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OPTIMAL CONTROL OF SINGULARLY PERTURBED ADVECTION-DIFFUSION-REACTION PROBLEMS

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In this paper, we consider the numerical analysis of quadratic optimal control problems governed by a linear advection-diffusion-reaction equation without control constraints. In the case of dominating advection, the Galerkin discretization is stabilized via the one- or two-level variant of the local projection approach which leads to a symmetric optimality system at the discrete level. The optimal control problem simultaneously covers distributed and Robin boundary control. In the singularly perturbed case, the boundary control at inflow and/or characteristic parts of the boundary can be seen as regularization of a Dirichlet boundary control. Some numerical tests illustrate the analytical results.

Keywords: Optimal control; stabilized finite element method; singularly perturbed elliptic problem; error estimates; regularized Dirichlet control.

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1. Introduction

In this paper, we consider some aspects of the numerical analysis of the quadratic optimal control problem

$$\min J(u, q_\Omega, q_\Gamma) = \frac{\lambda_\Omega}{2}\|u - \tilde{u}_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2}\|u - \tilde{u}_\Gamma\|^2_{L^2(\Gamma_R)} + \frac{\alpha_\Omega}{2}\|q_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Gamma}{2}\|q_\Gamma\|^2_{L^2(\Gamma_R)}, \tag{1.1}$$

where $(u, q_\Omega, q_\Gamma) \in V \times Q_\Omega \times Q_\Gamma$ with

$$V = \{v \in H^1(\Omega) : u|_{\Gamma_D} = 0\}, \quad Q_\Omega = L^2(\Omega), \quad Q_\Gamma = L^2(\Gamma_R)$$

is subject to the linear mixed boundary value problem of advection-diffusion-reaction type

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u &= f + q_\Omega \quad \text{in } \Omega, \\ \varepsilon\nabla u \cdot \mathbf{n} + \beta u &= g + q_\Gamma \quad \text{on } \Gamma_R \\ u &= 0 \qquad\quad\;\; \text{on } \Gamma_D. \end{aligned} \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ is a bounded polyhedral domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_R \cup \Gamma_D}, \Gamma_D \cap \Gamma_R = \emptyset$ and outer normal unit vector $\mathbf{n}$. We assume that $\varepsilon > 0$ and $\sigma \geq 0$ are constants and that the advective field $\mathbf{b}$ is divergence-free.

In (1.1), the desired states are $\tilde{u}_\Omega$ in Ω and $\tilde{u}_\Gamma$ on Γ_R. The constants $\lambda_\Omega, \lambda_\Gamma \geq 0$ with $\lambda_\Omega^2 + \lambda_\Gamma^2 > 0$ describe the weights of different parts whereas $\alpha_\Omega, \alpha_\Gamma \geq 0$ with $\alpha_\Omega^2 + \alpha_\Gamma^2 > 0$ serve as regularization parameters. The state equation (1.2) describes the dependence of the state u on the control (q_Ω, q_Γ). Here we consider the problem without restrictions of the control. The problem with box-constraints for the control will be considered elsewhere.

The linear-quadratic optimal control problem (1.1)−(1.2) with $\Gamma_R = \emptyset$, without and with control constraints has been considered by Becker and Vexler in Ref. 5 for the singularly perturbed case with $0 < \varepsilon \ll 1$, see also the references therein. One goal of the present paper is to consider problem (1.1)−(1.2) simultaneously for distributed and (Robin) boundary control. Notably, in the singularly perturbed case $0 < \varepsilon \ll 1$, the Robin control at inflow and/or characteristic parts of the boundary can be seen as regularization of a Dirichlet boundary control.

The standard Galerkin discretization is stabilized as in Ref. 5 via the local projection approach (LPS for short below) which leads to a symmetric optimality system at the discrete level. This implies that the operations "discretization" and "optimization" commute as opposed to residual-based stabilization techniques like the standard streamline-diffusion method. Another aim of this paper is a more general LPS approach, including the so-called two-level variant (as in Ref. 5) and a one-level variant introduced in Ref. 13.

