Stabilized finite element methods for thermally coupled incompressible flows

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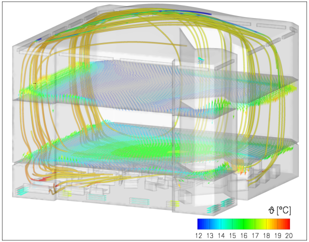

Lecture I in DK-Seminar ”Numerical Simulations in Technical Sciences”

TU Graz, March 10-12, 2009

Outline

1. Mathematical model
2. Time discretization. Linearization. Decoupling.
3. Residual-based stabilization of linearized problems: Isotropic meshes
4. Residual-based stabilization of linearized problems: Hybrid meshes
5. Application to buoyancy-driven flows

Joint work with: G. Rapin (Göttingen), T. Knopp (DLR Göttingen), Th. Apel (Munich), M. Rösler, R. Gritzki, J. Seifert (TU Dresden)

Outline

Non-isothermal incompressible Navier-Stokes model

Find velocity $\vec{u}$, pressure p and temperature θ in $(0,T)\times\Omega$:

Fluid motion (Navier-Stokes + continuity eq.) $\nu = \sqrt{Pr/Ra}$

$$\partial_t \vec{u} - \vec{\nabla}\cdot\big(\nu\,\underbrace{(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T)}_{=:\,2S(\vec{u})}\big) + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p \;=\; \vec{f}$$

$$\nabla\cdot\vec{u} \;=\; 0$$

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$$\nabla\cdot\vec{u} = 0$$

Heat transfer (advection-diffusion eq.) $a = \frac{\nu}{Pr} = 1/\sqrt{Ra\cdot Pr}$

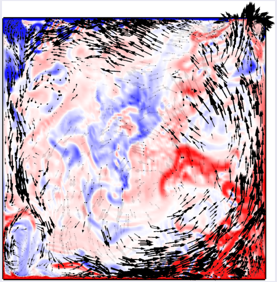
$$\partial_t\theta - \nabla\cdot(a\nabla\theta) + (\vec{u}\cdot\nabla)\theta = \dot{q}^V/c_p$$

- Boussinesq approximation $\vec{f} = -\beta\theta\vec{g}$
- Turbulence occurs for Rayleigh numbers $Ra \succeq 10^8 \ldots 10^9$

Examples of thermally coupled incompressible flows

Example 1: Rayleigh-Benard problem

- Natural convection problem in a box (heating from below, cooling from top, insulation on lateral sides)
- Numerical simulation with FVM-Code at DLR Göttingen (Shishkina/ Wagner JFM 2008)
- $Ra \sim 10^9 \dots 10^{10}$, $Pr = 5.4$ (water)



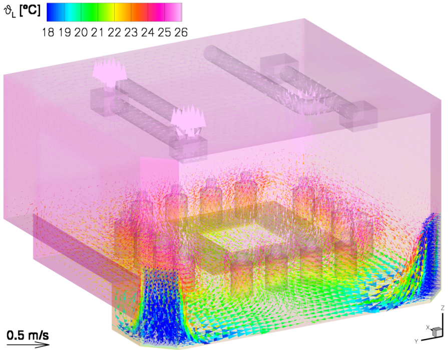
Project:

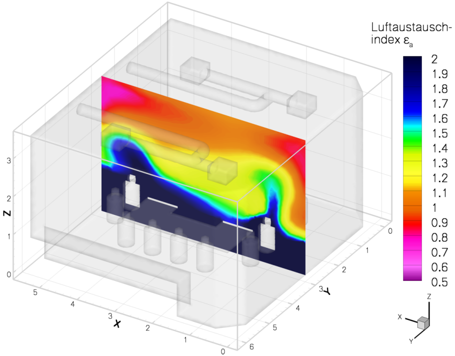
- Thermal convection experiment with rotation (up to $Ra = 10^{15}$) at MPI DS Göttingen
- Numerical simulations with FVM-code of DLR and FEM-Code of NAM Göttingen

Examples of thermally coupled incompressible flows II

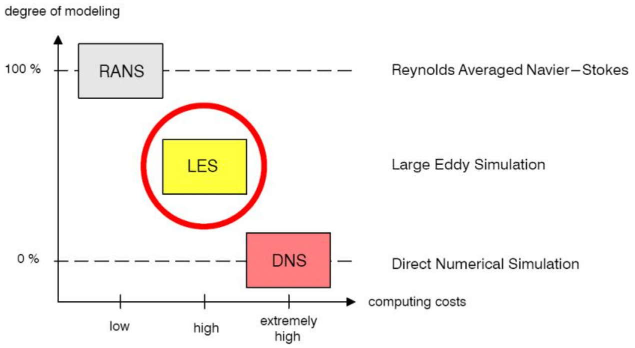
Example 2: Indoor air flow

- Mixed convection problem in a room: natural + forced convection
- Numerical simulation with FEM-code (TU Dresden/ NAM Göttingen) at $Ra \sim 10^{10}$, $Pr = 0.7$ (air)





Numerical simulation of turbulent flows



Computational costs of DNS, LES and (U)RANS, cf. BREUER [2004]

- **DNS:** *almost unfeasible for high Re- and Ra-numbers*
- **Unsteady RANS (URANS) model:** *as current industrial standard*
- **LES** or **Detached-eddy simulation (DES):** *as reasonable (but still very expensive) compromise*

Non-isothermal URANS model

Statistical turbulence model: $\Rightarrow$ consider averaged values $\vec{u}, p, \theta$

$$\begin{aligned} \partial_t \vec{u} - \vec{\nabla} \cdot (2\nu_e S(\vec{u})) + (\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla} p &= -\beta\theta\vec{g} \\ \vec{\nabla} \cdot \vec{u} &= 0 \\ \partial_t \theta - \vec{\nabla} \cdot (a_e \vec{\nabla}\theta) + (\vec{u} \cdot \vec{\nabla})\theta &= \dot{q}^V / c_p \end{aligned}$$

