Analysis of a kinematic dynamo model with FEM–BEM coupling

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Abstract

We consider a kinematic dynamo model in a bounded interior simply connected region Ω and in an insulating exterior region $\Omega^c := \mathbb{R}^3 \backslash \overline{\Omega}$. In the so–called direct problem, the magnetic field $\mathbf{B}$ and the electric field $\mathbf{E}$ are unknown and are driven by a given incompressible flow field $\mathbf{w}$. After eliminating $\mathbf{E}$, a vector and a scalar potential ansatz for $\mathbf{B}$ in the interior and exterior domains, respectively, are applied, leading to a coupled interface problem. We apply a finite element approach in the bounded interior domain Ω, whereas a symmetric boundary element approach in the unbounded exterior domain Ω^c is used. We present results on the well–posedness of the continuous coupled variational formulation, prove well–posedness and stability of the semi–discretized and fully discretized schemes, and we provide quasi–optimal error estimates for the fully discretized scheme.

1 Introduction

Large–scale planetary and stellar magnetic activities are driven by magnetohydrodynamic dynamo processes in the interior of planets and stars. In the convectively unstable case, they consist of large–scale global circulations and small–scale turbulent flows. A widely accepted theory for the generation of large–scale magnetic fields through the effect of small–scale turbulences in a conducting field is the mean–field dynamo theory [21] which will be considered here.

Starting point are the full Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \quad \nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial}{\partial t}\mathbf{D}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = \varrho,$$

where the dielectric displacement $\mathbf{D}$ and the magnetic field intensity $\mathbf{H}$ are expressed through the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$, respectively, by the constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

with the magnetic permeability μ and the electric permittivity ε.
If $\mathbf{w}$ is a given incompressible flow field, we can write an extended Ohm's law as

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{w} \times \mathbf{B} + \frac{Rf}{1 + s|\mathbf{B}|^2}\mathbf{B} \right),$$

where σ is the electric conductivity. The nonlinear saturation term is a turbulence modelling term from the mean–field dynamo theory [9, Section 6.2] where the turbulent electromotive force $\mathcal{E} = \langle \hat{\mathbf{w}} \times \hat{\mathbf{B}} \rangle$ is a key quantity. Here $\langle \cdot \rangle$ indicates an average in the dynamo domain, and $\hat{\mathbf{w}}$ and $\hat{\mathbf{B}}$ denote the fluctuating small–scale velocity and magnetic fields. Following [7] we consider the α–quenching approximation

$$\mathcal{E} \approx \alpha \mathbf{B} = \frac{Rf}{1 + s|\mathbf{B}|^2}\mathbf{B}$$

with a model–oriented function f, and positive parameters s and R. Moreover, following [9, Section 2.5] we neglect the temporal change of the dielectric displacements. Therefore Maxwell's equations decouple to some extend. As we will see in Sect. 2.1, we do not need Gauss law to determine the magnetic field $\mathbf{B}$. In [1, Sect. 8], an asymptotic analysis of a linear eddy current problem is carried out. Finally we assume that the electric conductivity σ and the magnetic permeability μ do not depend on time but satisfy $\mu \geq \mu_{\min} > 0$ and $0 < \sigma_{min} \leq \sigma \leq \sigma_{\max}$ in a bounded, simply connected, and polyhedral Lipschitz domain Ω. Since the exterior domain $\Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}$ is assumed to be insulating, we have $\sigma = 0$ and $\mu = \mu_0$ is constant in Ω^c. Hence we end up with the transient non-linear eddy current problem

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \quad \nabla \times \frac{1}{\mu}\mathbf{B} = \sigma \left(\mathbf{E} + \mathbf{w} \times \mathbf{B} + \frac{Rf}{1 + s|\mathbf{B}|^2}\mathbf{B} \right), \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \Omega, \tag{1.1}$$

coupled with

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \quad \nabla \times \frac{1}{\mu_0}\mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \Omega^c, \tag{1.2}$$

together with the transmission conditions, see e.g. [22]

$$[\mathbf{n} \times \mathbf{E}]_{|\Gamma} = 0, \quad [\mathbf{H} \times \mathbf{n}]_{|\Gamma} = 0, \quad [\mathbf{B} \cdot \mathbf{n}]_{|\Gamma} = 0 \quad \text{on } \Gamma := \partial\Omega, \tag{1.3}$$

where $[\cdot]_{|\Gamma}$ denotes the jump of a function across Γ, and $\mathbf{n}$ is the exterior normal vector. Moreover, we have some initial condition $\mathbf{B}(0) = \mathbf{B}_0$, $\nabla \cdot \mathbf{B}_0 = 0$.
Let us briefly comment on the literature. The majority of stellar and planetary dynamo models employs spectral methods with spherical harmonics which are not feasible in the case of variable data. For bounded domains, the standard approach for the finite element solution of Maxwell's system is to use curl conforming elements, e.g., Nédélec elements, see [5, 6]. Alternatively, a saddle point approach with a (vanishing) pressure like Lagrange multiplier allows to apply Lagrangian finite elements [7]. In the case of discontinuous data, an interior penalty approach together with Lagrange finite elements is addressed in [10, 11, 18].
For the full space $\mathbb{R}^3$, a usual approach is a coupling of finite element (FEM) or finite volume (FVM) methods in a bounded domain with boundary element (BEM) methods in the remaining exterior domain. For a FVM–BEM coupling we refer, e.g., to [15, 29]. The idea of a symmetric FEM–BEM coupling for the stationary linear Maxwell system can be found in [17] and [13] which will be used in this paper as well. An hp–adaptive FEM–BEM coupling is considered in [27]. A comparison of the interior penalty finite element approach and a FVM–BEM coupling is considered in [12].
In the present paper we consider the coupled problem (1.1)–(1.3) where we first eliminate the electric field $\mathbf{E}$ and apply a vector potential ansatz for the magnetic field $\mathbf{B}$ in the interior domain Ω, and a scalar potential ansatz for $\mathbf{B}$ in the exterior domain Ω^c. The coupling of both subproblems at the interface is accomplished by using both boundary integral equations of the exterior Calderón projection where we discuss two different approaches. Then we analyze the well–posedness of the arising continuous coupled problem which is formed by a time–dependent nonlinear problem in the interior, and a quasi–stationary elliptic problem in the exterior domain. For the spatial discretization we first introduce an approximation of the exterior Dirichlet to Neumann map, and apply lowest order Nédélec elements for a finite element discretization. For simplicity we only consider an implicit Euler scheme for the time discretization, but we prove well–posedness of the discrete system arising within each time step. Finally we present a priori error estimates of the fully discretized system.

2 Variational formulation

In this section, we specify the mathematical model and derive a variational formulation of the coupled problem (1.1)–(1.3). Later on, and following the approach in [17], this will be the basis of a symmetric coupling of finite element methods (FEM) and boundary element methods (BEM) in the discrete case. We start by using a vector potential ansatz to describe solutions of the interior problem (1.1), and a scalar potential ansatz for the exterior problem (1.2). Thus we end up with an $\mathbf{A} - \Phi$ formulation, which is widely used in eddy current modeling, see e.g. [16, 17].

2.1 Interior problem

In the interior bounded domain Ω we use a vector potential ansatz to describe the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. From (1.1) we then obtain

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad \frac{\partial}{\partial t}\mathbf{B} + \nabla \times \mathbf{E} = \nabla \times \left(\frac{\partial}{\partial t}\mathbf{A} + \mathbf{E} \right) = \mathbf{0},$$

from which we further conclude

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla \phi$$

for some scalar potential ϕ. Hence we may introduce the new potential

$$\mathbf{u} := \mathbf{A} + \int_0^t \nabla \phi \, ds,$$

from which we conclude

$$\mathbf{B} = \nabla \times \mathbf{u}, \quad \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{u}.$$

Inserting these expressions into the remaining equation of (1.1) we finally obtain the partial differential equation to be solved

$$\begin{aligned} \sigma(x)\frac{\partial}{\partial t}\mathbf{u}(x,t) + \nabla_x \times \frac{1}{\mu(x)}(\nabla_x \times \mathbf{u}(x,t)) \\ -\sigma(x)\,\mathbf{w}(x,t) \times (\nabla_x \times \mathbf{u}(x,t)) - \frac{\sigma(x) R f(x,t)}{1 + s\,|\nabla_x \times \mathbf{u}(x,t)|^2}\nabla_x \times \mathbf{u}(x,t) \;=\; 0. \end{aligned} \tag{2.1}$$

In order to prove ellipticity of the spatial linear second order partial differential operator in (2.1) we will use a scaling argument. Let $\kappa \in \mathbb{R}_+$ be some parameter to be specified later in (3.4). We multiply the partial differential equation (2.1) with the nonnegative function $e^{-\kappa t}$, apply the product rule, and introduce $\widehat{\mathbf{u}}(x,t) := e^{-\kappa t}\mathbf{u}(x,t)$ to obtain

$$\begin{aligned} \sigma(x)\frac{\partial}{\partial t}\widehat{\mathbf{u}}(x,t) + \sigma(x)\kappa\widehat{\mathbf{u}}(x,t) + \nabla_x \times \frac{1}{\mu(x)}(\nabla_x \times \widehat{\mathbf{u}}(x,t)) \\ -\sigma(x)\,\mathbf{w}(x,t) \times (\nabla_x \times \widehat{\mathbf{u}}(x,t)) - \frac{\sigma(x) R f(x,t)}{1 + s e^{2\kappa t}\,|\nabla_x \times \widehat{\mathbf{u}}(x,t)|^2}\nabla_x \times \widehat{\mathbf{u}}(x,t) \;=\; 0. \end{aligned} \tag{2.2}$$

To derive a variational formulation which is related to (2.2) we first introduce the function space

$$H(\mathrm{curl};\Omega) := \left\{ \mathbf{v} \in [L^2(\Omega)]^3 \mid \mathrm{curl}\,\mathbf{v} \in [L^2(\Omega)]^3 \right\}$$

with the graph norm

$$\|\mathbf{v}\|^2_{H(\mathrm{curl};\,\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla \times \mathbf{v}\|^2_{L^2(\Omega)}.$$

Moreover, the duality pairing between the dual space $[H(\mathrm{curl};\Omega)]^*$ and $H(\mathrm{curl};\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$, while the inner product in $[L^2(\Omega)]^3$ is denoted by $(\cdot,\cdot)$. In addition we use $\langle\cdot,\cdot\rangle_\Gamma$ for the duality pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. For the treatment of the time dependent partial differential equation (2.2), we consider the standard vector valued Lebesgue and Sobolev spaces $L^q(0,T;W)$ and $H^1(0,T;W) := W^{1,2}(0,T;W)$ with an appropriate Banach space W. Multiplication of the partial differential equation (2.2) with a test function $\mathbf{v} \in H(\mathrm{curl};\Omega)$, integration over the interior domain Ω, and integration by parts, see, e.g., [2], lead to a variational formulation to find $\widehat{\mathbf{u}} \in H^1(0,T;H(\mathrm{curl}^2;\Omega))$ such that for all test functions $\mathbf{v} \in H(\mathrm{curl};\Omega)$ and almost all $t \in (0,T)$ there holds

$$\begin{aligned} 0 = \Big(\sigma\frac{\partial}{\partial t}\widehat{\mathbf{u}}(t),\mathbf{v}\Big) + \kappa\Big(\sigma\widehat{\mathbf{u}}(t),\mathbf{v}\Big) + \Big(\frac{1}{\mu}\nabla\times\widehat{\mathbf{u}}(t),\nabla\times\mathbf{v}\Big) - \Big\langle\frac{1}{\mu}\gamma_N\widehat{\mathbf{u}}(t),\gamma_D\mathbf{v}\Big\rangle_\tau \\ - \Big(\sigma\mathbf{w}(t)\times(\nabla\times\widehat{\mathbf{u}}(t)),\mathbf{v}\Big) - \Big(\frac{\sigma R f(t)}{1+se^{2\kappa t}|\nabla\times\widehat{\mathbf{u}}(t)|^2}\nabla\times\widehat{\mathbf{u}}(t),\mathbf{v}\Big) \end{aligned} \tag{2.3}$$

together with the initial condition $\widehat{\mathbf{u}}(0) = \mathbf{u}_0$, i.e. $\mathbf{B}_0 = \mathrm{curl}\,\mathbf{u}_0$. The tangential surface trace $\gamma_D\mathbf{v}$ is defined for $\mathbf{v} \in [C(\overline{\Omega})]^3$ by $\gamma_D\mathbf{v}(x) := \mathbf{n}(x)\times(\mathbf{v}(x)\times\mathbf{n}(x))$ for almost all $x \in \Gamma$ and can be generalized to a continuous and surjective mapping $\gamma_D : H(\mathrm{curl};\Omega) \to H_\perp^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$. The Neumann trace is defined by $\gamma_N\mathbf{v} := (\nabla\times\mathbf{v})\times\mathbf{n}$ for smooth $\mathbf{v}$ and can be generalized to a continuous and surjective mapping $\gamma_N : H(\mathrm{curl}^2;\Omega) \to H_\parallel^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$. $\langle\cdot,\cdot\rangle_\tau$ denotes the duality pairing between $H_\perp^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$ and $H_\parallel^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$. For details on the definition of the Sobolev spaces and the operators see [2, 3, 4, 13].

