (extremal points of $\mathcal{M}(T_0)$) in a *unique* way, $\mathcal{M}(T_0)$ is a simplex of dimension $\in [0, d-1]$.

The Farey tree

Definition: A *Farey interval* is an interval $(\frac{p}{q}, \frac{p'}{q'})$ with $p, p', q, q' \in \mathbb{Z}$, $q, q' \geqslant 1$, $p \wedge q = p' \wedge q' = 1$, $p'q - pq' = 1$.

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The slow homogeneous version of the continuous fraction algorithm

Given a pair $\lambda = (\lambda_L, \lambda_R)$ of **distinct** positive numbers, we set

$$\widetilde{\lambda} = \begin{cases} (\lambda_L, \lambda_R - \lambda_L) & \text{if } \lambda_L < \lambda_R, \\ (\lambda_L - \lambda_R, \lambda_R) & \text{if } \lambda_L > \lambda_R, \end{cases}$$

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In any case, the algorithm determines the sequence of Farey intervals containing α (and thus α itself).

The usual (fast inhomogeneous) version of the continuous fraction algorithm

For $\alpha \in (0,1)$, we define the Gauss map by

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Starting with $\alpha = \alpha_0 \in (0,1)$, we define

$$\alpha_{n+1} := G(\alpha_n), \quad a_{n+1} = \lfloor \alpha_n^{-1} \rfloor$$

as long as $\alpha_n \neq 0$ and write

$$\alpha = [a_1, \ldots, a_n, \ldots].$$

Convergents of α

One has, for $n \geqslant 0$

$$\alpha = \frac{p_n + p_{n-1}\alpha_n}{q_n + q_{n-1}\alpha_n}, \quad \alpha_n = -\frac{q_n\alpha - p_n}{q_{n-1}\alpha - p_{n-1}}$$

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where the *convergents* $\frac{p_n}{q_n}$ satisfy the recurrence relations

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1},$$

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Relation between the two versions of the continuous fraction algorithm

Let us say than an iteration $\lambda \mapsto \widetilde{\lambda}$ of the slow algorithm is of *left type* if $\lambda_L > \lambda_R$, of *right type* if $\lambda_R > \lambda_L$.

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Concatenating operations of the same type correspond to Euclidean division with rest: for left type

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Setting $\alpha(\lambda) = \min(\frac{\lambda_L}{\lambda_R}, \frac{\lambda_R}{\lambda_L})$, we have (in the case of left type)

$$\alpha(\widehat{\lambda}) = \frac{\widehat{\lambda}_L}{\lambda_R} = \frac{\lambda_L}{\lambda_R} - N = G(\alpha(\lambda)),$$

and similarly for right type.

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(cf. formula of the slow algorithm).

Rotation number of circle homeomorphisms

Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of an orientation-preserving homeomorphism f of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, i.e an homeomorphism of $\mathbb{R}$ commuting with $R_1(x) = x + 1$.

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If $F(0) = F^{-1}(1)$, we have $F^2(0) = 1$ and $\rho(F) = \frac{1}{2}$.
Otherwise, we consider the first return map T_1 of T_0 on the interval $I^{(0)} = (0, \max(F(0), F^{-1}(1))$. Observe that $1 \geqslant \rho(F) \geqslant \frac{1}{2}$ if $F(0) \geqslant F^{-1}(1))$ and $0 \leqslant \rho(F) \leqslant \frac{1}{2}$ if $F(0) \leqslant F^{-1}(1))$.

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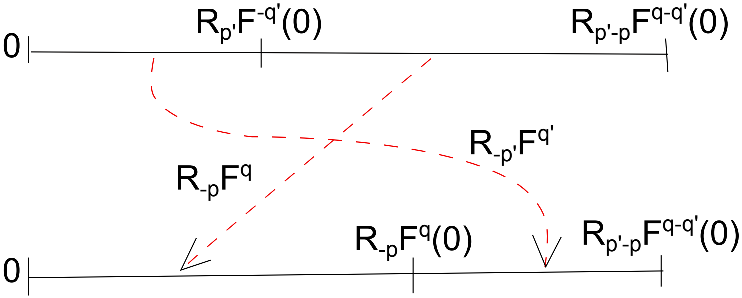

We have a generalized i.e.m T_n on the interval $I^{(n)} := (0, R_{p'-p}F^{q-q'}(0))$ with with $T_n = R_{-p} \circ F^q$ on $(0, R_{p'}F^{-q'}(0))$ and $R_{-p'} \circ F^{q'}$ on $(R_{p'}F^{-q'}(0), R_{p'-p}F^{q-q'}(0))$.