Non-isothermal URANS model

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Eddy viscosity ansatz for turbulent effects:

$$\nu_e = \nu + \nu_t \, , \qquad a_e = a + \nu_t / Pr_t$$

Variants of URANS models for turbulent viscosity ν_t:

- k-ϵ, k-ω model + wall functions — Codina/Soto '99, Lew et al. '01, Turek et al. '05
- k-ϵ-$\overline{v^2}$-f-model (Durbin '94) — Corsini et al. '05, Hadziabdic '06
- **Final part:** φ-$\bar{f}$-version of k-ϵ-$\overline{v^2}$-f-model (Hanjalic '04)

Non-isothermal k-ϵ turbulence model

Turbulent viscosity: $\nu_t = C_\mu \, k^2/\epsilon \quad \rightsquigarrow \quad \nu_k = \nu + \frac{\nu_t}{Pr_k}, \quad \nu_\epsilon = \nu + \frac{\nu_t}{Pr_\epsilon}$

$$\begin{aligned} \partial_t k - \vec{\nabla} \cdot (\nu_k \vec{\nabla} k) + (\vec{u} \cdot \vec{\nabla}) k &= P_k + G - \epsilon \\ \partial_t \epsilon - \vec{\nabla} \cdot (\nu_\epsilon \vec{\nabla} \epsilon) + (\vec{u} \cdot \vec{\nabla}) \epsilon + C_2 \frac{\epsilon}{k} \epsilon &= C_1 \frac{\epsilon}{k} (P_k + G) \end{aligned}$$

Non-isothermal k-ϵ turbulence model

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Production/ destruction terms and model constants:

- $P_k = 2\nu_t |S(\vec{u})|^2$,
- $G = \beta C_t \frac{k}{\epsilon} \sum_{i,j} g_i \left[\frac{2}{3} k \delta_{ij} - \nu_t (u_{i,j} + u_{j,i}) \right] \theta_{,j}$
- $C_\mu = 0.09, \ \ C_1 = 1.44, \ \ C_2 = 1.92, \ \ Pr_t = 0.9, \ \ Pr_k = 1.0, Pr_\epsilon = 1.3$

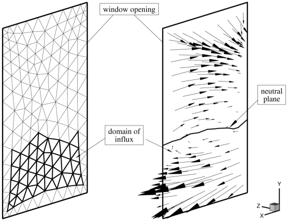
Outlook: Final part of lecture I

Extension to non-isothermal k-ϵ-φ-$\bar{f}$ turbulence model LAURENCE ET AL. '04

Boundary conditions: $\sigma = 2\nu_e S(\vec{u}) - (p + \frac{2}{3}k)I$

Split boundary $\partial\Omega$ into $\Gamma_-(\vec{u})$, $\Gamma_+(\vec{u})$, and $\Gamma_0(\vec{u})$ depending on

- $\text{sign}(\vec{n} \cdot \vec{u})$



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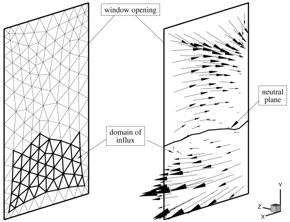

	inlet Γ_-	wall $\Gamma_0 = \Gamma_W$ or on $\Gamma_{W+\delta}$	outlet Γ_+
$\vec{u}$, p	$\vec{u} = \vec{u}_{in}$ or $\sigma \cdot \vec{n} = \sigma_n \vec{n}$	$\vec{u} = \vec{0}$	$\sigma \cdot \vec{n} = 0$
θ	$\theta = \theta_{in}$	$\theta = \theta_W$	$\nabla\theta \cdot \vec{n} = 0$
k	$k = \frac{3}{2}(T_u\|\vec{u}\|)^2$	$k = 0$	$\vec{\nabla} k \cdot \vec{n} = 0$
ϵ	$\epsilon = c_\mu^{3/4} k^{3/2}/L$	$\epsilon = \frac{2\nu}{\delta^2} k_\delta$ on $\Gamma_{W+\delta}$	$\vec{\nabla}\epsilon \cdot \vec{n} = 0$

with characteristic turbulence length scale L

Summary: Mathematical model

- Unsteady Reynolds-averaged Navier-Stokes (URANS) model for turbulent non-isothermal flow

- **So far:** Standard non-isothermal k-ε model

  $\rightsquigarrow$ Strongly coupled system of nonlinear PDE

- Well-posedness of coupled nonlinear model:
  $\rightsquigarrow$ MOHAMMADI/ PIRONNEAU 1994

- **Later on:** k-ε-φ-f model = improvement of standard k-ε model

Outline

Time discretization: Non-isothermal Navier-Stokes block

Linearized BDF(2) scheme: with unconditional A-stability

$$\partial_t \theta|_{t=t_m} \approx \frac{3\theta^m - 4\theta^{m-1} + \theta^{m-2}}{2\Delta t_m}; \quad \vec{U}^m = 2\vec{u}^{m-1} - \vec{u}^{m-2}, \quad T^m = 2\theta^{m-1} - \theta^{m-2}$$