2.2 Exterior domain

In the exterior domain Ω^c, the partial differential equations (1.2) reduce to the solution of

$$\nabla\times\mathbf{B} = 0, \quad \nabla\cdot\mathbf{B} = 0 \quad \text{in } \Omega^c,$$

with an appropriate radiation condition. Hence we introduce the scalar potential $\mathbf{B} = \nabla\Phi$ satisfying

$$\nabla\times\mathbf{B} = \nabla\times\nabla\Phi = 0, \quad \nabla\cdot\mathbf{B} = \nabla\cdot\nabla\Phi = \Delta\Phi = 0 \quad \text{in } \Omega^c, \quad \Phi(x) = \mathcal{O}\Big(\frac{1}{|x|}\Big) \text{ as } |x|\to\infty.$$

Any solution of the Laplace equation in the exterior domain Ω^c can be described by the representation formula

$$\Phi(x) = -\int_\Gamma U^*(x,y)\,\gamma_1^{\mathrm{ext}}\Phi(y)ds_y + \int_\Gamma(\mathbf{n}_y\cdot\nabla_y U^*(x,y))\gamma_0^{\mathrm{ext}}\Phi(y)ds_y \quad \text{for } x\in\Omega^c, \tag{2.4}$$

where

$$U^*(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} \quad \text{for } x,y\in\mathbb{R}^3, x\neq y,$$

is the related fundamental solution. $\gamma_0^{\text{ext}} : H^1_{\text{loc}}(\Omega^c) \to H^{1/2}(\Gamma)$ denotes the standard trace operator where $\gamma_0^{\text{ext}}\Psi = \Psi_{|\Gamma}$ for $\Psi \in C(\overline{\Omega^c})$. $\gamma_1^{\text{ext}} : H^1_L(\Omega^c) \to H^{-1/2}(\Gamma)$ is the normal derivative where $H^1_L(\Omega^c) := \left\{\Psi \in H^1_{\text{loc}}(\Omega^c) : \Delta\Psi \in L^2_{\text{comp}}(\Omega^c)\right\}$ and $\gamma_1^{\text{ext}}\Psi = \mathbf{n}\cdot\gamma_0^{\text{ext}}(\nabla\Psi)$ for $\Psi \in C^1(\overline{\Omega^c})$.
From (2.4) we obtain a system of boundary integral equations which can be written by using the exterior Calderon projection

$$\begin{pmatrix} \gamma_0^{\text{ext}}\Phi \\ \gamma_1^{\text{ext}}\Phi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K' \end{pmatrix} \begin{pmatrix} \gamma_0^{\text{ext}}\Phi \\ \gamma_1^{\text{ext}}\Phi \end{pmatrix} \tag{2.5}$$

where for $x \in \Gamma$ we use the single layer integral operator

$$(V\Psi)(x) = \int_\Gamma U^*(x,y)\Psi(y)ds_y, \quad V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

the double layer integral operator

$$(K\Phi)(x) = \int_\Gamma (\mathbf{n}_y \cdot \nabla_y U^*(x,y))\Phi(y)ds_y, \quad K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

its adjoint

$$(K'\Psi)(x) = \int_\Gamma \mathbf{n}_x \cdot \nabla_x U^*(x,y)\Psi(y)ds_y, \quad K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

and the hypersingular boundary integral operator

$$(D\Phi)(x) = -\mathbf{n}_x \cdot \nabla_x \int_\Gamma (\mathbf{n}_y \cdot \nabla_y U^*(x,y))\Phi(y)ds_y, \quad D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

The mapping properties of the boundary integral operators are well established, see, e.g., [8, 14, 20, 26].

2.3 Transmission problem

For the coupling of the interior partial differential equation (1.1) with the exterior problem (1.2) we need to take care of the transmission conditions (1.3). By using the vector potential ansatz $\mathbf{B} = \nabla \times \mathbf{u}$ in the interior domain Ω and the scalar potential ansatz $\mathbf{B} = \nabla\Phi$ in the exterior domain Ω^c, the transmission conditions (1.3) read

$$\frac{1}{\mu}\gamma_N \mathbf{u} = \frac{1}{\mu_0}\gamma_\tau(\nabla\Phi), \quad \gamma_n(\nabla \times \mathbf{u}) = \gamma_1^{\text{ext}}\Phi \quad \text{on } \Gamma. \tag{2.6}$$

The twisted tangential surface trace $\gamma_\tau \mathbf{v}$ is defined by $\gamma_\tau \mathbf{v} := \mathbf{v} \times \mathbf{n}$ for smooth $\mathbf{v}$ and can be generalized to continuous and surjective mapping $\gamma_\tau : H(\text{curl};\Omega) \to H_\parallel^{-1/2}(\text{div}_\Gamma;\Gamma)$. The normal trace $\gamma_n \mathbf{v}$ is defined by $\gamma_n \mathbf{v} := \mathbf{v} \cdot \mathbf{n}$ for smooth $\mathbf{v}$ and can be generalized

to continuous and surjective mapping $\gamma_n : H(\text{div};\Omega) \to H^{-1/2}(\Gamma)$. For details on the generalizations see [2, 3, 4, 13, 23].
Using the first transmission condition of (2.6), and integration by parts twice, see e.g. [22] we have

$$\left\langle \frac{1}{\mu}\gamma_N \mathbf{u}, \gamma_D \mathbf{v} \right\rangle_\tau = \frac{1}{\mu_0}\langle \gamma_\tau(\nabla\Phi), \gamma_D \mathbf{v}\rangle_\tau = \frac{1}{\mu_0}\langle \gamma_0^{\text{ext}}\Phi, \gamma_n(\nabla\times\mathbf{v})\rangle_\Gamma \tag{2.7}$$

for $\mathbf{v} \in H(\text{curl};\Omega)$. Next, we will use the boundary integral equation in (2.5) to derive two representations of the Neumann to Dirichlet map $\mathcal{B}$ for the exterior Laplace problem.

First boundary integral representation of the Neumann to Dirichlet map $\mathcal{B}$

For the ease of presentation, we introduce the operator $T_n : H(\text{curl};\Omega) \to H^{-1/2}(\Gamma)$ with $T_n\mathbf{v} := \gamma_n(\nabla\times\mathbf{v})$ for $\mathbf{v} \in H(\text{curl};\Omega)$. Note that there holds

$$\|T_n\mathbf{v}\|_{H^{-1/2}(\Gamma)} \le c\|\nabla\times\mathbf{v}\|_{H(\text{div};\Omega)} \le c_T\,\|\mathbf{v}\|_{H(\text{curl};\Omega)}. \tag{2.8}$$

Using the first boundary integral equation in (2.5), and the second transmission condition of (2.6) in (2.7), we have

$$\begin{aligned}\left\langle \frac{1}{\mu}\gamma_N \mathbf{u}, \gamma_D \mathbf{v} \right\rangle_\tau &= \frac{1}{\mu_0}\langle \gamma_0^{\text{ext}}\Phi, T_n\mathbf{v}\rangle_\Gamma = -\frac{1}{\mu_0}\langle V\gamma_1^{\text{ext}}\Phi - (\frac{1}{2}I + K)\gamma_0^{\text{ext}}\Phi, T_n\mathbf{v}\rangle_\Gamma \\ &= -\frac{1}{\mu_0}\langle V(T_n\mathbf{u}) - (\frac{1}{2}I + K)\gamma_0^{\text{ext}}\Phi, T_n\mathbf{v}\rangle_\Gamma.\end{aligned}$$

In addition, we consider the second boundary integral equation in (2.5), which together with the second transmission condition of (2.6) results in a variational formulation to find $\gamma_0^{\text{ext}}\Phi \in H^{1/2}(\Gamma)$ such that

$$\langle D\gamma_0^{\text{ext}}\Phi, \Psi\rangle_\Gamma = -\langle(\frac{1}{2}I + K')\gamma_1^{\text{ext}}\Phi, \Psi\rangle_\Gamma = -\langle(\frac{1}{2}I + K')T_n\mathbf{u}, \Psi\rangle_\Gamma$$

is satisfied for all $\Psi \in H^{1/2}(\Gamma)$. Since the hypersingular boundary integral operator D is only semi–elliptic, and since the scalar potential $\gamma_0^{\text{ext}}\Phi$ is only unique up to an additive constant, we consider the stabilized variational formulation to find $\gamma_0^{\text{ext}}\Phi \in H^{1/2}(\Gamma)$ such that

$$\langle \widetilde{D}\gamma_0^{\text{ext}}\Phi, \Psi\rangle_\Gamma := \langle D\gamma_0^{\text{ext}}\Phi, \Psi\rangle_\Gamma + \langle \gamma_0^{\text{ext}}\Phi, 1\rangle_\Gamma\langle \Psi, 1\rangle_\Gamma = -\langle(\frac{1}{2}I + K')T_n\mathbf{u}, \Psi\rangle_\Gamma \tag{2.9}$$

is satisfied for all $\Psi \in H^{1/2}(\Gamma)$. Although we have fixed the constant part of the scalar potential $\gamma_0^{\text{ext}}\Phi$ when solving the modified variational problem (2.9), the constant part is eliminated when applying $\frac{1}{2}I + K$. Since the stabilized hypersingular integral operator $\widetilde{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is $H^{1/2}(\Gamma)$–elliptic [24], we obtain for the unique solution of the variational formulation (2.9) the representation

$$\gamma_0^{\text{ext}}\Phi = -\widetilde{D}^{-1}(\frac{1}{2}I + K')T_n\mathbf{u}.$$

By using the transformations $\widehat{\mathbf{u}} := e^{-\kappa t}\mathbf{u}$ and $\widehat{\Phi} := e^{-\kappa t}\Phi$ we finally obtain for the coupling term as used in (2.3)

$$\left\langle \frac{1}{\mu}\gamma_N \widehat{\mathbf{u}}, \gamma_D \mathbf{v} \right\rangle_\tau = -\frac{1}{\mu_0}\langle (V + (\frac{1}{2}I + K)\widetilde{D}^{-1}(\frac{1}{2}I + K'))T_n\widehat{\mathbf{u}}, T_n\mathbf{v}\rangle_\Gamma \tag{2.10}$$

for all $\mathbf{v} \in H(\text{curl}; \Omega)$.