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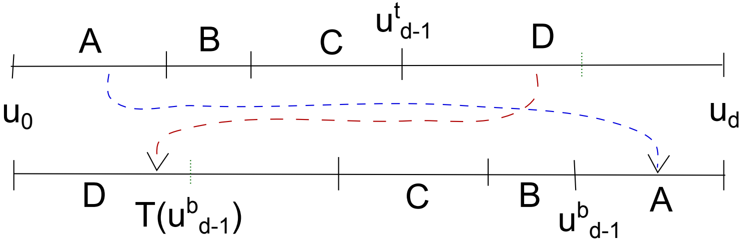
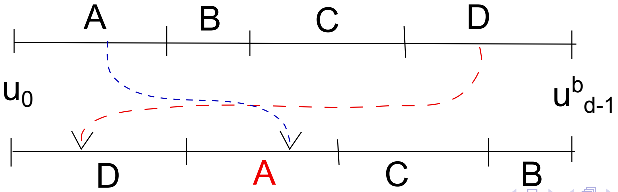
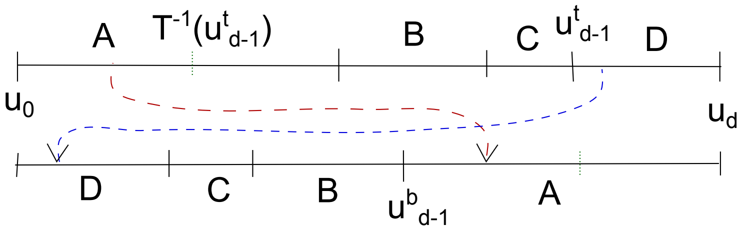
The step $T \to \widehat{T}$ is of **top** type if $u^t_{d-1} < u^b_{d-1}$, of **bottom** type if $u^t_{d-1} > u^b_{d-1}$.

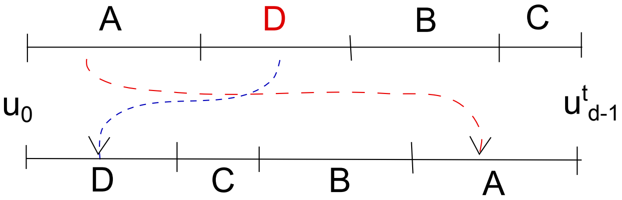
The elementary step of the Rauzy-Veech algorithm: Example for top type





The elementary step of the Rauzy-Veech algorithm: Example for bottom type





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The formulas in the case of bottom type are symmetric.

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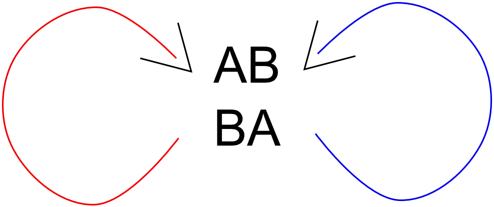
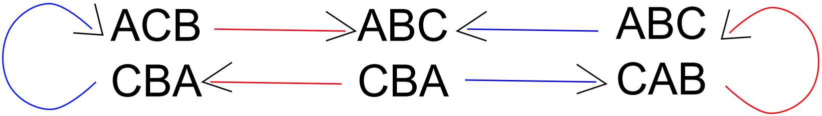
Iterating the Rauzy-Veech algorithm for a generalized i.e.m T with combinatorial data π and no connection produces an infinite path $\rho(T)$ starting at π in the Rauzy diagram having π as a vertex.