$\rightsquigarrow$

Non-isothermal Navier-Stokes block:

$$\begin{aligned} -\vec{\nabla}\cdot(2\nu_e^m S(\vec{u}^m)) + (\vec{U}^m\cdot\vec{\nabla})\vec{u}^m + \frac{3\vec{u}^m}{2\Delta t_m} + \vec{\nabla} p^m &= -\beta T^m \vec{g} + \frac{4\vec{u}^{m-1} - \vec{u}^{m-2}}{2\Delta t_m} \\ \vec{\nabla}\cdot\vec{u}^m &= 0 \\ -\vec{\nabla}\cdot(a_e^m \vec{\nabla}\theta^m) + (\vec{U}^m\cdot\vec{\nabla})\theta^m + \frac{3}{2\Delta t_m}\theta^m &= c_p^{-1}\dot{q}^{Vm} + \frac{4\theta^{m-1} - \theta^{m-2}}{2\Delta t_m} \end{aligned}$$

similarly for k-ε-turbulence block

Treatment of turbulence block

Nonlinear advection-diffusion-reaction system:

$$-\vec{\nabla}\cdot(\nu_k^m\vec{\nabla}k^m)+\vec{U}^m\cdot\vec{\nabla}k^m+\left(\frac{\epsilon^m}{k^m}+\frac{3}{2\Delta t_m}\right)k^m \;=\; P_k^m+G^m+\frac{4k^{m-1}-k^{m-2}}{2\Delta t_m}$$

$$-\vec{\nabla}\cdot(\nu_\epsilon^m\vec{\nabla}\varepsilon^m)+\vec{U}^m\cdot\vec{\nabla}\varepsilon^m+\left(\frac{C_2\varepsilon^m}{k^m}+\frac{3}{2\Delta t_m}\right)\varepsilon^m \;=\; \frac{C_1}{T}(P_k^m+G^m)+\frac{4\varepsilon^{m-1}-\varepsilon^{m-2}}{2\Delta t_m}$$

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Positivity-preserving formulation (!) LEW et.al '01

- "Freeze" non-negative (!) reaction and diffusion coefficients and R.H.S.'s
- (Continuous) maximum principle valid for linearized problems !
- Maximum principle preserved after semidiscretization with BDF(2) !

Linearization cycle within each BDF(2) time step

(A): Solve (semidiscretized) non-isothermal Navier-Stokes equations:

via block Gauss-Seidel method with iterative decoupling

- Update turbulent viscosities ν_t^m and a_t^m
- Linearized Navier-Stokes problem for $\vec{u}^m, p^m$
- Linearized advection-diffusion-reaction problem for θ^m.

Linearization cycle within each BDF(2) time step

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(B): Solve (semidiscretized) equations for turbulence quantities:

via block Gauss-Seidel method with iterative decoupling

- Update non-negative (!) reaction- and diffusion coefficients and R.H.S.'s
- Solve for k^m, ϵ^m (until convergence)

(C): Stopping criterion:

Goto (A) if stopping-criterion for $\vec{u}^m, p^m, \theta^m$ not yet fulfilled. Otherwise goto next time step.

Summary: Time discretization. Linearization. Decoupling

- Alternative time discretizations: SDIRK methods etc.
- A-priori analysis for fully coupled nonlinear problem unrealistic ?!
- **Alternative:** A-posteriori analysis required
- **Open:** Interplay of (accurate, robust) linearization and maximum principle

Outline

Linearized advection-diffusion-reaction model

Advection-diffusion-reaction model: $u \in \{\theta, k, \epsilon\}$

$$Lu := -\vec{\nabla} \cdot (a\vec{\nabla} u) + (\vec{b} \cdot \vec{\nabla})u + cu \; = \; f \qquad \text{in } \Omega; \qquad u = 0 \;\text{ on }\; \partial\Omega$$

Assumptions: $a, c \in L^\infty(\Omega), \quad \vec{b} \in (H^1(\Omega))^d \cap (L^\infty(\Omega))^d, \quad f \in L^2(\Omega)$

$$a(x) \geq a_0 > 0, \quad (\nabla \cdot \vec{b})(x) = 0, \quad \frac{1}{\Delta t} + \tilde{c}(x) \sim c(x) \geq 0 \quad \text{a.e. in } \Omega$$

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Variational formulation:

$$\text{Find } u \in V := H_0^1(\Omega) \quad \text{s.t.} \quad A(u, v) = l(v) \qquad \forall v \in V$$

$$\begin{aligned} A(u, v) &:= (a\vec{\nabla} u, \vec{\nabla} v)_\Omega + (\vec{b} \cdot \vec{\nabla} u + cu, v)_\Omega, \\ l(v) &:= (f, v)_\Omega. \end{aligned}$$

Residual-based stabilized FEM

Basic SUPG-stabilized FEM: on admissible triangulation $\mathcal{T}_h$ of Ω

$$\text{Find } u \in V_h := \{v \in H_0^1(\Omega) \cap C(\bar{\Omega}) \mid v|_T \in P_r(T)\ \forall T \in \mathcal{T}_h\}$$

$$A_{rbs}(u,v) = L_{rbs}(v) \qquad \forall v \in V_h.$$

$$A_{rbs}(u,v) := (a\vec{\nabla}u, \vec{\nabla}v)_\Omega + \left(\vec{b}\cdot\vec{\nabla}u + cu, v\right)_\Omega + \sum_{T\in\mathcal{T}_h} \delta_T\,(\hat{L}u, \vec{b}\cdot\vec{\nabla}v)_T$$

$$L_{rbs}(v) := (f,v)_\Omega + \sum_{T\in\mathcal{T}_h} \delta_T\,(f, \vec{b}\cdot\vec{\nabla}v)_T$$

with orthogonal projection $\Pi_T : [L^2(T)]^d \to [P_r(T)]^d$ and

$$\hat{L}u|_T := -\vec{\nabla}\cdot\Pi_T(a\vec{\nabla}u) + (\vec{b}\cdot\vec{\nabla})u + cu$$

Analysis of SUPG-FEM - Stability and continuity

Stabilized norm: $|||v||| := \left(\sum_{T\in\mathcal{T}_h} \left(\|\sqrt{a}\vec{\nabla} v\|^2_{(L^2(T))^d} + \|\sqrt{c}v\|^2_{L^2(T)} + \delta_T \|\vec{b}\cdot\vec{\nabla} v\|^2_{L^2(T)} \right) \right)^{\frac{1}{2}}$