Second boundary integral representation of the Neumann to Dirichlet map $\mathcal{B}$

Before we introduce the variational formulation of the transmission problem (1.1)–(1.3) we will present an alternative boundary integral operator representation for the exterior problem, which is probably more suitable for the numerical implementation. As in (2.7), we write the interface term as

$$\left\langle \frac{1}{\mu}\gamma_N \mathbf{u}, \gamma_D \mathbf{v} \right\rangle_\tau = \frac{1}{\mu_0}\langle \gamma_0^{\text{ext}}\Phi, T_n\mathbf{v}\rangle_\Gamma$$

where we now keep the first equation of (2.5) separately,

$$V(\gamma_1^{\text{ext}}\Phi) = (-\frac{1}{2}I + K)\gamma_0^{\text{ext}}\Phi, \quad \text{i.e.} \quad \gamma_1^{\text{ext}}\Phi = V^{-1}(-\frac{1}{2}I + K)\gamma_0^{\text{ext}}\Phi.$$

Inserting into the stabilized version of the second equation of (2.5) and using the second transmission condition of (2.6) this gives

$$T_n\mathbf{u} = \gamma_1^{\text{ext}}\Phi = -\widetilde{D}\gamma_0^{\text{ext}}\Phi + (\frac{1}{2}I - K')\gamma_1^{\text{ext}}\Phi = -\left(\widetilde{D} + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\right)\gamma_0^{\text{ext}}\Phi.$$

Due to the ellipticity and boundedness of $\widetilde{D}$ and V, see e.g. [26], we find

$$\gamma_0^{\text{ext}}\Phi = -\left(\widetilde{D} + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\right)^{-1} T_n\mathbf{u},$$

and therefore we obtain the alternative representation

$$\left\langle \frac{1}{\mu}\gamma_N \widehat{\mathbf{u}}, \gamma_D \mathbf{v} \right\rangle_\tau = -\frac{1}{\mu_0}\left\langle \left(\widetilde{D} + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\right)^{-1} T_n\widehat{\mathbf{u}}, T_n\mathbf{v} \right\rangle_\Gamma. \tag{2.11}$$

Coupled variational formulation

Unifying (2.10) and (2.11) we finally write for the coupling term

$$\left\langle \frac{1}{\mu}\gamma_N \widehat{\mathbf{u}}, \gamma_D \mathbf{v} \right\rangle_\tau = -\frac{1}{\mu_0}\langle \mathcal{B}T_n\widehat{\mathbf{u}}, T_n\mathbf{v}\rangle_\Gamma,$$

where $\mathcal{B} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is defined by

$$\mathcal{B} = V + (\frac{1}{2}I + K)\widetilde{D}^{-1}(\frac{1}{2}I + K') = \Big(\widetilde{D} + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\Big)^{-1}. \tag{2.12}$$

By using the mapping properties of all boundary integral operators we conclude

$$\langle \mathcal{B}\eta, \eta\rangle_\Gamma \geq c_1^{\mathcal{B}} \|\eta\|^2_{H^{-1/2}(\Gamma)}, \quad \|\mathcal{B}\eta\|_{H^{1/2}(\Gamma)} \leq c_2^{\mathcal{B}} \|\eta\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \eta \in H^{-1/2}(\Gamma). \tag{2.13}$$

Now we are in a position to introduce the variational formulation of the transmission problem (1.1)–(1.3).

Problem 2.1 *Find* $\widehat{\mathbf{u}} \in H^1(0,T; H(\mathrm{curl};\Omega))$ *such that for all test functions* $\mathbf{v} \in H(\mathrm{curl};\Omega)$ *and almost all* $t \in (0,T)$ *there holds*

$$\begin{aligned} 0 = &\Big(\sigma\frac{\partial}{\partial t}\widehat{\mathbf{u}}(t), \mathbf{v}\Big) + \kappa\Big(\sigma\widehat{\mathbf{u}}(t), \mathbf{v}\Big) + \Big(\frac{1}{\mu}\nabla\times\widehat{\mathbf{u}}(t), \nabla\times\mathbf{v}\Big) \\ &- \Big(\sigma\mathbf{w}(t)\times(\nabla\times\widehat{\mathbf{u}}(t)), \mathbf{v}\Big) - \Big(\frac{\sigma R f(t)}{1+se^{2\kappa t}|\nabla\times\widehat{\mathbf{u}}(t)|^2}\nabla\times\widehat{\mathbf{u}}(t), \mathbf{v}\Big) + \frac{1}{\mu_0}\langle\mathcal{B}T_n\widehat{\mathbf{u}}(t), T_n\mathbf{v}\rangle_\Gamma \end{aligned} \tag{2.14}$$

together with the initial condition $\widehat{\mathbf{u}}(0) = \mathbf{u}_0$.

Using the dual operator T_n' of T_n and the following definitions

$$\begin{aligned} \langle\mathcal{A}(t)\mathbf{u}, \mathbf{v}\rangle &:= \Big(\frac{1}{\mu}\nabla\times\mathbf{u}, \nabla\times\mathbf{v}\Big) - \Big(\sigma\mathbf{w}(t)\times(\nabla\times\mathbf{u}), \mathbf{v}\Big) + \kappa\Big(\sigma\mathbf{u}, \mathbf{v}\Big), \\ \langle\mathcal{A}_{nl}(\mathbf{u}), \mathbf{v}\rangle &:= \Big(\frac{\sigma R f(t)}{1+se^{2\kappa t}|\nabla\times\mathbf{u}|^2}\nabla\times\mathbf{u}, \mathbf{v}\Big), \\ \langle\mathcal{S}(t)\mathbf{u}, \mathbf{v}\rangle &:= \langle\mathcal{A}(t)\mathbf{u} - \mathcal{A}_{nl}(\mathbf{u}) + \frac{1}{\mu_0}T_n'\mathcal{B}T_n\mathbf{u}, \mathbf{v}\rangle \end{aligned} \tag{2.15}$$

for all $\mathbf{u}, \mathbf{v} \in H(\mathrm{curl};\Omega)$, we can reformulate Problem 2.1 in the following way:

Problem 2.2 *Find* $\widehat{\mathbf{u}} \in H^1(0,T; H(\mathrm{curl};\Omega))$ *such that for almost all* $t \in (0,T)$ *there holds*

$$\sigma\frac{d}{dt}\widehat{\mathbf{u}}(t) + \mathcal{S}(t)\widehat{\mathbf{u}}(t) = 0$$

together with the initial condition $\widehat{\mathbf{u}}(0) = \mathbf{u}_0$.

3 Well–posedness of the coupled formulation

For the subsequent analysis, we apply the following result on nonlinear evolution problems, cf. [30, Theorem 30.A].

Theorem 3.1 *Let $V \subset H \subset V^*$ be an evolution triple with* $\dim V = \infty$. *Let $0 < T < \infty$, and let the following assumptions to be satisfied:*

(a) *For each $t \in (0,T)$, the operator $\mathcal{S}(t) : V \to V^*$ is monotone and hemicontinuous.*

(b) *For each $t \in (0,T)$, the operator $\mathcal{S}(t)$ is coercive, i.e., there exist constants $M > 0$ and $\Lambda \geq 0$ such that*

$$\langle \mathcal{S}(t)v, v\rangle_{V^* \times V} \geq M \; \|v\|_V^2 - \Lambda \quad \textit{for all}\, v \in V.$$

(c) *There exist a nonnegative function $K_1 \in L^2(0,T)$ and a constant $K_2 > 0$ such that*

$$\|\mathcal{S}(t)v\|_{V^*} \leq K_1(t) + K_2 \|v\|_V \quad \textit{for all}\, v \in V, t \in (0,T).$$

(d) *The function $t \mapsto \mathcal{S}(t)$ is weakly measurable, i.e., the function $t \mapsto \langle \mathcal{S}(t)u, v\rangle$ is measurable on $(0,T)$ for all $u, v \in V$.*

(e) *Let $u_0 \in H$ be given.*

Then there exists a unique solution to the problem:

Find $u \in \{v \in L^2(0,T;V) \mid \exists v' \in L^2(0,T;V^*)\}$ which fulfills

$$\frac{d}{dt}u(t) + \mathcal{S}(t)u(t) = 0 \quad \textit{for almost all}\, t \in (0,T), \quad u(0) = u_0.$$

Although we will not give all details on the application of Theorem 3.1, we will sketch two important properties, Lipschitz continuity and strong monotonicity.

Lemma 3.2 *Assume $f \in L^\infty(0,T;L^\infty(\Omega))$, $\mathbf{w} \in L^\infty(0,T;[L^\infty(\Omega)]^3)$. The operator $\mathcal{S}(t)$ as defined in* (2.15) *is Lipschitz continuous, i.e. for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in H(\mathrm{curl};\Omega)$ and for all $\kappa \in \mathbb{R}_+$ and almost all $t \in (0,T)$ there holds*

$$\big\langle \mathcal{S}(t)\mathbf{u}_1 - \mathcal{S}(t)\mathbf{u}_2, \mathbf{v}\big\rangle \;\leq\; c_L^{\mathcal{S}} \, \|\mathbf{u}_1 - \mathbf{u}_2\|_{H(\mathrm{curl};\Omega)} \|\mathbf{v}\|_{H(\mathrm{curl};\Omega)} \tag{3.1}$$

with

$$c_L^{\mathcal{S}} := \max\left\{ \frac{1}{\mu_{\min}}, \sigma_{\max} \operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{w}(t)\|_{L^\infty(\Omega)}, \sigma_{\max}\kappa, \frac{c_2^{\mathcal{B}}}{\mu_0} c_T^2, 3\sigma_{\max}\, R \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\|_{L^\infty(\Omega)} \right\}.$$

Proof. By using (2.13) and (2.8) we first have for all $\mathbf{u},\mathbf{v}\in H(\mathrm{curl};\Omega)$, $t\in(0,T)$,

$$
\begin{aligned}
&\left\langle \left(\mathcal{A}(t)+\frac{1}{\mu_0}T_n'\mathcal{B}T_n\right)\mathbf{u},\mathbf{v}\right\rangle\\
&\quad=\left(\frac{1}{\mu}\nabla\times\mathbf{u},\nabla\times\mathbf{v}\right)-\Big(\sigma\mathbf{w}(t)\times(\nabla\times\mathbf{u}),\mathbf{v}\Big)+\kappa\Big(\sigma\mathbf{u},\mathbf{v}\Big)+\frac{1}{\mu_0}\Big\langle\mathcal{B}T_n\mathbf{u},T_n\mathbf{v}\Big\rangle_\Gamma\\
&\quad\le\frac{1}{\mu_{\min}}\|\nabla\times\mathbf{u}\|_{L^2(\Omega)}\|\nabla\times\mathbf{v}\|_{L^2(\Omega)}+\sigma_{\max}\|\mathbf{w}(t)\|_{L^\infty(\Omega)}\|\nabla\times\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)}\\
&\qquad\qquad+\sigma_{\max}\kappa\,\|\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)}+\frac{c_2^{\mathcal{B}}}{\mu_0}\|T_n\mathbf{u}\|_{H^{-1/2}(\Gamma)}\|T_n\mathbf{v}\|_{H^{-1/2}(\Gamma)}\\
&\quad\le\max\left\{\frac{1}{\mu_{\min}},\sigma_{\max}\operatorname*{ess\,sup}_{t\in(0,T)}\|\mathbf{w}(t)\|_{L^\infty(\Omega)},\sigma_{\max}\kappa,\frac{c_2^{\mathcal{B}}}{\mu_0}c_T^2\right\}\|\mathbf{u}\|_{H(\mathrm{curl};\Omega)}\|\mathbf{v}\|_{H(\mathrm{curl};\Omega)}.
\end{aligned}
$$

It remains to consider the nonlinear term $\mathcal{A}_{nl}$. For $\varrho,\tau\in\mathbb{R}_+$ we define

$$\psi_\varrho(\tau):=\frac{1}{1+\varrho\,\tau^2},\qquad\text{thus}\quad|\psi_\varrho(\tau)|\le1.$$