The Rauzy diagram for $d = 2$



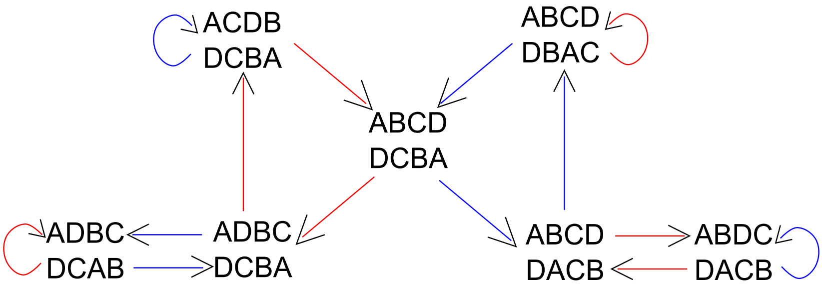
d=2, g=1, s=1

The Rauzy diagram for $d = 3$



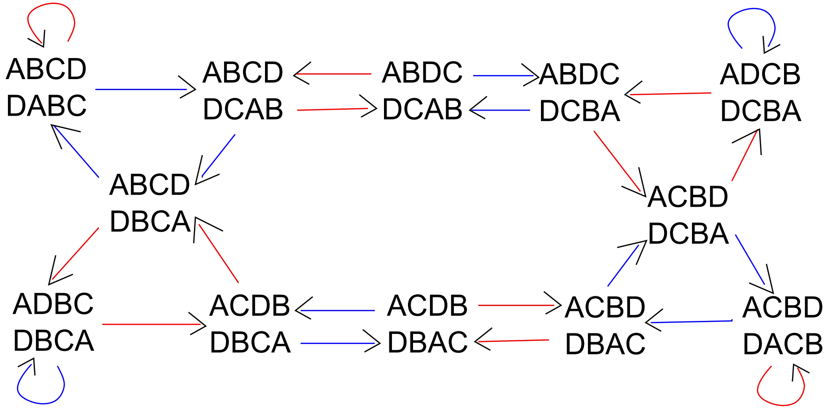
d=3, g=1, s=2

A Rauzy diagram for $d = 4$



d=4, g=2, s=1

The other Rauzy diagram for $d = 4$



d=4, g=1, s=3

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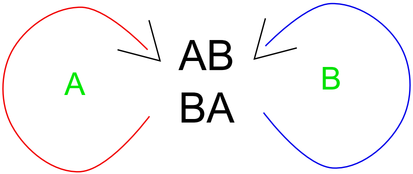
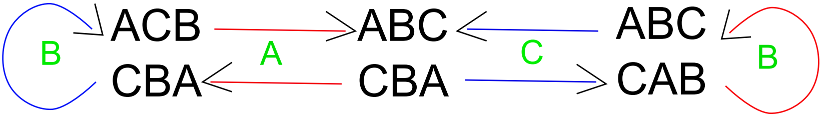


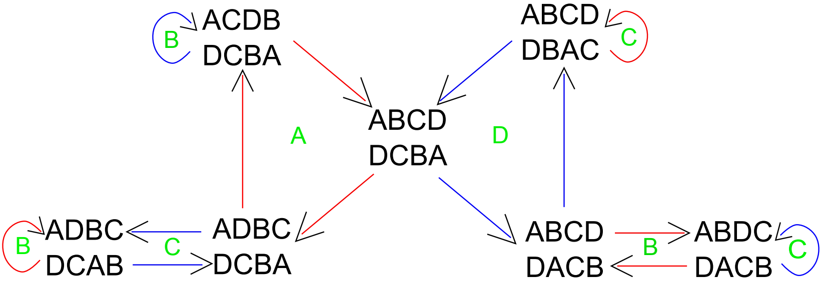
d=2, g=1, s=1

Winners in the Rauzy diagram for $d = 3$



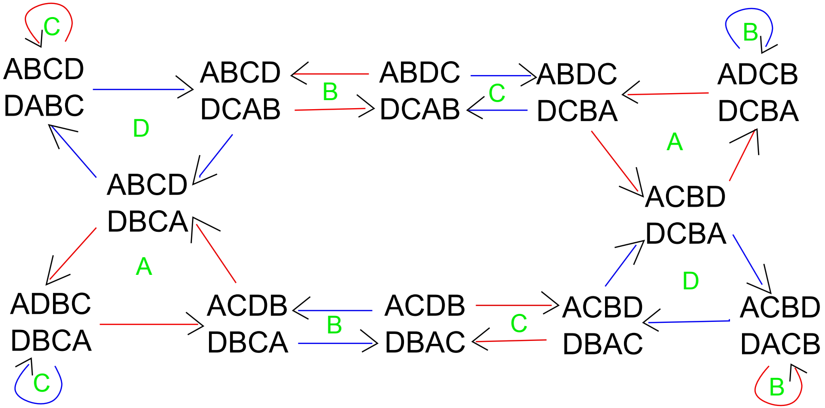
d=3, g=1, s=2

Winners in the first Rauzy diagram for $d = 4$



d=4, g=2, s=1

Winners in the second Rauzy diagram for $d = 4$



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∞-complete paths in Rauzy diagrams