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Theorem: L., Rapin: CMAME 2006

Set

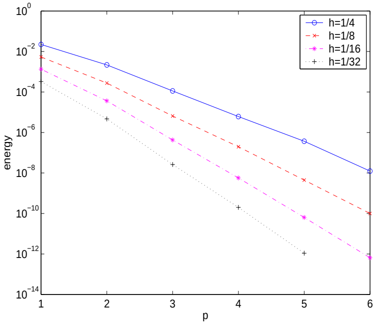
$$\delta_T \sim \min\left\{ \frac{h_T}{r\|\vec{b}\|_{L^\infty(T)}} ; \frac{1}{\|c\|_{L^\infty(T)}} ; \frac{h_T^2}{r^4 \mu_{inv}^2 \|a\|_{L^\infty(T)}} \right\}$$

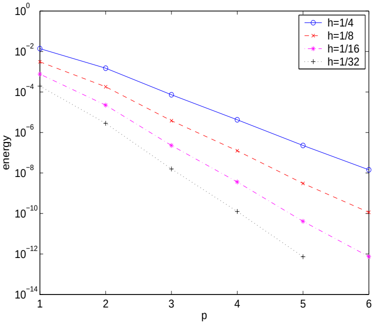
- **Stability:** $A_{rbs}(v,v) \geq \frac{1}{2}|||v|||^2 \quad \forall v \in V_h$
- **Galerkin orthogonality:** $A_{rbs}(u-u_h,v) = 0 \quad \forall v \in V_h$
- **A-priori error estimate:**

$$|||u-u_h|||^2 \leq C \sum_{T\in\mathcal{T}_h} \frac{h_T^{2(l-1)}}{r^{2(k-1)}} M_T^{opt} \|u\|^2_{H^k(T)}, \qquad l = \min(r+1,k)$$

$$M_T^{opt} := \|a\|_{L^\infty(T)} \Bigg(1 + \underbrace{\frac{h_T \|\vec{b}\|_{(L^\infty(T))^d}}{r\|a\|_{L^\infty(T)}}}_{=:Pe_T} + \underbrace{\frac{\|c\|_{L^\infty(T)} h_T^2}{r^2 \|a\|_{L^\infty(T)}}}_{=:\Gamma_T} + \frac{\|a\|^2_{W^{k-1,\infty}(T)}}{\|a\|^2_{L^\infty(T)}} \Bigg)$$

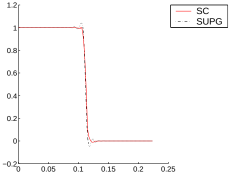
Numerical results for SUPG scheme on isotropic meshes:





- Spectral convergence for fixed (isotropic) mesh width h
- Results for $a = 10^{-6}, c = 0$ (left) and $a = 10^{-6}, c = 10^3$ (right)

Crosswind-stabilization of SUPG method



Problem of SUPG:

Spurious local (!) oscillations in shear layers

$\rightsquigarrow$ Spurious turbulence quantities !!

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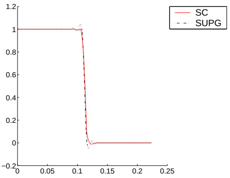

Framework of crosswind-stabilized variants:

$$\text{Find } U_h \in V_h : \quad A_{rbs}(U_h, v) + \sum_{T\in\mathcal{T}_h} (\tau_T(U_h) D_{sc} \nabla U_h, \nabla v)_T = L_s(v) \;\; \forall v \in V_h$$

Crosswind diffusion schemes with almost linear dependence on $R_T^*(w)$:

$$D_{sc}^{cd} := \begin{cases} I - \frac{\vec{b}\oplus\vec{b}}{|\vec{b}|^2}, & \vec{b} \neq 0 \\ 0, & \vec{b} = 0 \end{cases}, \qquad \tau_T^{cd}(w) := l_T^{cd}(w) \underbrace{\frac{\|\hat{L}w - f\|_{L^2(T)}}{|w|_{H^1(T)} + \kappa_T}}_{=:R_T^*(w)}$$

A-priori analysis of crosswind-stabilized SUPG

Theorem: L./ Rapin CMAME '06

- Restriction on limiter function: $0 \leq l_T^{cd}(w) \leq \rho \delta_T R_T^*(w), \forall w \in V_h$, with appropriate $\rho > 0$
- $(\nabla \cdot (a \nabla u))|_T \in L^2(T)$ and $u \in H^k(T)$, $k > \frac{d}{2}$ for all $T \in \mathcal{T}_h$,

⇝ **A-priori estimate for crosswind-stabilized SUPG-scheme:**

$$|||u - U_h|||^2 + \underbrace{\sum_{T \in \mathcal{T}_h} \|\tau_T^{cd}(U_h)\, D_{sc}^{cd\,\frac{1}{2}} \nabla (u - U_h)\|_{L^2(T)}^2}_{\text{additional crosswind control}} \preceq \quad \text{R.H.S. of SUPG estimate}$$

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Theorem: L./ RAPIN CMAME '06

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Remark:

- Semi-implicit treatment of nonlinearity within linearization loop !