The Lipschitz continuity

$$|\psi_\varrho(\tau_1)\,\tau_1-\psi_\varrho(\tau_2)\,\tau_2|\le|\tau_1-\tau_2|\quad\text{for all }\tau_1,\tau_2,\varrho\in\mathbb{R}_+\tag{3.2}$$

follows from

$$\left|\frac{d}{d\tau}\left(\frac{\tau}{1+\varrho\,\tau^2}\right)\right|\le1.$$

For $s,t\in\mathbb{R}_+$ we define $\varrho:=se^{2\kappa t}$ and set $\widetilde{\mathbf{u}}_i=\nabla\times\mathbf{u}_i$ to obtain

$$
\begin{aligned}
&\left(\frac{\nabla\times\mathbf{u}_1}{1+se^{2\kappa t}|\nabla\times\mathbf{u}_1|^2}-\frac{\nabla\times\mathbf{u}_2}{1+se^{2\kappa t}|\nabla\times\mathbf{u}_2|^2},\mathbf{v}\right)=\left(\frac{\widetilde{\mathbf{u}}_1}{1+\varrho|\widetilde{\mathbf{u}}_1|^2}-\frac{\widetilde{\mathbf{u}}_2}{1+\varrho|\widetilde{\mathbf{u}}_2|^2},\mathbf{v}\right)\\
&\quad=\Big(\psi_\varrho(|\widetilde{\mathbf{u}}_1|)\widetilde{\mathbf{u}}_1-\psi_\varrho(|\widetilde{\mathbf{u}}_2|)\widetilde{\mathbf{u}}_2,\mathbf{v}\Big)\\
&\quad=\Big(\psi_\varrho(|\widetilde{\mathbf{u}}_1|)(\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2),\mathbf{v}\Big)+\Big(\big(\psi_\varrho(|\widetilde{\mathbf{u}}_1|)-\psi_\varrho(|\widetilde{\mathbf{u}}_2|)\big)\widetilde{\mathbf{u}}_2,\mathbf{v}\Big)\\
&\quad=\int_\Omega\psi_\varrho(|\widetilde{\mathbf{u}}_1|)(\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2)\cdot\mathbf{v}\,dx+\int_\Omega\big(\psi_\varrho(|\widetilde{\mathbf{u}}_1|)-\psi_\varrho(|\widetilde{\mathbf{u}}_2|)\big)\,\widetilde{\mathbf{u}}_2\cdot\mathbf{v}\,dx\\
&\quad\le\int_\Omega|\psi_\varrho(|\widetilde{\mathbf{u}}_1|)|\,|\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2|\,|\mathbf{v}|\,dx+\int_\Omega|\psi_\varrho(|\widetilde{\mathbf{u}}_1|)-\psi_\varrho(|\widetilde{\mathbf{u}}_2|)|\,|\widetilde{\mathbf{u}}_2|\,|\mathbf{v}|\,dx\\
&\quad\le\|\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)}+\int_\Omega|\psi_\varrho(|\widetilde{\mathbf{u}}_1|)|\,\big(|\widetilde{\mathbf{u}}_2|-|\widetilde{\mathbf{u}}_1|\big)\,|\mathbf{v}|\,dx\\
&\qquad\qquad+\int_\Omega\Big(|\psi_\varrho(|\widetilde{\mathbf{u}}_1|)|\,|\widetilde{\mathbf{u}}_1|-|\psi_\varrho(|\widetilde{\mathbf{u}}_2|)|\,|\widetilde{\mathbf{u}}_2|\Big)\,|\mathbf{v}|\,dx\\
&\quad\overset{(3.2)}{\le}\|\widetilde{\mathbf{u}}_1-\widetilde{\mathbf{u}}_2\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)}+\big\||\widetilde{\mathbf{u}}_1|-|\widetilde{\mathbf{u}}_2|\big\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)}+\big\||\widetilde{\mathbf{u}}_1|-|\widetilde{\mathbf{u}}_2|\big\|_{L^2(\Omega)}\|\mathbf{v}\|_{L^2(\Omega)}.
\end{aligned}
$$

By using the triangle inequality $\big||\widetilde{\mathbf{u}}_1| - |\widetilde{\mathbf{u}}_2|\big| \le |\widetilde{\mathbf{u}}_1 - \widetilde{\mathbf{u}}_2|$ we finally obtain

$$\langle A_{nl}(\mathbf{u}_1) - A_{nl}(\mathbf{u}_2), \mathbf{v}\rangle \le 3\,\sigma_{\max} R\, \|f(t)\|_{L^\infty(\Omega)} \|\nabla\times(\mathbf{u}_1-\mathbf{u}_2)\|_{L^2(\Omega)} \|\mathbf{v}\|_{L^2(\Omega)}, \tag{3.3}$$

which together with the boundedness of the linear part concludes the proof. ■

Let us finally consider the strong monotonicity of $\mathcal{S}$.

Lemma 3.3 *Let $\mathcal{S}(t)$ be given as in (2.15), $f \in L^\infty(0,T;L^\infty(\Omega))$, $\mathbf{w} \in L^\infty(0,T;[L^\infty(\Omega)]^3)$, and let $\kappa \in \mathbb{R}_+$ be chosen such that*

$$\kappa \ge \mu_{\max}\sigma_{\max} \operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + 3R\|f(t)\|_{L^\infty(\Omega)}\Big)^2 \tag{3.4}$$

is satisfied. Then there holds the monotonicity estimate

$$\langle \mathcal{S}(t)\mathbf{u} - \mathcal{S}(t)\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \ge c_M \|\mathbf{u}-\mathbf{v}\|^2_{H(\mathrm{curl};\Omega)} \quad \text{for all } \mathbf{u},\mathbf{v} \in H(\mathrm{curl};\Omega), \tag{3.5}$$

for almost all $t \in (0,T)$ with some positive constant c_M.

Proof. By neglecting the positive definite operator $\mathcal{B}$ (see (2.13)), and by using the Lipschitz continuity (3.3), we have

$$\begin{aligned}
&\langle \mathcal{S}(t)\mathbf{u} - \mathcal{S}(t)\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \ge \langle \mathcal{A}(t)(\mathbf{u}-\mathbf{v}) - \mathcal{A}_{nl}(\mathbf{u}) + \mathcal{A}_{nl}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle \\
&= \Big(\frac{1}{\mu}\nabla\times(\mathbf{u}-\mathbf{v}), \nabla\times(\mathbf{u}-\mathbf{v})\Big) - \Big(\sigma\mathbf{w}(t)\times(\nabla\times(\mathbf{u}-\mathbf{v})), \mathbf{u}-\mathbf{v}\Big) \\
&\quad + \kappa\Big(\sigma(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\Big) - \Big(\sigma f(t) R\Big(\frac{\nabla\times\mathbf{u}}{1+se^{2\kappa t}|\nabla\times\mathbf{u}|^2} - \frac{\nabla\times\mathbf{v}}{1+se^{2\kappa t}|\nabla\times\mathbf{v}|^2}\Big), \mathbf{u}-\mathbf{v}\Big) \\
&\ge \frac{1}{\mu_{\max}}\|\nabla\times(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} + \kappa\|\sqrt{\sigma}(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} \\
&\quad - \sqrt{\sigma_{\max}} \operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + 3R\|f(t)\|_{L^\infty(\Omega)}\Big) \|\nabla\times(\mathbf{u}-\mathbf{v})\|_{L^2(\Omega)} \|\sqrt{\sigma}(\mathbf{u}-\mathbf{v})\|_{L^2(\Omega)} \\
&\ge \frac{1}{\mu_{\max}}\|\nabla\times(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} + \kappa\|\sqrt{\sigma}(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} \\
&\quad - \frac{1}{2}\sqrt{\sigma_{\max}} \operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + 3R\|f(t)\|_{L^\infty(\Omega)}\Big) \\
&\qquad\qquad \cdot \Big(\gamma\|\nabla\times(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|\sqrt{\sigma}(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)}\Big)
\end{aligned}$$

with $\gamma > 0$. By setting

$$\gamma = \frac{1}{\mu_{max}\sqrt{\sigma_{max}} \operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + 3R\|f(t)\|_{L^\infty(\Omega)}\Big)}$$

for

$$\kappa \geq \mu_{\max}\sigma_{max} \operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + 3R\|f(t)\|_{L^\infty(\Omega)} \Big)^2$$

we obtain

$$\begin{aligned} \langle \mathcal{S}(t)\mathbf{u} - \mathcal{S}(t)\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle &\geq \frac{1}{2\mu_{\max}} \|\nabla \times (\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|\sqrt{\sigma}(\mathbf{u}-\mathbf{v})\|^2_{L^2(\Omega)} \\ &\geq c_M \|\mathbf{u}-\mathbf{v}\|^2_{H(\mathrm{curl};\Omega)} \end{aligned} \tag{3.6}$$

with the positive constant

$$c_M := \min\left(\frac{1}{2\mu_{\max}}; \frac{\mu_{\max}\sigma_{max}}{2\sigma_{min}} \operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + 3R\|f(t)\|_{L^\infty(\Omega)} \Big)^2 \right).$$

■

All other assumptions of Theorem 3.1 follow in a similar way, so that we can conclude unique solvability of Problem 2.2.

4 Discretization of the coupled problem

In this section we describe the spatial and temporal discretization of Problem 2.2. First, let $\widetilde{\mathcal{B}}$ some bounded and at least positive semi–definite approximation of $\mathcal{B}$ which results from the approximate solution of the involved boundary integral equations. Specific approximations will be given in Sect. 4.3. Such an approximation implies an approximate operator

$$\widetilde{\mathcal{S}}(t) := \mathcal{A}(t) - \mathcal{A}_{nl}(\cdot) + \frac{1}{\mu_0} T_n' \widetilde{\mathcal{B}} T_n.$$

Instead of Problem 2.2 we now consider a perturbed evolution equation.

Problem 4.1 *Find $\widetilde{\mathbf{u}} \in H^1(0,T;H(\mathrm{curl};\Omega))$ such that for almost all $t \in (0,T)$ there holds*

$$\sigma \frac{d}{dt}\widetilde{\mathbf{u}} + \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}} = 0, \quad \widetilde{\mathbf{u}}(0) = \mathbf{u}_0.$$

As for $\mathcal{S}$ we can prove all required assumptions of Theorem 3.1 for the perturbed operator $\widetilde{\mathcal{S}}$, in particular, the Lipschitz continuity (3.1) (with a Lipschitz constant $c_L^{\widetilde{\mathcal{S}}}$) and the monotonicity estimate (3.5) (with the same constant c_M) remain true. Hence we conclude the well–posedness of Problem 4.1. It remains to estimate the error due to the use of the approximation $\widetilde{\mathcal{B}}$.

Lemma 4.1 *Let $\widetilde{\mathcal{B}} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ be some bounded and positive semi–definite approximation of $\mathcal{B}$. Let $\widehat{\mathbf{u}}, \widetilde{\mathbf{u}} \in L^2(0,T;H(\mathrm{curl};\Omega))$ be the unique solutions of Problems 2.2 and 4.1, respectively. Then there hold the error estimates*

$$\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \widetilde{\mathbf{u}}(t))\|^2_{L_2(\Omega)} + c_M\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \leq \frac{c_T^2}{c_M \mu_0^2} \|(\widetilde{\mathcal{B}} - \mathcal{B})T_n\widehat{\mathbf{u}}\|^2_{L^2(0,t,H^{1/2}(\Gamma))}$$

for almost all $t \in (0, T]$ *and*

$$\|\sigma \frac{d}{dt}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|_{L^2(0,T;[H(curl;\Omega)]^*)} \le \frac{c_T}{\mu_0}\left(1 + \frac{c_L^{\tilde{S}}}{c_M}\right) \|(\mathcal{B} - \widetilde{\mathcal{B}})T_n\widehat{\mathbf{u}}\|_{L^2(0,T;H^{1/2}(\Gamma))}.$$

Proof. From Problems 2.2 and 4.1 we obtain

$$\sigma \frac{d}{dt}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}) + \mathcal{A}(t)(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}) + \frac{1}{\mu_0}T_n'\widetilde{\mathcal{B}}T_n(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}) - \mathcal{A}_{nl}(\widehat{\mathbf{u}}) + \mathcal{A}_{nl}(\widetilde{\mathbf{u}}) = \frac{1}{\mu_0}T_n'(\widetilde{\mathcal{B}} - \mathcal{B})T_n\widehat{\mathbf{u}},$$

and by using the monotonicity of $\widetilde{\mathcal{S}}$ and the continuity of T_n we conclude, for any $\gamma \in \mathbb{R}_+$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\sigma}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{L_2(\Omega)} + c_M\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{H(\mathrm{curl};\Omega)} &\le \frac{1}{\mu_0}\langle[\widetilde{\mathcal{B}} - \mathcal{B}]T_n\widehat{\mathbf{u}}, T_n(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\rangle_\Gamma \\ &\le \frac{1}{\mu_0}\|(\widetilde{\mathcal{B}} - \mathcal{B})T_n\widehat{\mathbf{u}}\|_{H^{1/2}(\Gamma)}\|T_n(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|_{H^{-1/2}(\Gamma)} \\ &\le \frac{c_T}{\mu_0}\|(\widetilde{\mathcal{B}} - \mathcal{B})T_n\widehat{\mathbf{u}}\|_{H^{1/2}(\Gamma)}\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|_{H(\mathrm{curl};\Omega)} \\ &\le \frac{1}{2}\frac{c_T}{\mu_0}\left[\frac{1}{\gamma}\|(\widetilde{\mathcal{B}} - \mathcal{B})T_n\widehat{\mathbf{u}}\|^2_{H^{1/2}(\Gamma)} + \gamma\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{H(\mathrm{curl};\Omega)}\right].\end{aligned}$$

In particular for $\gamma = c_M\mu_0/c_T$ we then obtain

$$\frac{d}{dt}\|\sqrt{\sigma}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{L_2(\Omega)} + c_M\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{H(\mathrm{curl};\Omega)} \le \frac{c_T^2}{c_M\mu_0^2}\|(\widetilde{\mathcal{B}} - \mathcal{B})T_n\widehat{\mathbf{u}}\|^2_{H^{1/2}(\Gamma)}.$$