Let us emphasize two aspects of the analysis in the present paper: Firstly, the regularity of the solution of the mixed boundary value problem (1.2) is taken into account by using Sobolev−Slobodeckij spaces and adapting the analysis of the LPS method. Secondly, the analysis is performed for shape regular meshes (as opposed to quasi-uniform meshes in Ref. 5) which allows for (isotropic) mesh refinement at corners or edges of the domain and in boundary layers.

An outline of the paper is as follows: In Sec. 2, we address the existence, uniqueness and regularity of the solution of problem (1.1)−(1.2). Then, in Sec. 3, we consider the discretization of the state equation by means of finite element methods (FEM) with local projection stabilization and derive the discretized optimality systems. In Sec. 4, we analyze the convergence properties of the discretized optimal control problem. In Sec. 5, we briefly address the interpretation of Robin boundary control as regularized Dirichlet control. Some numerical experiments will be presented in Sec. 6.

Throughout this paper, standard notations for Lebesgue and Sobolev spaces are used. In particular, the L^2-inner product and the corresponding norm in a domain $G \subseteq \Omega$ are denoted by $(\cdot,\cdot)_G$ and $\|\cdot\|_{0,G}$, respectively.

2. Continuous Optimal Control Problem

Here we consider the optimality system for the *continuous* optimal control problem (1.1)–(1.2).

2.1. *Solvability*

To this goal, we first consider the solvability of the state equation (1.2) with $\tilde{f} := f + q_\Omega$ and $\tilde{g} := g + q_\Gamma$. The variational form of problem (1.2) reads

$$\text{Find } u \in V := \{v \in H^1(\Omega): \quad v|_{\Gamma_D} = 0\}, \quad \text{s.t.} \quad a(u,v) = f(v) \quad \forall\, v \in V, \tag{2.1}$$

with

$$\begin{aligned} a(u,v) &:= \varepsilon(\nabla u, \nabla v)_\Omega + (\mathbf{b}\cdot\nabla u + \sigma u, v)_\Omega + (\beta u, v)_{\Gamma_R}, \\ f(v) &:= (\tilde{f}, v)_\Omega + (\tilde{g}, v)_{\Gamma_R}. \end{aligned}$$

The following result provides sufficient conditions for the unique solvability of (2.1).

Lemma 2.1. *Let the following assumptions be valid*:

(i) $\mathbf{b} \in (L^\infty(\Omega))^d$, $\tilde{f} \in L^2(\Omega)$, $\tilde{g} \in L^2(\Gamma_R), \beta \in L^\infty(\Gamma_R)$.
(ii) $\varepsilon > 0$, $\sigma \geq 0$ *and* $\nabla\cdot\mathbf{b} = 0$ *a.e. in* Ω.
(iii) $\beta \geq 0$ *and* $\tilde{\beta} := \beta + \frac{1}{2}(\mathbf{b}\cdot\mathbf{n}) \geq \beta_0 \geq 0$ on Γ_R.
(iv) *Let at least one of the following conditions be valid*:

  (a) $\mu_{n-1}(\Gamma_D) > 0$,
  (b) $\sigma > 0$ *and* $\beta_0 > 0$.

Then there exists a unique solution $u \in H^1(\Omega)$ *of the mixed boundary value problem* (2.1).