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Theorem *An infinite path is the path $\rho(T)$ associated to a* **standard irrational** *i.e.m T iff it is ∞-complete.*

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A closely related map is the two-to-one shift map $\bar{V}$ from the set $\mathcal{C}_\infty(\mathcal{D})$ of ∞-complete paths in $\mathcal{D}$ to itself. Indeed, the map $\rho : T \mapsto \rho_T$ from $\mathcal{N}C(\mathcal{D})$ to $\mathcal{C}_\infty(\mathcal{D})$ is onto and is a semiconjugacy between V and $\bar{V}$.

The Kontsevich-Zorich cocycle

To each arrow γ in a Rauzy diagram, with winner α_w, loser α_ℓ, we associate the matrix $B_\gamma \in SL(\mathbb{Z}^{\mathcal{A}})$ defined by

$$B_\gamma := \mathbf{1} + E_{\alpha_\ell \alpha_w},$$

where $E_{\alpha_\ell \alpha_w}$ is the elementary matrix having a single nonzero entry equal to 1 in position $\alpha_\ell \alpha_w$.

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The matrices $B_{\underline{\gamma}}$ belong to $SL(\mathbb{Z}^{\mathcal{A}})$ and have nonnegative coefficients.

The Kontsevich-Zorich cocycle

To each arrow γ in a Rauzy diagram, with winner α_w, loser α_ℓ, we associate the matrix $B_\gamma \in SL(\mathbb{Z}^{\mathcal{A}})$ defined by

$$B_\gamma := \mathbf{1} + E_{\alpha_\ell \alpha_w},$$

where $E_{\alpha_\ell \alpha_w}$ is the elementary matrix having a single nonzero entry equal to 1 in position $\alpha_\ell \alpha_w$.

For a path $\underline{\gamma}$ made of the successive arrows $\gamma_1, \ldots, \gamma_m$, we define

$$B_{\underline{\gamma}} := B_{\gamma_m} \ldots B_{\gamma_1}.$$

The matrices $B_{\underline{\gamma}}$ belong to $SL(\mathbb{Z}^{\mathcal{A}})$ and have nonnegative coefficients.

For an elementary step $T_{\pi,\lambda} \longrightarrow T_{\widehat{\pi},\widehat{\lambda}}$ of the algorithm associated to the arrow $\gamma : \pi \longrightarrow \widehat{\pi}$, one has

$$\lambda = {}^t B_\gamma \widehat{\lambda}, \quad \widehat{\delta} = B_\gamma \delta.$$

The Kontsevich-Zorich cocycle (over Rauzy-Veech dynamics) is given by

$$(\pi, \lambda, v) \mapsto (\widehat{\pi}, \widehat{\lambda}, B_\gamma v),$$

with γ as above and $v \in \mathbb{R}^{\mathcal{A}}$.

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Let $T_0 = T_{\pi,\lambda_0}$ be a normalized standard i.e.m. with no connection. Recall that $\Lambda(T_0)$ is the set of length vectors λ, normalized by $\sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1$, such that $T_{\pi,\lambda}$ is topologically conjugated to T_0 by an orientation-preserving homeomorphism of $I = [0, 1]$.

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Let $\rho(T_0) = (\gamma_1, \ldots, \gamma_m, \ldots)$ be the infinite path associated to T_0. For $m \geqslant 0$, let B_m be the matrix $B_{\gamma_m} \ldots B_{\gamma_1}$ associated to the initial part of $\rho(T_0)$.

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Proposition: The closed simplex $\Lambda(T_0)$ is equal to the set of normalized standard irrational i.e.m T with $\rho(T) = \rho(T_0)$, i.e to the intersection of $\bigcap_{m\geqslant 0} {}^tB_m(\mathbb{R}_+^{\mathcal{A}})$ with the normalizing hyperplane $\sum_{\alpha\in\mathcal{A}} \lambda_\alpha = 1$.

An important technical result

Proposition: Let $\underline{\gamma}$ be a path which is the concatenation of $2d - 3$ complete subpaths. Then all coefficients of $B_{\underline{\gamma}}$ are nonzero (positive).

Basic properties of the KZ-cocycle

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- Proof of the upper bound for the dimension of $\Lambda(T_0)$.