Integration in time gives the first estimate, where we use $\widehat{\mathbf{u}}(0) = \widetilde{\mathbf{u}}(0) = \mathbf{u}_0$.
On the other hand, we have

$$\sigma\frac{d}{dt}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}) = \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}} - \mathcal{S}(t)\widehat{\mathbf{u}}.$$

For $\mathbf{v} \in H(\mathrm{curl};\Omega)$ we then conclude

$$\begin{aligned}\langle\sigma\frac{d}{dt}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}), \mathbf{v}\rangle &= \langle\widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}} - \mathcal{S}(t)\widehat{\mathbf{u}}, \mathbf{v}\rangle = \langle\widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}} - \widetilde{\mathcal{S}}(t)\widehat{\mathbf{u}}, \mathbf{v}\rangle + \langle\widetilde{\mathcal{S}}(t)\widehat{\mathbf{u}} - \mathcal{S}(t)\widehat{\mathbf{u}}, \mathbf{v}\rangle \\ &\le c_L^{\tilde{S}}\,\|\widetilde{\mathbf{u}} - \widehat{\mathbf{u}}\|_{H(\mathrm{curl};\Omega)}\|\mathbf{v}\|_{H(\mathrm{curl};\Omega)} + \frac{1}{\mu_0}\|(\mathcal{B} - \widetilde{\mathcal{B}})T_n\widehat{\mathbf{u}}\|_{H^{1/2}(\Gamma)}\|T_n\mathbf{v}\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Using (2.8), dividing by $\|\mathbf{v}\|_{H(\mathrm{curl};\Omega)}$, and taking the supremum over all $\mathbf{v} \in H(\mathrm{curl};\Omega)$, $\mathbf{v} \not\equiv \mathbf{0}$, this gives

$$\|\sigma\frac{d}{dt}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|_{[H(\mathrm{curl};\Omega)]^*} \le c_L^{\tilde{S}}\|\widetilde{\mathbf{u}} - \widehat{\mathbf{u}}\|_{H(\mathrm{curl};\Omega)} + \frac{c_T}{\mu_0}\|(\mathcal{B} - \widetilde{\mathcal{B}})T_n\widehat{\mathbf{u}}\|_{H^{1/2}(\Gamma)}.$$

Integration in time and using the first estimate gives the second result. ■

4.1 Spatial discretization

Next we discuss the spatial discretization of the perturbed evolution Problem 4.1 where we start with the definition of appropriate discrete spaces. For the discretized vector potential in the interior domain Ω we use $X_h \subset H(\mathrm{curl};\Omega)$ which consists of lowest order Nédélec elements.

Problem 4.2 *Find $\widetilde{\mathbf{u}}_h \in H^1(0,T;X_h)$ such that for almost all $t \in (0,T)$ and all $\mathbf{v}_h \in X_h$ there holds*

$$\left\langle \sigma \frac{d}{dt}\widetilde{\mathbf{u}}_h(t), \mathbf{v}_h \right\rangle + \left\langle \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}}_h(t), \mathbf{v}_h \right\rangle = 0, \quad \left(\widetilde{\mathbf{u}}_h(0) - \mathbf{u}_0, \mathbf{v}_h\right) = 0.$$

Theorem 4.2 *Let $\widetilde{\mathbf{u}} \in H^1(0,T;H(\mathrm{curl};\Omega))$ and $\widetilde{\mathbf{u}}_h \in H^1(0,T;X_h)$ be the unique solutions of Problems 4.1 and 4.2, respectively. Let $\pi_h\widehat{\mathbf{u}} \in X_h$ be some approximation of the solution $\widehat{\mathbf{u}} \in H(\mathrm{curl};\Omega)$ of Problem 2.2. Then there holds the error estimate*

$$\begin{aligned}
&\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \widetilde{\mathbf{u}}_h(t))\|^2_{L^2(\Omega)} + c_M\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\
&\quad\le c(c_M, c_L^{\widetilde{\mathcal{S}}}) \Big(\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \pi_h\widehat{\mathbf{u}}(t))\|^2_{L^2(\Omega)} + \|\sqrt{\sigma}(\pi_h\mathbf{u}_0 - \mathbf{u}_0)\|^2_{L^2(\Omega)} \\
&\qquad\qquad + \|\sqrt{\sigma}(\mathbf{u}_0 - \widetilde{\mathbf{u}}_h(0))\|^2_{L^2(\Omega)} + \|\pi_h\widehat{\mathbf{u}} - \widehat{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\
&\qquad\qquad + \|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{L^2(0,t;[H(\mathrm{curl};\Omega)]^*)} + \|(\mathcal{B} - \widetilde{\mathcal{B}})T_n\widehat{\mathbf{u}}\|_{L^2(0,t;H^{1/2}(\Gamma))}\Big)
\end{aligned}$$

for almost all $t \in (0,T]$. Under our assumptions on μ, σ, f, and $\boldsymbol{w}$, the constant $c(c_M, c_L^{\widetilde{\mathcal{S}}})$ is independent of $t \in (0,T]$ but might depend on T via the constants c_M and $c_L^{\widetilde{\mathcal{S}}}$.

Proof. Since $X_h \subset H(\mathrm{curl};\Omega)$, we first obtain the Galerkin orthogonality

$$\left\langle \sigma\frac{d}{dt}(\widetilde{\mathbf{u}} - \widetilde{\mathbf{u}}_h), \mathbf{v}_h\right\rangle + \left\langle \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}} - \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}}_h, \mathbf{v}_h\right\rangle = 0 \quad \text{for all } \mathbf{v}_h \in X_h,$$

or equivalently

$$\left\langle \sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h), \mathbf{v}_h\right\rangle + \left\langle \widetilde{\mathcal{S}}(t)\pi_h\widehat{\mathbf{u}} - \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}}_h, \mathbf{v}_h\right\rangle = \left\langle \widetilde{\mathcal{S}}(t)\pi_h\widehat{\mathbf{u}} - \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}}, \mathbf{v}_h\right\rangle + \left\langle \sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}), \mathbf{v}_h\right\rangle$$

for all $\mathbf{v}_h \in X_h$. In particular for $\mathbf{v}_h = \pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h$ we conclude, by using the monotonicity and the Lipschitz continuity of $\widetilde{\mathcal{S}}$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h)\|^2_{L^2(\Omega)} + c_M\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{H(\mathrm{curl};\Omega)} \\
&\quad\le c_L^{\widetilde{\mathcal{S}}}\,\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|_{H(\mathrm{curl};\Omega)}\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|_{H(\mathrm{curl};\Omega)} \\
&\qquad\quad + \|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|_{[H(\mathrm{curl};\Omega)]^*}\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|_{H(\mathrm{curl};\Omega)} \\
&\quad\le \frac{1}{2}c_L^{\widetilde{\mathcal{S}}}\left(\frac{1}{\gamma_1}\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{H(\mathrm{curl};\Omega)} + \gamma_1\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{H(\mathrm{curl};\Omega)}\right) \\
&\qquad\qquad + \frac{1}{2}\left(\frac{1}{\gamma_2}\|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{[H(\mathrm{curl};\Omega)]^*} + \gamma_2\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{H(\mathrm{curl};\Omega)}\right)
\end{aligned}$$

for some positive constants γ_1, γ_2. In particular for $\gamma_1 = \frac{1}{2}c_M/c_L^{\widetilde{\mathcal{S}}}$ and $\gamma_2 = \frac{1}{2}c_M$ we obtain

$$\frac{d}{dt}\|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h)\|^2_{L^2(\Omega)} + c_M\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{H(\mathrm{curl};\Omega)} \\ \leq 2\frac{(c_L^{\widetilde{\mathcal{S}}})^2}{c_M}\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{H(\mathrm{curl};\Omega)} + \frac{2}{c_M}\|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{[H(\mathrm{curl};\Omega)]^*}.$$

Integration in time gives

$$\|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}}(t) - \widetilde{\mathbf{u}}_h(t))\|^2_{L^2(\Omega)} + c_M\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ \leq \|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}}(0) - \widetilde{\mathbf{u}}_h(0))\|^2_{L^2(\Omega)} + 2\frac{(c_L^{\widetilde{\mathcal{S}}})^2}{c_M}\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ +\frac{2}{c_M}\|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{L^2(0,t;[H(\mathrm{curl};\Omega)]^*)}.$$

Hence we find, by using the triangle inequality,

$$\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \widetilde{\mathbf{u}}_h(t))\|^2_{L^2(\Omega)} + c_M\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ \leq 2\Big(\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \pi_h\widehat{\mathbf{u}}(t))\|^2_{L^2(\Omega)} + \|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}}(t) - \widetilde{\mathbf{u}}_h(t))\|^2_{L^2(\Omega)}\Big) \\ +2c_M\Big(\|\widehat{\mathbf{u}} - \pi_h\widehat{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} + \|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))}\Big) \\ \leq 2\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \pi_h\widehat{\mathbf{u}}(t))\|^2_{L^2(\Omega)} + 2c_M\|\widehat{\mathbf{u}} - \pi_h\widehat{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ +2\|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}}(0) - \widetilde{\mathbf{u}}_h(0))\|^2_{L^2(\Omega)} + 4\frac{(c_L^{\widetilde{\mathcal{S}}})^2}{c_M}\|\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ +\frac{4}{c_M}\|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{L^2(0,t;[H(\mathrm{curl};\Omega)]^*)}.$$

Again by using the triangle inequality we further conclude

$$\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \widetilde{\mathbf{u}}_h(t))\|^2_{L^2(\Omega)} + c_M\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}_h\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ \leq 2\|\sqrt{\sigma}(\widehat{\mathbf{u}}(t) - \pi_h\widehat{\mathbf{u}}(t))\|^2_{L^2(\Omega)} + 2c_M\|\widehat{\mathbf{u}} - \pi_h\widehat{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ +4\|\sqrt{\sigma}(\pi_h\widehat{\mathbf{u}}(0) - \mathbf{u}_0)\|^2_{L^2(\Omega)} + 4\|\sqrt{\sigma}(\mathbf{u}_0 - \widetilde{\mathbf{u}}_h(0))\|^2_{L^2(\Omega)} \\ +8\frac{(c_L^{\widetilde{\mathcal{S}}})^2}{c_M}\|\pi_h\widehat{\mathbf{u}} - \widehat{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} + 8\frac{(c_L^{\widetilde{\mathcal{S}}})^2}{c_M}\|\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\|^2_{L^2(0,t;H(\mathrm{curl};\Omega))} \\ +\frac{8}{c_M}\|\sigma\frac{d}{dt}(\pi_h\widehat{\mathbf{u}} - \widehat{\mathbf{u}})\|^2_{L^2(0,t;[H(\mathrm{curl};\Omega)]^*)} + \frac{8}{c_M}\|\sigma\frac{d}{dt}(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}})\|^2_{L^2(0,t;[H(\mathrm{curl};\Omega)]^*)}$$

and the assertion follows by using Lemma 4.1. ■

4.2 Time discretization

We proceed with the time discretization of the semi–discrete Problem 4.2 which was to find $\widetilde{\mathbf{u}}_h \in H^1(0,T;X_h)$ such that for almost all $t \in (0,T)$ and all $\mathbf{v}_h \in X_h$ there holds

$$\left\langle \sigma \frac{d}{dt}\widetilde{\mathbf{u}}_h(t), \mathbf{v}_h \right\rangle + \left\langle \widetilde{\mathcal{S}}(t)\widetilde{\mathbf{u}}_h(t), \mathbf{v}_h \right\rangle = 0, \quad \Big(\widetilde{\mathbf{u}}_h(0) - \mathbf{u}_0, \mathbf{v}_h\Big) = 0.$$

For simplicity we only consider the implicit Euler scheme with a constant time step $\tau = \frac{T}{M}$, and with nodes $t_m = m\tau$, $m = 0, \ldots, M$. We do not consider the explicit Euler scheme which results in a linear problem for stability reasons. In addition, if we need many time steps for stability reasons the overall computational times will be very high as the solving of the coupled system is costly in each time step. If we evaluate a function $\mathbf{u}$ at time t_m we write $\mathbf{u}^m$. The discretized time derivative is $\Delta_\tau \mathbf{v}^m := \frac{1}{\tau}(\mathbf{v}^m - \mathbf{v}^{m-1})$, $m = 1, \ldots, M$. Hence we obtain the following fully discretized problem.