Proof. The continuity of $a(\cdot,\cdot)$ and $f(\cdot)$ follows via standard inequalities and (i)–(iii):

$$\begin{aligned} |a(u,v)| &= |\varepsilon(\nabla u, \nabla v)_\Omega + ((\mathbf{b}\cdot\nabla u + \sigma u, v)_\Omega + (\beta u, v)_{\Gamma_R}| \\ &\leq \left(\varepsilon + \sigma + \left(\sum_{i=1}^n \|b_i\|^2_{\infty;\Omega}\right)^{\frac{1}{2}} + C\|\beta\|_{\infty;\Gamma_R}\right)\|u\|_1\|v\|_1 \equiv M_a\|u\|_1\|v\|_1, \\ |f(v)| &= |(f,v)_\Omega + (g,v)_{\Gamma_R}| \leq (\|f\|_0 + C\|g\|_{0;\Gamma_R})\|v\|_1 \equiv M_f\|v\|_1. \end{aligned}$$

Integration by parts of the advective term together with assumption $\nabla \cdot \mathbf{b} = 0$ and the abbreviation $\tilde{\beta} := \beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n}$ yield H^1-ellipticity of a

$$a(v,v) = \varepsilon |v|_1^2 + \sigma \|v\|_0^2 + \|\tilde{\beta}^{\frac{1}{2}} v\|_{0;\Gamma_R}^2$$

as $\sqrt{a(v,v)}$ is equivalent to the standard norm on $H^1(\Omega)$ if one of the assumptions in (iv) is valid. Finally, the Lax−Milgram theorem delivers the assertion. □

The following existence result follows by standard arguments in optimal control.[14]

Theorem 2.1. *Under the assumptions of Lemma 2.1, the optimal control problem (1.1)−(1.2) admits a unique solution* $(\bar{u}, \bar{q}_\Omega, \bar{q}_\Gamma) \in V \times Q_\Omega \times Q_\Gamma$.

2.2. *Regularity*

For the convergence analysis below, statements on the regularity of the solution of (2.1) are required. In general, the solution of this mixed boundary value problem is not in $W^{2,p}(\Omega)$. A standard approach is to consider weighted Sobolev spaces. Let $\mathcal{S}$ be the set of points (for $d = 2$) or edges (for $d = 3$) which subdivide the polyhedral boundary $\partial\Omega$ into smooth disjoint connected components. The space $V_\beta^{k,p}(\Omega)$ denotes the closure of $C^\infty(\Omega)$ w.r.t.

$$\|v\|_{V_\beta^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega r^{p(\beta - k + |\alpha|)} |D^\alpha u|^p dx \right)^{\frac{1}{p}},$$

where $r = r(x) = \mathrm{dist}(x, \mathcal{S})$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$ and $p > 1$. The parameter β is defined via eigenvalues of certain eigenvalue problems (in local coordinate systems at parts of the set $\mathcal{S}$) being associated with the mixed boundary value problem. As it is not the goal of this paper to give sufficient conditions for the solution of problem (2.1) to belong to $V_\beta^{k,p}(\Omega)$, we refer to standard textbooks such as Refs. 8 and 11. Moreover, we do not intend to consider graded finite element meshes in the neighborhood of the set $\mathcal{S}$ although the forthcoming numerical analysis allows such kind of refinement. For this approach to optimal control problems for elliptic problems, we refer, e.g. to Refs. 1 and 2.

Here we consider on a subdomain $G \subseteq \Omega$, the Sobolev−Slobodeckij spaces

$$W^{k+\lambda,p}(G) := \{ v \in W^{k,p}(G) : \|v\|_{k+\lambda,p,G} < \infty \}, \tag{2.2}$$

$$\|v\|_{k+\lambda,p,G} := \left(\|v\|_{k,p,G}^p + \sum_{|\alpha|=k} \int_G \int_G \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x-y\|^{d+p\lambda}} dx\, dy \right)^{\frac{1}{p}} \tag{2.3}$$

with $k \in \mathbb{N}_0, \lambda \in [0,1), p \in (1,\infty)$ and the obvious modifications in case of $p = \infty$. The spaces $W^{k+\lambda,p}(\Gamma_R)$ are defined in a similar way.