Numerical experiments: Interior layer – skew to mesh

Example: DC/CD scheme CODINA '93, '99

- Limiter function $\quad l_T^{cd}(w) =:= \frac{1}{2} h_T \max \left\{ 0, \beta - \frac{2\|a\|_{L^\infty(T)}}{h_T R_T^*(w)} \right\}, \quad \kappa_T = 0$

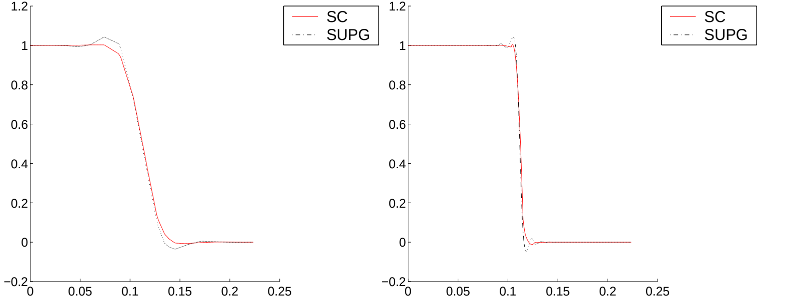

Figure: SUPG-FEM without/ with DC/CD for $h = \frac{1}{64}, r \in \{1, 4\}, \; \beta = 0.7, \; \kappa = 10^{-4}$

Linearized Navier-Stokes problem: Oseen-type problem

Assumptions: $\nu(x) > 0; \quad (\vec{\nabla} \cdot \vec{b})(x) = 0; \quad \frac{1}{\Delta t} \sim c = \text{const.} \geq 0$

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Assumptions: $\nu(x) > 0; \quad (\vec{\nabla} \cdot \vec{b})(x) = 0; \quad \frac{1}{\Delta t} \sim c = \text{const.} \geq 0$

Variational formulation:

Find $\quad U = \{\mathbf{u}, p\} \in \mathbf{W} := \mathbf{V} \times \mathbf{Q} := (H_0^1(\Omega))^d \times L_0^2(\Omega),$ s.t.

$$\mathcal{A}(\mathbf{b}; \mathbf{U}, \mathbf{V}) = \mathcal{L}(\mathbf{V}) \qquad \forall\, \mathbf{V} = \{\mathbf{v}, \mathbf{q}\} \in \mathbf{W}$$

with

$$\begin{aligned}
\mathcal{A}(\mathbf{b}; U, V) \quad &:= \quad \Big(2\nu S(\vec{u}), \vec{\nabla}\vec{v}\Big)_\Omega + \Big((\vec{b} \cdot \vec{\nabla})\vec{u} + c\vec{u},\ \vec{v}\Big)_\Omega \\
&\qquad - (p,\ \vec{\nabla} \cdot \vec{v})_\Omega + (q,\ \vec{\nabla} \cdot \vec{u})_\Omega \\
\mathcal{L}(V) \quad &:= \quad (\vec{f},\ \vec{v})_\Omega
\end{aligned}$$

Galerkin finite element discretization

- $\mathcal{T}_h$ – admissible triangulation of polyhedral domain Ω
- $X_h^r := \{v \in C(\bar{\Omega}) \mid v|_T \in P_r(T)\ \forall T \in \mathcal{T}_h\},\ r \in \mathbf{N}$
- Equal-order FE spaces for velocity/ pressure:

$$\mathbf{V}_h^r := \left[X_h^r \cap H_0^1(\Omega)\right]^d, \qquad \mathbf{Q}_h^r := X_h^r \cap L_0^2(\Omega)$$

$\rightsquigarrow$ no discrete LBB-condition !

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Galerkin FEM:

$$\text{Find} \quad U = \{\vec{u}, p\} \in \mathbf{W}_h^{r,r} := \mathbf{V}_h^r \times \mathbf{Q}_h^r, \text{ s.t.}$$
$$\mathcal{A}(\vec{b}; U, V) = \mathcal{L}(V) \quad \forall\, V = \{\vec{v}, q\} \in \mathbf{W}_h^{r,r}$$

Remedy: "Classical" residual-based stabilization

Residual-based scheme:

$$\text{Find } U = \{\vec{u}, p\} \in \mathbf{W}_h^{r,r} \ : \ \mathcal{A}_{rbs}(\vec{b}; U, V) \ = \ \mathcal{L}_{rbs}(V) \quad \forall\, V = \{\vec{v}, q\} \in \mathbf{W}_h^{r,r}$$

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$$\mathcal{A}_{rbs}(\vec{b}; U, V) = \mathcal{A}(\vec{b}; U, V) + \sum_T \underbrace{\delta_T \left(\hat{L}_{Os}(\vec{b}; \vec{u}, p), (\vec{b} \cdot \vec{\nabla})\vec{v} + \vec{\nabla} q \right)_T}_{SUPG- \ \text{and} \ PSPG-stabilization} + \underbrace{\gamma_T \left(\vec{\nabla} \cdot \vec{u}, \vec{\nabla} \cdot \vec{v} \right)_T}_{grad-div-stabilization}$$

$$\mathcal{L}_{rbs}(V) = \mathcal{L}(V) + \sum_T \overbrace{\delta_T \left(\vec{f}, (\vec{b} \cdot \vec{\nabla})\vec{v} + \vec{\nabla} q \right)_T}$$

with orthogonal L^2-projection $\Pi_T : [L^2(T)]^{d \times d} \rightarrow [P_r(T)]^{d \times d}$ and

$$\hat{L}_{Os}(\vec{b}; \vec{u}, p) := -\vec{\nabla} \cdot \Pi_T(\nu S(\vec{u})) + \vec{\nabla} p + (\vec{b} \cdot \vec{\nabla})\vec{u} + c\vec{u}$$

Analysis on isotropic meshes

$$|[V]|_{rbs}{}^2 := \|\sqrt{\nu}\vec{S}(\vec{v})\|^2_{L^2(\Omega)} + \|\sqrt{c}\vec{v}\|^2_{L^2(\Omega)} + \sum_T \left(\delta_T \|(\vec{b}\cdot\vec{\nabla})\vec{v} + \vec{\nabla}q\|^2_{L^2(T)} + \gamma_T \|\vec{\nabla}\cdot\vec{v}\|^2_{L^2(T)}\right)$$