For any $g \geqslant 1$, there exists T_0 of genus g s.t. $\Lambda(T_0)$ has dimension $g-1$ (cf. Keane, Keynes-Newton).

Almost sure unique ergodicity

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Theorem (Masur, Veech) *For any irreducible combinatorial data* π*, almost every length vector* $\lambda \in \mathbb{R}_+^{\mathcal{A}}$*, the standard i.e.m* $T_{\pi,\lambda}$ *is uniquely ergodic: normalized Lebesgue measure is the unique invariant probability measure.*

Zorich acceleration

Let $\mathcal{D}$ be a Rauzy diagram. Recall that a single iteration of the Rauzy-Veech algorithm defines a two-to-one map V from $\mathcal{N}C(\mathcal{D})$ to itself. Let V_Z be the map deduced from V by concatenating in a single step successive steps of the same type.

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Almost sure recurrence is the basis of the proof of the Masur-Veech theorem.

Rotation number and semiconjugacy

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Corollary: *Such an f is uniquely ergodic.*

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Corollary: If T_0 is uniquely ergodic (i.e uniquely defined up to scaling by T), then so is T.

Sketch of proof of semiconjugacy

Let $u_1^t < \ldots < u_{d-1}^t$, $u_1^b < \ldots < u_{d-1}^b$ be the respective singularities of T, T^{-1} and let similarly $v_1^t < \ldots < v_{d-1}^t$, $v_1^b < \ldots < v_{d-1}^b$ be the respective singularities of T_0, T_0^{-1}.

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As T, T_0 have no connection, the points in the definition of Z, Z_0 are distinct.

Therefore, there exists a one-to-one map h from Z onto Z_0 sending $T^m(u_i^b)$ to $T_0^m(v_i^b)$, and $T^{-n}(u_j^b)$ to $T_0^{-n}(v_j^b)$ for all m, n, i, j.

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The relation $h \circ T = T_0 \circ h$ holds on Z, and therefore on all of I.

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- In the last case, J is a *wandering interval*: one has $f^m(J) \cap f^n(J) = \emptyset$ for distinct $m, n \in \mathbb{Z}$.
  There are at most a countable number of maximal wandering intervals.
  The complement of the interior of the wandering intervals is an invariant Cantor set which is the limit set of every half-orbit of f.
- If there is no wandering interval, h is a homeomorphism and f is minimal.

Denjoy's theorem

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   In particular, a piecewise-C^2 diffeomorphism (for instance a piecewise-affine homeomorphism) with no periodic orbit is C^0-conjugated to an irrational rotation.

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Let T be a generalized i.e.m, let $I = \bigsqcup I_\alpha^t = \bigsqcup I_\alpha^b$ be the associated partitions, so that the restriction of T to I_α^t is an orientation-preserving homeomorphism onto I_α^b.

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We say that T is of class C^r if , for each $\alpha \in \mathcal{A}$, the restriction of T to I_α^t extends to a C^r-diffeomorphism from the closure of I_α^t onto the closure of I_α^b.

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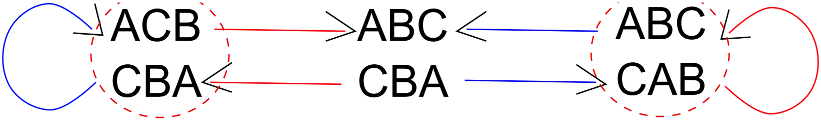
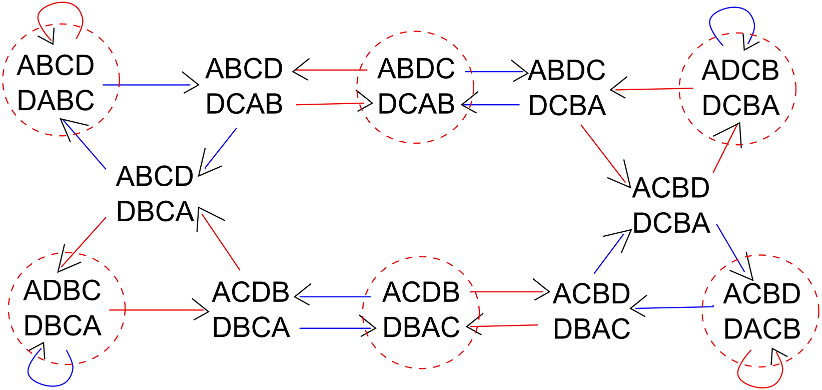
If T has no periodic orbit, then T has no wandering interval.
Proof = Exercise: Hint Any infinite path in a Rauzy diagram of genus 1 has to go infinitely many times through *rotation-like* vertices, i.e combinatorial data π such that, for some $0 < k < d$,one has $\pi_t(\alpha) = \pi_b(\alpha) + k \text{ mod. } d \quad \forall \alpha \in \mathcal{A}$.