Problem 4.3 *Given* $\widetilde{\mathbf{u}}_h^0$*, find for all* $m = 1, \ldots, M$ $\widetilde{u}_h^m \in X_h$ *such that*

$$\left\langle \sigma \frac{\widetilde{\mathbf{u}}_h^m - \widetilde{\mathbf{u}}_h^{m-1}}{\tau}, \mathbf{v}_h \right\rangle + \langle \widetilde{\mathcal{S}}(t_m)\widetilde{\mathbf{u}}_h^m, \mathbf{v}_h \rangle = 0 \quad \textit{for all } \mathbf{v}_h \in X_h.$$

The well–posedness of the fully discretized Problem 4.3 follows from the main theorem on strongly monotone operators, see, e.g., [30]. In addition we can prove the following stability estimate.

Lemma 4.3 *Let* $f(t) \in L^\infty(\Omega)$*,* $\mathbf{w}(t) \in L^\infty(\Omega)]^3$ *for all* $t \in (0,T]$ *and let* $\kappa \in \mathbb{R}_+$ *be chosen according to* (3.4)*. For the solution of the fully discrete Problem* 4.3 *there holds*

$$\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^M\|^2_{L^2(\Omega)} + \frac{\tau}{\mu_{\max}} \sum_{m=1}^{M} \|\nabla \times \widetilde{\mathbf{u}}_h^m\|^2_{L^2(\Omega)} \le \|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^0\|^2_{L^2(\Omega)}.$$

Proof. Choosing $\tau\widetilde{\mathbf{u}}_h^m$ as a test function in Problem 4.3 this gives

$$\langle \sigma\widetilde{\mathbf{u}}_h^m + \tau\mathcal{A}(t_m)\widetilde{\mathbf{u}}_h^m - \tau\mathcal{A}_{nl}(\widetilde{\mathbf{u}}_h^m) + \frac{\tau}{\mu_0} T_n' \widetilde{\mathcal{B}} T_n \widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^m \rangle = \langle \sigma\widetilde{\mathbf{u}}_h^{m-1}, \widetilde{\mathbf{u}}_h^m \rangle,$$

which results in

$$
\begin{aligned}
&\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 + \frac{\tau}{\mu_{\max}}\|\nabla\times\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 + \kappa\tau\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 \\
&\quad\le \|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^{m-1}\|_{L^2(\Omega)}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)} + \tau\Big(\sigma\mathbf{w}(t_m)\times(\nabla\times\widetilde{\mathbf{u}}_h^m), \widetilde{\mathbf{u}}_h^m\Big) \\
&\qquad\quad +\tau\left(\sigma\frac{Rf(t_m)}{1+\varrho e^{2\kappa t_m}|\nabla\times\widetilde{\mathbf{u}}_h^m|^2}\nabla\times\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^m\right) \\
&\quad\le \frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^{m-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 \\
&\qquad\quad +\tau\sqrt{\sigma_{\max}}\sup_{t\in(0,T]}\Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + R\|f(t)\|_{L^\infty(\Omega)}\Big)\|\nabla\times\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)} \\
&\quad\le \frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^{m-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2}\tau\sqrt{\sigma_{\max}}\sup_{t\in(0,T]}\Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + R\|f(t)\|_{L^\infty(\Omega)}\Big)\left(\gamma\|\nabla\times\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 + \frac{1}{\gamma}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2\right)
\end{aligned}
$$

for some positive constant γ. In particular for

$$
\gamma = \frac{1}{\mu_{\max}\sqrt{\sigma_{\max}}\sup_{t\in(0,T]}\Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + R\|f(t)\|_{L^\infty(\Omega)}\Big)}
$$

we conclude

$$
\begin{aligned}
&\frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 + \frac{\tau}{2\mu_{\max}}\|\nabla\times\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 \\
&\quad\le \frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^{m-1}\|_{L^2(\Omega)}^2 \\
&\qquad\qquad +\tau\left(\frac{1}{2}\mu_{\max}\sigma_{\max}\sup_{t\in(0,T]}\Big(\|\mathbf{w}(t)\|_{L^\infty(\Omega)} + R\|f(t)\|_{L^\infty(\Omega)}\Big)^2 - \kappa\right)\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 \\
&\quad\le \frac{1}{2}\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^{m-1}\|_{L^2(\Omega)}^2
\end{aligned}
$$

if we chose κ according to (3.4). Hence we obtain

$$
\|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 + \frac{\tau}{\mu_{\max}}\|\nabla\times\widetilde{\mathbf{u}}_h^m\|_{L^2(\Omega)}^2 \le \|\sqrt{\sigma}\widetilde{\mathbf{u}}_h^{m-1}\|_{L^2(\Omega)}^2
$$

and summation over $m = 1, \ldots, M$ gives the desired estimate. ■

4.3 Boundary element approximations

In this section we will introduce suitable approximations $\widetilde{\mathcal{B}}$ of the operator $\mathcal{B}$ as defined in (2.12).

First representation of the Neumann to Dirichlet map $\mathcal{B}$

Using the first representation $\mathcal{B}_1$ in (2.12), the application of $\mathcal{B}$ for a given $\psi \in H^{-1/2}(\Gamma)$ reads

$$\mathcal{B}_1\psi = V\psi + (\frac{1}{2}I + K)\widetilde{D}^{-1}(\frac{1}{2}I + K')\psi = V\psi + (\frac{1}{2}I + K)w,$$

where $w = \widetilde{D}^{-1}(\frac{1}{2}I + K')\psi \in H^{1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle \widetilde{D}w, v\rangle_\Gamma = \langle (\frac{1}{2}I + K')\psi, v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma). \tag{4.1}$$

Let $S_h^1(\Gamma) \subset H^{1/2}(\Gamma)$ be some boundary element space of, e.g., piecewise linear basis and globally continuous functions, which are defined with respect to an admissible and locally quasi–uniform boundary element mesh. From a theoretical point of view it is not required that the boundary element mesh matches the trace of the finite element mesh, but from a practical point of view such a matching will simplify the implementation.
The Galerkin discretization of the variational problem (4.1) is to find $w_h \in S_h^1(\Gamma)$ such that

$$\langle \widetilde{D}w_h, v_h\rangle_\Gamma = \langle (\frac{1}{2}I + K')\psi, v_h\rangle_\Gamma \quad \text{for all } v_h \in S_h^1(\Gamma). \tag{4.2}$$

Since the stabilized hypersingular boundary integral operator $\widetilde{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is $H^{1/2}(\Gamma)$–elliptic, stability and quasi–optimality follow, i.e.,

$$\|w_h\|_{H^{1/2}(\Gamma)} \le c_1\|\psi\|_{H^{-1/2}(\Gamma)}, \quad \|w - w_h\|_{H^{1/2}(\Gamma)} \le c \inf_{v_h \in S_h^1(\Gamma)} \|w - v_h\|_{H^{1/2}(\Gamma)}.$$

Hence we can define an approximate operator $\widetilde{\mathcal{B}}$ by considering

$$\widetilde{\mathcal{B}}_1\psi := V\psi + (\frac{1}{2}I + K)w_h. \tag{4.3}$$

The corresponding Galerkin matrix operator is

$$\widetilde{\mathcal{B}}_{1,h} = V_h + (\frac{1}{2}M_h + K_h)\widetilde{D}_h^{-1}(\frac{1}{2}M_h^\top + K_h^\top)$$

where the matrices are given by

$$\begin{aligned} V_h[i,j] &= \langle V\phi_j^0, \phi_i^0\rangle_\Gamma & \widetilde{D}_h[k,\ell] &= \langle \widetilde{D}\phi_\ell^1, \phi_k^1\rangle_\Gamma \\ M_h[i,\ell] &= \langle \phi_\ell^1, \phi_i^0\rangle_\Gamma & K_h[i,\ell] &= \langle K\phi_\ell^1, \phi_i^0\rangle_\Gamma \end{aligned}$$

for $i,j = 1,\ldots,N$; $k,\ell = 1,\ldots,M$. N and M denote the number of surface triangles and vertices, respectively. ϕ_i^0 are piecewise constant basis functions, while ϕ_ℓ^1 are the standard piecewise linear and globally continuous hat functions.
From the properties of the Galerkin approximation w_h we easily conclude the boundedness of $\widetilde{\mathcal{B}}_1$, as well as an approximation estimate, see, e.g., [26], i.e.,

$$\|\widetilde{\mathcal{B}}_1\psi\|_{H^{1/2}(\Gamma)} \le \widetilde{c}_1\|\psi\|_{H^{-1/2}(\Gamma)}, \quad \|(\mathcal{B} - \widetilde{\mathcal{B}}_1)\psi\|_{H^{1/2}(\Gamma)} \le \widetilde{c}_2 \inf_{v_h \in S_h^1(\Gamma)} \|w - v_h\|_{H^{1/2}(\Gamma)}.$$

Moreover, $\widetilde{\mathcal{B}}_1 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is $H^{-1/2}(\Gamma)$–elliptic. Hence we conclude, that $\widetilde{\mathcal{B}}_1$ defines an admissible approximation of $\mathcal{B}$.

Second representation of the Neumann to Dirichlet map $\mathcal{B}$

However, the use of the approximation (4.3) results in the Galerkin discretization of boundary integral operators by using boundary traces of Nédélec elements. For lowest order Nédélec elements we have piecewise constant traces. But the traces of higher order Nédélec elements on polyhedral elements do not match with the related higher order boundary elements on triangular elements. Thus the approximation (4.3) may become rather cumbersome or at least expensive on the side of the boundary element method. Hence we are interested in a formulation, where the finite and boundary element basis functions are only linked via a generalized mass matrix. Note that we can use non-matching grids, i.e., the trace of the finite element mesh does not have to coincide with the boudary element mesh. For this we consider the second representation $\mathcal{B}_2$ of $\mathcal{B}$ as given in (2.12). For a given $\psi \in H^{-1/2}(\Gamma)$, the application of $\mathcal{B}_2$ reads

$$\mathcal{B}_2\psi = \Big(\widetilde{D} + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\Big)^{-1}\psi =: w,$$

where $w \in H^{1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle(\widetilde{D} + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K))w, v\rangle_\Gamma = \langle\psi, v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma).$$

By introducing $\theta := V^{-1}(\frac{1}{2}I - K)w \in H^{-1/2}(\Gamma)$ this is equivalent to a variational problem to find $(w, \theta) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\langle\widetilde{D}w, v\rangle_\Gamma + \langle(\frac{1}{2}I - K')\theta, v\rangle_\Gamma = \langle\psi, v\rangle_\Gamma, \quad \langle V\theta, \eta\rangle_\Gamma - \langle(\frac{1}{2}I - K)w, \eta\rangle_\Gamma = 0 \tag{4.4}$$

is satisfied for all $(v, \eta) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. In addition to $S_h^1(\Gamma) \subset H^{1/2}(\Gamma)$ we now consider a second boundary element space $S_h^0(\Gamma) \subset H^{-1/2}(\Gamma)$ of, e.g., piecewise constant basis functions, which, for convenience, are defined with respect to the same boundary element mesh as $S_h^1(\Gamma)$. The Galerkin discretization of the variational problem (4.4) is to find $(w_h, \theta_h) \in S_h^1(\Gamma) \times S_h^0(\Gamma)$ such that

$$\langle\widetilde{D}w_h, v_h\rangle_\Gamma + \langle(\frac{1}{2}I - K')\theta_h, v_h\rangle_\Gamma = \langle\psi, v_h\rangle_\Gamma, \quad \langle V\theta_h, \eta_h\rangle_\Gamma - \langle(\frac{1}{2}I - K)w_h, \eta_h\rangle_\Gamma = 0 \tag{4.5}$$

is satisfied for all $(v_h, \eta_h) \in S_h^1(\Gamma) \times S_h^0(\Gamma)$. The Galerkin solution w_h finally implies the approximation

$$\widetilde{\mathcal{B}}_2\psi := w_h. \tag{4.6}$$

The corresponding matrix operator is

$$\widetilde{\mathcal{B}}_{2,h} = \widehat{M}_h \left(\widetilde{D}_h + (\frac{1}{2}M_h^\top - K_h^\top)V_h^{-1}(\frac{1}{2}M_h - K_h)\right)^{-1} \widehat{M}_h^\top, \tag{4.7}$$

where $\widehat{M}_h = M_h$ in the considered special case of matching grids. For non-matching grids $\widehat{M}_h$ is the mass matrix of $S_h^1(\Gamma)$ and the T_n trace of the finite element space X_h.