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Theorem: GL/G. Rapin M³AS 2006

- $\delta_T^u = \delta_T^p \sim \min\{\frac{h_T^2}{r^2\|\nu\|_{L^\infty(T)}};\ \Delta t;\ \frac{h_T}{r\|\mathbf{b}\|_{L^\infty(T)}}\}$ (SUPG/PSPG)
- $\gamma_T \sim \frac{h_T^2}{r^2\delta_T}$ (grad-div) ;

$\rightsquigarrow$

- **Stability:** $\mathcal{A}_{rbs}(\vec{b}; V, V) \geq \frac{1}{2}|[V]|_{rbs}{}^2, \quad \forall V = \{\vec{v}, q\} \in \mathbf{V}_h^r \times \mathbf{Q}_h^r$
- **Galerkin orthogonality:** $\mathcal{A}_{rbs}(\vec{b}; U - U_h, V_h) = 0 \quad \forall V_h \in \mathbf{V}_h^r \times \mathbf{Q}_h^r$
- **A-priori estimate:** $|[U - U_h]|_{rbs}{}^2 \preceq \sum_{T\in\mathcal{T}_h} M_T \frac{h_T^{2(l-1)}}{r^{2(k-1)}} \left(\|\vec{u}\|^2_{H^k(T)} + \|p\|^2_{H^k(T)}\right)$

  with $M_T := \left(\|\nu\|_{L^\infty(T)} + \frac{\|\nu\|^2_{W^{k-1,\infty}(T)}}{\|\nu\|_{L^\infty(T)}} + \frac{\|\vec{b}\|_{L^\infty(T)} h_T}{r} + \frac{h_T^2}{r^2\Delta t}\right)$

Outline

Application of wall functions

Simplification of nonisothermal boundary layer (BL) equations:

$$
\begin{aligned}
-\frac{d}{dy}\left(\nu_e^{BL}\frac{du_x^{BL}}{dy}\right) &= -\beta\theta^{BL}g_x, \quad u_x^{BL}|_{y=0} = 0, \quad u_x^{BL}|_{y=y(\delta)} = u_x(y_\delta) \\
-\frac{d}{dy}\left(a_e^{BL}\frac{d_\theta^{BL}}{dy}\right) &= 0, \qquad\qquad \theta^{BL}|_{y=0} = \theta_W, \quad \theta^{BL}|_{y=y_\delta} = \theta(y_\delta)
\end{aligned}
$$

Overlapping domain decomposition approach:

- Matching of BL-solutions with global solution at Γ_δ
- Replace matching conditions at Γ_δ with wall boundary condition

$$
\nu_e\frac{du_x^{BL}}{dy}|_{y=0} = R, \qquad a_e\frac{d\theta^{BL}}{dy}|_{y=0} = S
$$

- Application of shooting method to IVP (on a layer-adapted 1D-grid) until convergence to matching conditions at Γ_δ

Non-isothermal k-ϵ-φ-$\bar{f}$ turbulence model LAURENCE ET AL. '04

Turbulent viscosity: $\nu_t = C_\mu\, T\, k\varphi \quad \rightsquigarrow \quad \nu_k = \nu + \frac{\nu_t}{Pr_k}, \quad \nu_\epsilon = \nu + \frac{\nu_t}{Pr_\epsilon}$

$$\partial_t k - \vec{\nabla}\cdot(\nu_k \vec{\nabla} k) + (\vec{u}\cdot\vec{\nabla})k = P_k + G - \epsilon$$

$$\partial_t \epsilon - \vec{\nabla}\cdot(\nu_\epsilon \vec{\nabla} \epsilon) + (\vec{u}\cdot\vec{\nabla})\epsilon + \frac{C_{\epsilon 2}}{T}\epsilon = \frac{C_{\epsilon 1}}{T}(P_k + G)$$

$$\partial_t \varphi - \vec{\nabla}\cdot(\nu_k \vec{\nabla} \varphi) + (\vec{u}\cdot\vec{\nabla})\varphi + \frac{P_k + G}{k}\varphi = \bar{f} + \frac{2\nu_k}{k}\vec{\nabla}\varphi\cdot\vec{\nabla}k$$

$$-L^2\Delta\bar{f} + \bar{f} = \frac{(D_1 - 1)(\frac{2}{3} - \varphi)}{T} + \frac{D_2(P_k + G)}{k} + \frac{2\nu}{k}\vec{\nabla}\varphi\cdot\vec{\nabla}k + \nu\Delta\varphi$$

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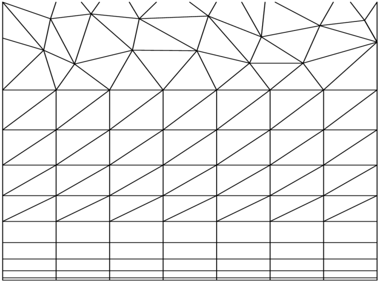
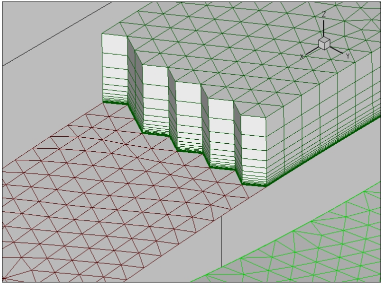
- turbulence time scale $\quad T = \max\left[\min\left(\frac{k}{\epsilon}\,;\,\frac{\alpha_r}{C_\mu\sqrt{6}\,\varphi|S(\vec{u})|}\right)\,;\,6\sqrt{\frac{\nu}{\epsilon}}\right]$
- turbulence length scale $\quad L = C_L \max\left[\min\left(\frac{\sqrt{k^3}}{\epsilon}\,;\,\frac{\alpha_r\sqrt{k}}{c_\mu\sqrt{6}\,\varphi|S(\vec{u})|}\right)\,;\,C_\eta\left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}\right]$
- production terms $\quad P_k = 2\nu_t|S(\vec{u})|^2, \quad G = \beta c_0\frac{k}{\epsilon}\sum_{i,j} g_i\left[\frac{2}{3}k\delta_{ij} - \nu_t(u_{i,j} + u_{j,i})\right]\theta_{,j}$