d=3, g=1, s=2

The other Rauzy diagram for $d = 4$



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Affine i.e.m

Definition: A generalized i.e.m T is affine if the restriction of T to each I^t_α is affine. The *log-slope vector* $w \in \mathbb{R}^{\mathcal{A}}$ is s.t. the slope of T on I^t_α is $\exp w_\alpha$.

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The parameter space is thus $2d - 2$-dimensional.

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where μ is the T-invariant probability measure.

Corollary: Let T_0 be a standard irrational uniquely ergodic i.e.m. Let $w \in \mathbb{R}^{\mathcal{A}}$. Then there exists an affine i.e.m T with log-slope vector w and $\rho(T) = \rho(T_0)$ iff

$$\sum_{\alpha} w_\alpha \lambda_\alpha^0 = 0$$

where λ^0 is the length vector of T_0.

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Previous results by Levitt, Camelier-Gutierez, Bressaud-Hubert-Maass.

Roth type irrationals

- An irrational number α is of *Roth type* if, for every $\tau > 0$, there exists $C = C(\tau)$ such that

$$|q\alpha - p| \geqslant Cq^{-1-\tau}$$

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- Almost every α is of Roth type.

Smooth conjugacy for Roth type rotation numbers

Theorem (Arnold; Herman; Y.; Khanin-Sinai; Katznelson-Ornstein) *Let f be an orientation preserving diffeomorphism of $\mathbb{T}$ of class C^r, $r > 2$, having an irrational rotation number α of Roth type. Then the conjugacy h between f and R_α is a diffeomorphism of class C^s for all $s < r - 1$.*

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On the other hand, if α is of Liouville type (for every τ, there exists $q \geqslant 1$, $p \in \mathbb{Z}$ with $|q\alpha - p| < q^{-1-\tau}$), there exists $f \in \mathit{Diff}_+^\infty(\mathbb{T})$ with rotation number α such that the conjugacy h is not of class C^1 (Herman).

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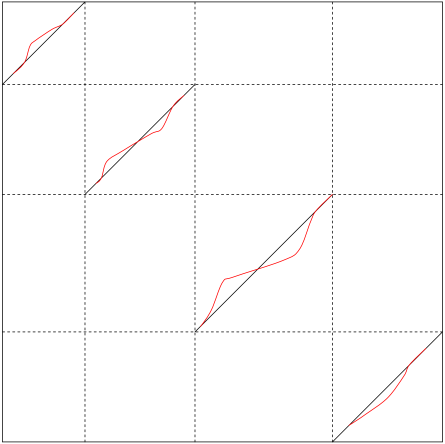

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Smooth linearization of generalized i.e.m: the result

Theorem (Marmi-Moussa-Y.) *Let* π *be combinatorial data,* g *the associated genus,* s *the number of marked points. For almost every standard i.e.m* T_0 *with combinatorial data* π*, for any integer* $r \geqslant 2$*, amongst the* C^{r+3} *simple deformations of* T_0*, those which are* C^r*-conjugated to* T_0 *by a diffeomorphism* C^r*-close to the identity form a* C^1*-submanifold of codimension* $(2r+1)(g-1)+s$*.*

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- Question: Is the C^0-conjugacy class of T_0 equal to the C^1-conjugacy class for almost all T_0?
- The "almost all T_0" in the statement of the theorem corresponds to a diophantine condition on the rotation number of T_0 (called *restricted Roth type*) which coincides for $g = 1$ with the Roth type condition seen earlier for circle diffeomorphisms.

J-C.Y. Interval exchange maps and Translation surfaces, p.1-71, in Homogeneous flows, Moduli spaces and Arithmetics, Einsiedler and al. eds, Proceedings of the Clay Summer School, Pisa 2007, Clay Math. Proceedings, Vol. 10 (2009)

These notes contain a full set of references. A preliminary version can be downloaded from my page on the Collège de France website.