Using standard arguments, see, e.g., [26], we can prove boundedness of $\widetilde{\mathcal{B}}_2$, as well as an approximation property, i.e.,

$$\begin{aligned}\|\widetilde{\mathcal{B}}_2\psi\|_{H^{1/2}(\Gamma)} &\leq c_1\|\psi\|_{H^{-1/2}(\Gamma)},\\ \|(\mathcal{B}-\widetilde{\mathcal{B}}_2)\psi\|_{H^{1/2}(\Gamma)} &\leq c_2\left\{\inf_{v_h\in S_h^1(\Gamma)}\|w-v_h\|_{H^{1/2}(\Gamma)}+\inf_{\eta_h\in S_h^0(\Gamma)}\|\theta-\eta_h\|_{H^{-1/2}(\Gamma)}\right\}.\end{aligned}$$

Note that $\widetilde{\mathcal{B}}_2 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ may fail to be $H^{1/2}(\Gamma)$–elliptic, but $\widetilde{\mathcal{B}}_2$ is $H^{1/2}(\Gamma)$–semi–elliptic. This is due to the mixed approximation scheme as used in the definition (4.6), which results in the circumjacent mass matrices $\widehat{M_h}$ and $\widehat{M_h^\top}$ with non-zero kernel in (4.7), while the inner matrix operator is positive definite. As in mixed finite element methods, stability would require an appropriate inf–sup condition, see e.g. [28]. But we conclude, that $\mathcal{B}_2$ in (4.6) also defines an admissible approximation as we have seen in the proof of the monotonicity estimate (3.5) that semi–ellipticity of $\widetilde{\mathcal{B}}$ is sufficient.

4.4 Convergence results

To present a final convergence result for the solution of the fully discrete Problem 4.3 we first cite an approximation property of the lowest order Nédélec ansatz space.

Theorem 4.4 [22, Theorem 5.41] *Let $X_h \subset H(curl;\Omega)$ be the space of lowest order Nédélec elements which is defined with respect to a regular and globally quasi uniform finite element mesh. For $\mathbf{u} \in [H^1(\Omega)]^3$ and $\nabla\times\mathbf{u} \in [H^1(\Omega)]^3$ let $r_h\mathbf{u} \in X_h$ be the interpolation. Then there holds the error estimate*

$$\|\mathbf{u}-r_h\mathbf{u}\|_{L^2(\Omega)}+\|\nabla\times(\mathbf{u}-r_h\mathbf{u})\|_{L_2(\Omega)} \leq c\,h\left(\|\mathbf{u}\|_{H^1(\Omega)}+\|\nabla\times\mathbf{u}\|_{H^1(\Omega)}\right). \tag{4.8}$$

Next we define $\pi_h\mathbf{u} \in X_h$ as the $H(\mathrm{curl};\Omega)$ projection of $\mathbf{u} \in H(\mathrm{curl};\Omega)$. By using standard techniques and (4.8) we conclude the error estimate

$$\|\mathbf{u}-\pi_h\mathbf{u}\|_{L^2(\Omega)}+\|\nabla\times(\mathbf{u}-\pi_h\mathbf{u})\|_{L_2(\Omega)} \leq c\,h\,\|\mathbf{u}\|_{H^1(\mathrm{curl};\Omega)} \tag{4.9}$$

for $\mathbf{u} \in [H^1(\Omega)]^3$ and $\nabla\times\mathbf{u} \in [H^1(\Omega)]^3$ as well as the stability estimate

$$\|\pi_h\mathbf{v}\|_{H(\mathrm{curl};\Omega)} \leq c_\pi\,\|\mathbf{v}\|_{H(\mathrm{curl};\Omega)} \quad \text{for all } \mathbf{v} \in H(\mathrm{curl};\Omega). \tag{4.10}$$

Now we are in the position to state the final convergence result.

Theorem 4.5 *Let $\widehat{\mathbf{u}} \in L^2(0,T;H(curl;\Omega))$ and $\widetilde{\mathbf{u}}_h^m \in X_h$ for all $m=1,\ldots,M$ be the unique solutions of Problem 2.2 and Problem 4.3, respectively. Assume*

$$\widehat{\mathbf{u}} \in L^2(0,T;H^1(\mathrm{curl};\Omega)) \cap L^\infty(0,T;H^1(\mathrm{curl};\Omega))$$

and

$$\frac{d}{dt}\widehat{\mathbf{u}}, \frac{d^2}{dt^2}\widehat{\mathbf{u}} \in L^2(0,T;[H(\mathrm{curl};\Omega)]^*), \quad \frac{d}{dt}\widehat{\mathbf{u}} \in H^1(\mathrm{curl},\Omega), \quad \mathbf{u}_0 \in H^1(\mathrm{curl};\Omega).$$

Then there holds the error estimate

$$\|\widehat{u}^M - \widetilde{u}_h^M\|^2_{L^2(\Omega)} + \tau \sum_{m=1}^{M} \|\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m\|^2_{H(\text{curl};\Omega)} \le c(\widehat{\mathbf{u}}) \left(h^2 + \tau^2\right).$$

Remark 4.1 *Note that in the case of reduced regularity involving $H^s(curl;\Omega)$, $s \in [0,1]$, the order of approximation is reduced to s accordingly.*

Proof. The proof follows partly the lines of the proof of Theorem 4.2. We will use the triangle inequality to estimate the error of an auxiliary approximation $\pi_h \widehat{\mathbf{u}}^m$ and the additional errors of the time discretization and the approximation of $\widetilde{S}$ on the discrete level. We start investigating the latter. We subtract the variational formulation of Problem 4.3 from Problem 4.1 at time $t = t_m$ and obtain

$$\left\langle \sigma \frac{d}{dt}\widetilde{\mathbf{u}}^m + \widetilde{S}(t_m)\widetilde{\mathbf{u}}^m, \mathbf{v}_h \right\rangle = \left\langle \sigma \frac{\widetilde{\mathbf{u}}_h^m - \widetilde{\mathbf{u}}_h^{m-1}}{\tau} + \widetilde{S}(t_m)\widetilde{\mathbf{u}}_h^m, \mathbf{v}_h \right\rangle \quad \text{for all } \mathbf{v}_h \in X_h,$$

or equivalently

$$\begin{aligned} &\langle \sigma\left(\pi_h \widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m\right), \mathbf{v}_h\rangle + \tau \left\langle \widetilde{S}(t_m)\pi_h\widehat{\mathbf{u}}^m - \widetilde{S}(t_m)\widetilde{\mathbf{u}}_h^m, \mathbf{v}_h \right\rangle \qquad (4.11)\\ &= \underbrace{\left\langle \sigma\left(\pi_h \widehat{\mathbf{u}}^{m-1} - \widetilde{\mathbf{u}}_h^{m-1}\right)\right), \mathbf{v}_h\right\rangle}_{=:A} + \underbrace{\left\langle \sigma\left(\pi_h\widehat{\mathbf{u}}^m - \pi_h\widehat{\mathbf{u}}^{m-1} - \tau\frac{d}{dt}\widetilde{\mathbf{u}}(t_m)\right), \mathbf{v}_h \right\rangle}_{=:B} \\ &\quad \underbrace{+\tau \left\langle \widetilde{S}(t_m)\pi_h\widehat{\mathbf{u}}^m - \widetilde{S}(t_m)\pi_h\widetilde{\mathbf{u}}^m, \mathbf{v}_h \right\rangle}_{=:C} + \underbrace{\tau \left\langle \widetilde{S}(t_m)\pi_h\widetilde{\mathbf{u}}^m - \widetilde{S}(t_m)\widetilde{\mathbf{u}}^m, \mathbf{v}_h \right\rangle}_{=:D}. \end{aligned}$$

We set $\mathbf{v}_h = \psi_h^m := \pi_h\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m$ and estimate the different terms of the error equation (4.11). The monotonicity of $\widetilde{S}$ gives for the left hand side terms

$$\begin{aligned} \left\langle \sigma\left(\pi_h\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m\right) + \tau\left(\widetilde{S}(t_m)\pi_h\widehat{\mathbf{u}}^m - \widetilde{S}(t_m)\widetilde{\mathbf{u}}_h^m\right), \pi_h\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m \right\rangle \\ \ge \|\sqrt{\sigma}\psi_h^m\|^2_{L^2(\Omega)} + \tau c_M \|\psi_h^m\|^2_{H(\text{curl};\Omega)}. \end{aligned}$$

The first right hand side term of (4.11) can be estimated as

$$A = \left\langle \sigma\psi_h^{m-1}, \psi_h^m \right\rangle \le \|\sqrt{\sigma}\psi_h^{m-1}\|_{L^2(\Omega)}\|\sqrt{\sigma}\psi_h^m\|_{L^2(\Omega)} \le \frac{1}{2}\|\sqrt{\sigma}\psi_h^{m-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sqrt{\sigma}\psi_h^m\|^2_{L^2(\Omega)}.$$

The Lipschitz continuity of $\widetilde{S}$ and the continuity (4.10) of the projection π_h enable the estimate of the third right hand side term of (4.11):

$$\begin{aligned} C &= \tau \left\langle \widetilde{S}(t_m)\pi_h\widehat{\mathbf{u}}^m - \widetilde{S}(t_m)\pi_h\widetilde{\mathbf{u}}^m, \psi_h^m \right\rangle \le \tau c_L^{\tilde{S}} c_\pi \|\pi_h(\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m)\|_{H(\text{curl};\Omega)}\|\psi_h^m\|_{H(\text{curl};\Omega)} \\ &\le \frac{1}{2}\gamma_1\tau\|\psi_h^m\|^2_{H(\text{curl};\Omega)} + \tau\frac{(c_L^{\tilde{S}})^2 c_\pi^2}{2\gamma_1}\|\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|^2_{H(\text{curl};\Omega)} \end{aligned}$$

for some positive constant γ_1. The Lipschitz continuity of $\widetilde{S}$ implies for the fourth right hand side term of (4.11), and for some positive constant γ_2:

$$
\begin{aligned}
D &= \tau \left\langle \widetilde{S}(t_m)\pi_h \widetilde{\mathbf{u}}^m - \widetilde{S}(t_m)\widetilde{\mathbf{u}}^m, \psi_h^m \right\rangle \leq \tau c_L^{\tilde{S}} \|\pi_h \widetilde{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|_{H(\mathrm{curl};\Omega)} \|\psi_h^m\|_{H(\mathrm{curl};\Omega)} \\
&\leq \frac{1}{2}\gamma_2 \tau \|\psi_h^m\|^2_{H(\mathrm{curl};\Omega)} + \tau \frac{(c_L^{\tilde{S}})^2}{2\gamma_2} \|\pi_h \widetilde{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)}.
\end{aligned}
$$

For the second right hand side term of (4.11) we obtain, for some positive constant γ_3,

$$
\begin{aligned}
B &= \left\langle \sigma \left(\pi_h \widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^{m-1} - \tau \frac{d}{dt} \widetilde{\mathbf{u}}(t_m) \right), \psi_h^m \right\rangle \\
&\leq \frac{1}{2}\tau\gamma_3 \|\psi_h^m\|^2_{H(\mathrm{curl};\Omega)} + \frac{1}{2\gamma_3 \tau} \left\| \sigma \Big(\pi_h \widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^{m-1} - \tau \frac{d}{dt} \widetilde{\mathbf{u}}(t_m) \Big) \right\|^2_{[H(\mathrm{curl};\Omega)]^*}.
\end{aligned}
$$

We now collect the estimates of (4.11), set $\gamma_i = \frac{1}{3} c_M$, $i = 1, \ldots, 3$, and rearrange the terms:

$$
\begin{aligned}
\|\sqrt{\sigma}\psi_h^m\|^2_{L^2(\Omega)} + c_M \tau \|\psi_h^m\|^2_{H(\mathrm{curl};\Omega)} &\leq \|\sqrt{\sigma}\psi_h^{m-1}\|^2_{L^2(\Omega)} + \frac{3(c_L^{\tilde{S}})^2 c_\pi^2}{c_M} \tau \|\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)} \\
&+ \tau \frac{3(c_L^{\tilde{S}})^2}{c_M} \|\pi_h \widetilde{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)} + \frac{3}{c_M \tau} \left\| \sigma \Big(\pi_h \widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^{m-1} - \tau \frac{d}{dt} \widetilde{\mathbf{u}}(t_m) \Big) \right\|^2_{[H(\mathrm{curl};\Omega)]^*}.
\end{aligned}
$$