Resolution of boundary layers with hybrid meshes

Assume: Shear layer located at wall (here: at $x_d = 0$)





Examples of hybrid meshes for $d = 2$ and $d = 3$

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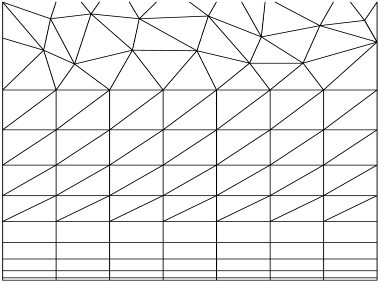

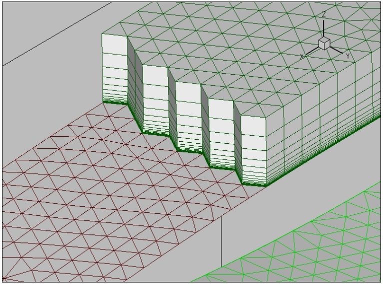

Examples of hybrid meshes for $d = 2$ and $d = 3$

- $\mathcal{T}_h^g$ - (unstructured) isotropic mesh away from wall layers
- $\mathcal{T}_h^{bl}$ - structured anisotropic mesh of tensor product type
- isotropic transition region between $\mathcal{T}_h^g$ and $\mathcal{T}_h^{bl}$

Analysis on hybrid meshes

- Tensor product type mesh in layer zone with refinement in x_d-direction s.t.

$$\text{aspect ratio} \quad \frac{h_{max,T}}{h_{min,T}} \sim \frac{1}{\text{characteristic length scale}} \quad \text{at wall} \;\; x_d = 0$$

- Analysis requires local (anisotropic) interpolation: APEL '99

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- Analysis requires local (anisotropic) interpolation: APEL '99

Modified design for advection-dominated case: $\tilde{h}_T \in [h_{min,T}, h_{max,T}]$

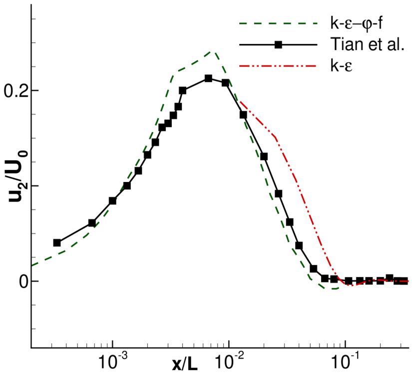
$$\delta_T \sim \min\left(\frac{h^2_{min,T}}{\mu^2_{inv}\nu};\ \Delta t;\ \frac{\tilde{h}_T}{\|\mathbf{b}\|_{(L^\infty(T))^d}} \right), \qquad \gamma_T \sim \frac{h^2_{max,T}}{\delta_T}$$

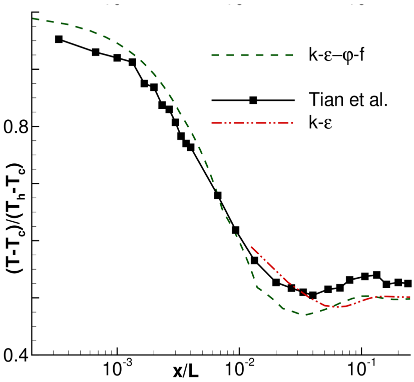
- **Isotropic region:** Standard design with $\tilde{h}_T \sim h_{max,T}$
- **Anisotropic region:** Influence of length $\tilde{h}_T$ (not very sensitive)
  - $\tilde{h}_T = |T|^{\frac{1}{d}}$: $\rightsquigarrow$ reasonable compromise between accuracy and costs

A-priori error estimates: APEL, KNOPP, L. APNUM 2008

Not shown: Similar approach to advection-diffusion-reaction problems

Turbulent flow at $Ra = 1.58 \times 10^9$ in a closed cavity





Left: Boundary layer profiles for vertical velocity component $\frac{u_2}{u_0}$ near hot left wall at $y/L = 0.5$ and experimental data of [TIAN/ KARAYIANNIS '00]

Right: Vertical temperature profile $\frac{T-T_c}{T_h-T_c}$ at $x/L = 0.5$ and experimental data of [TIAN/ KARAYIANNIS '00]

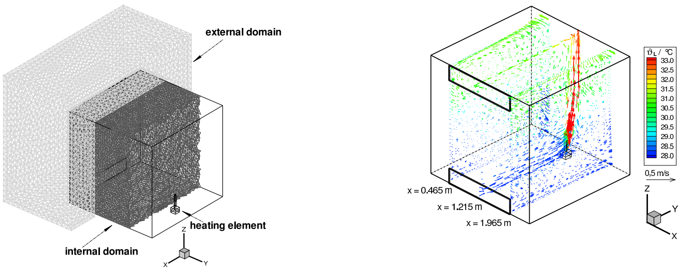
Summary: Residual-based stabilization of linearized problems

- **Residual-based stabilization of advection–diffusion-reaction model:**
  - Robust a-priori analysis on isotropic meshes (h- and p-version)
  - Attempt to diminish spurious oscillations of discrete solutions
- **Residual-based stabilization of linearized Navier-Stokes model:**
  - Equal-order stabilization of velocity/pressure requires stabilization
  - Robust a-priori analysis on isotropic meshes (h- and p-version)
  - Extension of a-priori analysis to hybrid meshes with layer refinement
- **Basic problems:**
  - Implementation rather expensive due to velocity/pressure coupling
  - Diminishing spurious discrete oscillations for higher order elements
  - Robust and scalable algebraic preconditioners (w.r.t. to ν, h, Δt)