Remark 2.1. The following embeddings $V_\beta^{2,2}(\Omega) \subset W^{\frac{d}{2}+\kappa,2}(\Omega) \subset C(\overline{\Omega})$ are valid for $\beta < 2 - \frac{d}{2} + \kappa$ with $\kappa > 0$, cf. Ref. 11. In particular, for the Dirichlet case $\partial\Omega = \Gamma_D$ in polyhedral domains there holds $\beta \le \frac{1}{2} + \kappa, \kappa > 0$.

2.3. *Optimality system*

As problem (2.1) admits a unique solution, see Lemma 2.1, we may define the affine linear continuous solution operator

$$S : L^2(\Omega) \times L^2(\Gamma_R) \to V, \quad u = S(q_\Omega + f, q_\Gamma + g).$$

Due to the linearity of problem (1.2) we can split S into its linear and affine linear parts. Inserting $u = S(q_\Omega + f, q_\Gamma + g) = S(q_\Omega, q_\Gamma) + S(f, g)$ in (1.1), we obtain (with trace operator γ) and the definitions $u_\Omega := \tilde{u}_\Omega - S(f, g)$ and $u_\Gamma := \tilde{u}_\Gamma - \gamma \circ S(f, g)$ the reduced cost functional:

$$\begin{aligned} j(q_\Omega, q_\Gamma) = J(S(q_\Omega, q_\Gamma), q_\Omega, q_\Gamma) &= \frac{\lambda_\Omega}{2} \|S(q_\Omega, q_\Gamma) - u_\Omega\|_{0;\Omega}^2 \\ &\quad + \frac{\lambda_\Gamma}{2} \|\gamma \circ S(q_\Omega, q_\Gamma) - u_\Gamma\|_{0;\Gamma_R}^2 \\ &\quad + \frac{\alpha_\Omega}{2} \|q_\Omega\|_{0;\Omega}^2 + \frac{\alpha_\Gamma}{2} \|q_\Gamma\|_{0;\Gamma_R}^2. \end{aligned} \tag{2.4}$$

Now the reduced optimization problem reads

$$\text{Minimize } j(q_\Omega, q_\Gamma), \quad (q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma. \tag{2.5}$$

Henceforth we denote the optimal control of the problem by $(\bar{q}_\Omega, \bar{q}_\Gamma)$ and the corresponding optimal state by $\bar{u} = S(\bar{q}_\Omega + f, \bar{q}_\Gamma + g)$. The reduced cost functional j is continuously differentiable.

Lemma 2.2. *The first-order derivatives of the reduced cost functional j are given by*

$$D_{q_\Omega} j(q_\Omega, q_\Gamma) \cdot k_\Omega = (\alpha_\Omega q_\Omega + p, k_\Omega)_\Omega, \quad D_{q_\Gamma} j(q_\Omega, q_\Gamma) \cdot k_\Gamma = (\alpha_\Gamma q_\Gamma + p, k_\Gamma)_{\Gamma_R}, \tag{2.6}$$

where the adjoint state $p \in V$ is the solution of the adjoint state problem

$$\text{Find } p \in V\colon \quad a_{\text{adj}}(p, v) = \lambda_\Omega(\bar{u} - u_\Omega)_\Omega + \lambda_\Gamma(\bar{u} - u_\Gamma, v)_{\Gamma_R} \quad \forall\, v \in V, \tag{2.7}$$

with

$$a_{\text{adj}}(p, v) := \varepsilon(\nabla p, \nabla v)_\Omega - (\mathbf{b} \cdot \nabla p, v)_\Omega + \sigma(p, v)_\Omega + ((\beta + \mathbf{b} \cdot \mathbf{n})p, v)_{\Gamma_R}.$$

Proof. Formula (2.6) follows via standard arguments, see Ref. 14. The solvability of the adjoint state problem (2.7) is shown as in the proof of Lemma 2.1. □