Summarizing over $m = 1, \ldots, M$ leads to

$$
\|\sqrt{\sigma}(\pi_h \widehat{\mathbf{u}}^M - \widetilde{\mathbf{u}}_h^M)\|^2_{L^2(\Omega)} + c_M \tau \sum_{m=1}^{M} \|\pi_h \widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m\|^2_{H(\mathrm{curl};\Omega)} \leq F
$$

with

$$
\begin{aligned}
F := \|\sqrt{\sigma}(\pi_h \widehat{\mathbf{u}}^0 - \widetilde{\mathbf{u}}_h^0)\|^2_{L^2(\Omega)} &+ \frac{3(c_L^{\tilde{S}})^2 c_\pi^2}{c_M} \tau \sum_{m=1}^{M} \|\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)} \\
&+ \tau \frac{3(c_L^{\tilde{S}})^2}{c_M} \sum_{m=1}^{M} \|\pi_h \widetilde{\mathbf{u}}^m - \widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)} \\
&+ \frac{3}{c_M \tau} \sum_{m=1}^{M} \left\| \sigma \Big(\pi_h \widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^{m-1} - \tau \frac{d}{dt} \widetilde{\mathbf{u}}(t_m) \Big) \right\|^2_{[H(\mathrm{curl};\Omega)]^*}.
\end{aligned}
$$

Inserting the approximation $\pi_h \widehat{\mathbf{u}}^m$ and application of the triangle inequality yields for the total error

$$
\begin{aligned}
&\|\sqrt{\sigma}(\widehat{\mathbf{u}}^M - \widetilde{\mathbf{u}}_h^M)\|_{L^2(\Omega)}^2 + c_M \tau \sum_{m=1}^{M} \|\widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m\|_{H(\mathrm{curl};\Omega)}^2 \\
&\le 2\left(\|\sqrt{\sigma}\left(\widehat{\mathbf{u}}^M - \pi_h \widehat{\mathbf{u}}^M\right)\|_{L^2(\Omega)}^2 + \|\sqrt{\sigma}\left(\pi_h \widehat{\mathbf{u}}^M - \widetilde{\mathbf{u}}_h^M\right)\|_{L^2(\Omega)}^2\right) \\
&\quad + 2 c_M \tau \sum_{m=1}^{M} \left(\|\widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^m\|_{H(\mathrm{curl};\Omega)}^2 + \|\pi_h \widehat{\mathbf{u}}^m - \widetilde{\mathbf{u}}_h^m\|_{H(\mathrm{curl};\Omega)}^2\right) \\
&\le 2\|\sqrt{\sigma}\left(\widehat{\mathbf{u}}^M - \pi_h \widehat{\mathbf{u}}^M\right)\|_{L^2(\Omega)}^2 + 2 c_M \tau \sum_{m=1}^{M} \|\widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^m\|_{H(\mathrm{curl};\Omega)}^2 + 2F. \qquad (4.12)
\end{aligned}
$$

Based on the approximation properties (4.9) of the $H(\mathrm{curl};\Omega)$ projection π_h, the following terms of the right hand side of equation (4.12) can be bounded by

$$
\begin{aligned}
&2\|\sqrt{\sigma}\left(\widehat{\mathbf{u}}^M - \pi_h \widehat{\mathbf{u}}^M\right)\|_{L^2(\Omega)}^2 + 2 c_M \tau \sum_{m=1}^{M} \|\widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^m\|_{H(\mathrm{curl};\Omega)}^2 + 2\|\sqrt{\sigma}(\pi_h \widehat{\mathbf{u}}^0 - \widetilde{\mathbf{u}}_h^0)\|_{L^2(\Omega)}^2 \qquad (4.13) \\
&\le c h^2 \left(\|\widehat{\mathbf{u}}\|_{L^\infty(0,T;H^1(\mathrm{curl};\Omega))}^2 + \|\nabla \times \widehat{\mathbf{u}}\|_{L^2(0,T;H^1(\mathrm{curl};\Omega))}^2 + \|\mathbf{u}_0\|_{H^1(\mathrm{curl};\Omega)}^2\right).
\end{aligned}
$$

For the last term of $2F$ we conclude, by using the triangle inequality and a Taylor expansion, requiring $\widehat{\mathbf{u}}$ to be twice continuously differentiable in time,

$$
\begin{aligned}
&\frac{6}{c_M \tau} \sum_{m=1}^{M} \left\|\sigma\left(\pi_h \widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^{m-1} - \tau \frac{d}{dt} \widetilde{\mathbf{u}}(t_m)\right)\right\|_{[H(\mathrm{curl};\Omega)]^*}^2 \\
&\quad \le \frac{18}{c_M \tau} \sum_{m=1}^{M} \left\|\sigma\left(\pi_h \widehat{\mathbf{u}}^m - \pi_h \widehat{\mathbf{u}}^{m-1} - \widehat{\mathbf{u}}^m + \widehat{\mathbf{u}}^{m-1}\right)\right\|_{[H(\mathrm{curl};\Omega)]^*}^2 \qquad (4.14) \\
&\qquad + \frac{18}{c_M \tau} \sum_{m=1}^{M} \left(\left\|\sigma\left(\widehat{\mathbf{u}}^m - \widehat{\mathbf{u}}^{m-1} - \tau \frac{d}{dt} \widehat{\mathbf{u}}(t_m)\right)\right\|_{[H(\mathrm{curl};\Omega)]^*}^2\right. \\
&\qquad \left. + \tau^2 \left\|\sigma \frac{d}{dt}\left(\widehat{\mathbf{u}}(t_m) - \widetilde{\mathbf{u}}(t_m)\right)\right\|_{[H(\mathrm{curl};\Omega)]^*}^2\right) \\
&\quad \le c_1 h^2 \left\|\frac{d}{dt} \widehat{\mathbf{u}}\right\|_{L^2(0,T;H^1(\mathrm{curl};\Omega))}^2 + c_2 \tau^2 \left\|\frac{d^2}{dt^2} \widehat{\mathbf{u}}\right\|_{L^2(0,T;[H(\mathrm{curl};\Omega)]^*)}^2 \\
&\qquad + c_3 \tau^2 \left\|\sigma \frac{d}{dt}\left(\widehat{\mathbf{u}} - \widetilde{\mathbf{u}}\right)\right\|_{L^2(0,T;[H(\mathrm{curl};\Omega)]^*)}^2.
\end{aligned}
$$

For the last step we applied the equivalence of the discrete L^2 norm and the $L^2(0,T)$ norm. In particular, we have used the mean value theorem for the first term to estimate

$$
\begin{aligned}
&\frac{1}{\tau}\sum_{m=1}^{M}\Big\|\sigma\Big(\pi_h\widehat{\mathbf{u}}^m-\pi_h\widehat{\mathbf{u}}^{m-1}-\widehat{\mathbf{u}}^m+\widehat{\mathbf{u}}^{m-1}\Big)\Big\|^2_{[H(\mathrm{curl};\Omega)]^*}\\
&=\frac{1}{\tau}\sum_{m=1}^{M}\Big\|\sigma\tau\Big(\pi_h\left(\frac{\widehat{\mathbf{u}}^m-\widehat{\mathbf{u}}^{m-1}}{\tau}\right)-\frac{\widehat{\mathbf{u}}^m-\widehat{\mathbf{u}}^{m-1}}{\tau}\Big)\Big\|^2_{[H(\mathrm{curl};\Omega)]^*}\\
&=\frac{1}{\tau}\sum_{m=1}^{M}\Big\|\sigma\tau\Big(\pi_h\frac{d}{dt}\widehat{\mathbf{u}}(\xi)-\frac{d}{dt}\widehat{\mathbf{u}}(\xi)\Big)\Big\|^2_{[H(\mathrm{curl};\Omega)]^*}\\
&\leq ch^2\sum_{m=1}^{M}\tau\Big\|\frac{d}{dt}\widehat{\mathbf{u}}(\xi)\Big\|^2_{H^1(\mathrm{curl};\Omega)}.
\end{aligned}
$$

For the second term we applied a Taylor expansion to obtain

$$
\frac{1}{\tau}\sum_{m=1}^{M}\Big\|\sigma\Big(\widehat{\mathbf{u}}^m-\widehat{\mathbf{u}}^{m-1}-\tau\frac{d}{dt}\widehat{\mathbf{u}}(t_m)\Big)\Big\|^2_{[H(\mathrm{curl};\Omega)]^*}\leq\sigma_{\max}\tau^3\sum_{m=1}^{M}\Big\|\frac{d^2}{dt^2}\widehat{\mathbf{u}}(\xi_m)\Big\|^2_{[H(\mathrm{curl};\Omega)]^*}.
$$

Then Lemma 4.1 and the estimates of the BEM approximations in Subsect. 4.3 yield

$$
\begin{aligned}
\Big\|\sigma\frac{d}{dt}(\widehat{\mathbf{u}}-\widetilde{\mathbf{u}})\Big\|_{L^2(0,T;[H(\mathrm{curl};\Omega)]^*)}&\leq\frac{c_T}{\mu_0}(1+\frac{c_L^{\tilde{S}}}{c_M})\,\|(\mathcal{B}-\widetilde{\mathcal{B}})T_n\widehat{\mathbf{u}}\|_{L^2(0,T;H^{1/2}(\Gamma))}\\
&\leq c\,h\,\|T_n\widehat{\mathbf{u}}\|_{L^2(0,T;H^{1/2}_{\mathrm{pw}}(\Gamma))}\\
&\leq c\,h\,\|\nabla\times\widehat{\mathbf{u}}\|_{L^2(0,T;H^1(\Omega))}. \qquad (4.15)
\end{aligned}
$$

The same kind of estimate holds true for the remaining part of the term $2F$, i.e.

$$
\begin{aligned}
&6\frac{(c_L^{\tilde{S}})^2c_\pi^2}{c_M}\,\tau\sum_{m=1}^{M}\big(c_\pi^2\|\widehat{\mathbf{u}}^m-\widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)}+\|\pi_h\widetilde{\mathbf{u}}^m-\widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)}\big)\\
&\leq 6\frac{(c_L^{\tilde{S}})^2c_\pi^2}{c_M}\,\tau\sum_{m=1}^{M}\big((c_\pi^2+2(1+c_\pi)^2)\|\widehat{\mathbf{u}}^m-\widetilde{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)}+\|\pi_h\widehat{\mathbf{u}}^m-\widehat{\mathbf{u}}^m\|^2_{H(\mathrm{curl};\Omega)}\big)\\
&\leq c_1\,\|(\mathcal{B}-\widetilde{\mathcal{B}})T_n\widehat{\mathbf{u}}\|^2_{L^2(0,T;H^{1/2}(\Gamma))}+c_2\,\|\pi_h\widehat{\mathbf{u}}-\widehat{\mathbf{u}}\|^2_{L^2(0,T;H(\mathrm{curl};\Omega))} \qquad (4.16)\\
&\leq c_1\,h^2\,\|T_n\widehat{\mathbf{u}}\|_{L^2(0,T;H^{1/2}_{\mathrm{pw}}(\Gamma))}+c_2\,h^2\,\|\widehat{\mathbf{u}}\|^2_{L^2(0,T;H^1(\mathrm{curl};\Omega))}\\
&\leq c\,h^2\,\|\widehat{\mathbf{u}}\|_{L^2(0,T;H^1(\mathrm{curl};\Omega))}. \qquad (4.17)
\end{aligned}
$$

Now the assertion follows after summarizing all previous estimates. ■

5 Conclusions

In the present paper we have given a numerical analysis of a coupled finite and boundary element formulation to model a kinematic dynamo in $\mathbb{R}^3$. While the analysis is given for the nonlinear model, the numerical examples [19] cover only a simplified linear model. Therefore we omit the numerical examples. The implementation can be extended to the nonlinear model straight forward. Solving non-linear problems and large scale problems require a more efficient solution approach for the linearized model. But such efficient solvers for the related saddle point problem are beyond the scope of the this paper.
The model as used in this paper is based on the knowledge of the given velocity field $\mathbf{w}$. A more general model will incorporate the Navier–Stokes equations to describe this velocity field as an additional unknown. However, the solution of these direct problems may serve as the basis of the inverse problem, i.e., the determination of the velocity field in the interior domain from measurements on the boundary.

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