Outline

Natural ventilation in a cavity I: $Ra \approx 2 \cdot 10^{10}$

Lube/ Knopp/ Gritzki/ Rösler/ Seifert: *Intern. J. Comput. Math.* 85 (2008) 10



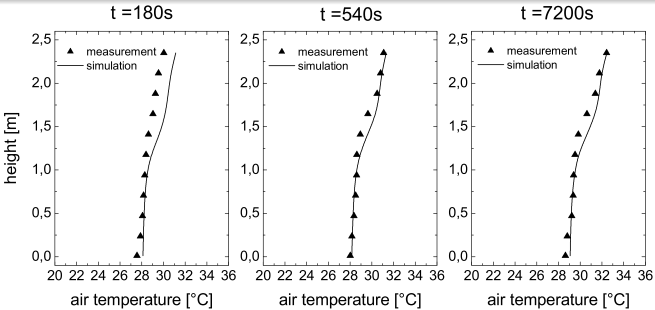
Sketch of cavity with extended domain (left) and prediction for θ at three cross-sections (right)

- Thermally insulated cavity with openings
- flow induced by temperature difference, heating rod and gravity
- simulates displacement ventilation with open windows

Natural ventilation in a cavity II

Non-overlapping domain decomposition:

- Enlarge domain for careful numerical prediction of flow at openings
- Domain decomposition into internal flow domain (cavity) and external domain
- DD-interface conditions instead of inflow/ outflow conditions as inflow field unknown

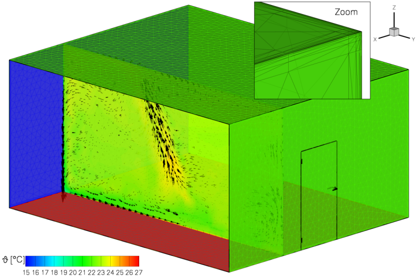


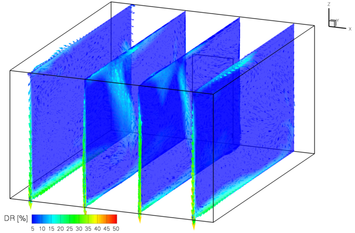
Comparison of temperature $\theta = \theta(z)$ measured and calculated at $x = 1.215$ m, $y = 1.10$ m and $t = 180\,\mathrm{s}/540\,\mathrm{s}/7200\,\mathrm{s}$

- Quasi-steady solution in reasonably well agreement with experimental data over long-time period of 7200s by HASLAVSKY et.al. '04

Simulation with ϕ-$\overline{f}$-model for floor-heating

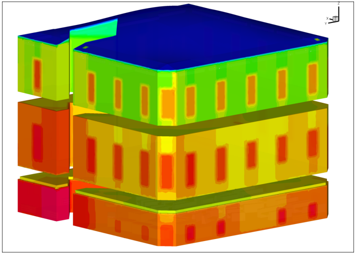
- Room with size $5 \times 6 \times 3$ [m] with heating from below
- Simulation with front of cold air
- Application of full ϕ-$\overline{f}$-model with anisotropic boundary layer resolution

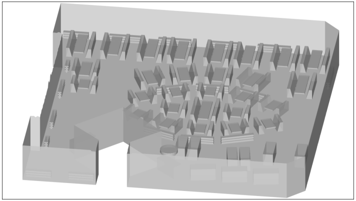




Draught risk avoided !

Indoor-air flow in atrium I

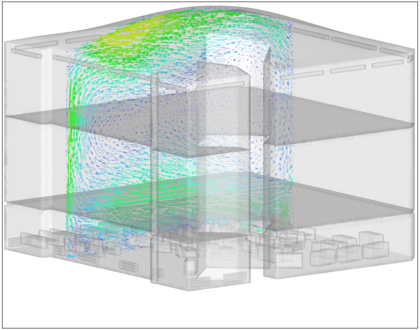


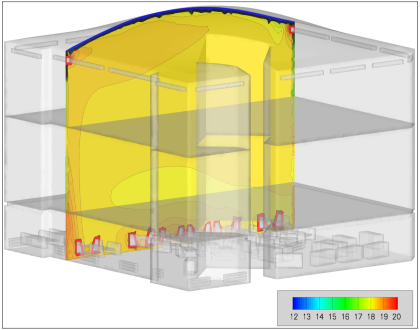


Numerical simulation in atrium with cafeteria:

- Atrium of size $22m \times 22.5m \times 17.2m$
- **Formerly:** Unpleasant air flow/ temperature conditions (caused by curved glass roof)
- **Boundary conditions:** from flow simulation of surrounding buildings
- **Domain decomposition:** into 3 subdomains and 1.2×10^6 tetrahedra (with $\approx 2 \times 10^6$ unknowns)
- Numerical simulation under winter conditions over two hours

Indoor-air flow in atrium II - Flow and temperature fields





"Optimization" with additional heating system under the roof yields reduction of maximal velocity from 1.5 m/s to 0.5 m/s and almost optimized temperature field

Summary. Outlook

Summary

- Simulation of thermally driven turbulent flows via URANS approach
- Application to real life problems (ventilation/ heating systems)
- A-priori analysis of FEM with residual-based stabilization for linearized problems
- Extension to layer-adapted meshes

Outlook

- Application of FEM with local-projection stabilization
- Application of LES/DES approach to thermally driven turbulent flows
- A-posteriori analysis of fully coupled model

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THANKS FOR YOUR ATTENTION !