The necessary (and here also sufficient) optimality conditions for the reduced control problem (2.5) read

$$D_{q_\Omega} j(\bar{q}_\Omega, \bar{q}_\Gamma) \cdot (k_\Omega - \bar{q}_\Omega) = (\alpha_\Omega \bar{q}_\Omega + \bar{p}, k_\Omega - \bar{q}_\Omega)_\Omega = 0, \quad \forall\, k_\Omega \in Q_\Omega, \tag{2.8}$$

$$D_{q_\Gamma} j(\bar{q}_\Omega, \bar{q}_\Gamma) \cdot (k_\Gamma - \bar{q}_\Gamma) = (\alpha_\Gamma \bar{q}_\Gamma + \bar{p}, k_\Gamma - \bar{q}_\Gamma)_{\Gamma_R} = 0, \quad \forall\, k_\Gamma \in Q_\Gamma, \tag{2.9}$$

where $\bar{p}$ is the associated adjoint state to $(\bar{q}_\Omega, \bar{q}_\Gamma)$. This leads to

$$\alpha_\Omega \bar{q}_\Omega + \bar{p} = 0, \quad \text{in } \Omega \qquad \alpha_\Gamma \bar{q}_\Gamma + \bar{p} = 0 \quad \text{on } \Gamma_R. \tag{2.10}$$

The optimality system (KKT-system) for the optimal control problem (1.1)–(1.2) is formed by (2.10) together with the state problem (1.2) and the adjoint state problem (2.7).

The second-order derivatives of $j(q_\Omega, q_\Gamma)$ do not depend on (q_Ω, q_Γ) and admit the estimates

$$D_{q_\Omega q_\Omega} j(q_\Omega, q_\Gamma) \cdot (k_\Omega, k_\Omega) \geq \alpha_\Omega \|k_\Omega\|_{0,\Omega}^2, \quad \forall\, k_\Omega \in Q_\Omega, \tag{2.11}$$

$$D_{q_\Gamma q_\Gamma} j(q_\Omega, q_\Gamma) \cdot (k_\Gamma, k_\Gamma) \geq \alpha_\Gamma \|k_\Gamma\|_{0,\Gamma_R}^2, \quad \forall\, k_\Gamma \in Q_\Gamma. \tag{2.12}$$

Motivated by Remark 2.1, we make the following regularity assumption for the solution of the optimal control problem which later on allows Lagrangian interpolation of the solution.

Assumption 2.1. The optimal solution $(\bar{u}, \bar{p}, \bar{q}_\Omega, \bar{q}_\Gamma)$ of the optimal control problem (1.1)–(1.2) belongs to $[W^{1+\lambda,2}(\Omega)]^3 \times W^{\frac{1}{2}+\lambda,2}(\Gamma_R)$ with $1 + \lambda > \frac{d}{2}$.

Assume that $\alpha_\Omega, \alpha_\Gamma > 0$. Then assumption 2.1 is valid if the solution u of (2.1) belongs to $W^{1+\lambda,2}(\Omega)$, $1 + \lambda > \frac{d}{2}$, eventually for sufficiently smooth data $\tilde{f}, \tilde{g}, \beta$. Then the same statement is valid for the solution p of (2.7) for sufficiently smooth data u_Ω, u_Γ. Moreover, the regularity of $\bar{q}_\Omega$ and $\bar{q}_\Gamma$ follows via (2.10).

3. Stabilized Discrete Optimality System

In this section, we introduce the discretized optimal control problem corresponding to (1.1)–(1.2). In particular, we apply a more general approach to the discretization as in Ref. 5 by considering shape-regular finite element meshes and a more flexible stabilization concept.

3.1. *Finite element spaces*

Consider a family of shape-regular, admissible decompositions $\mathcal{T}_h$ of Ω into d-dimensional simplices, quadrilaterals for $d = 2$ or hexahedra for $d = 3$. Let h_T be the diameter of a cell $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$. Let $\hat{T}$ be a reference element of the decomposition $\mathcal{T}_h$. Assume that, for each $T \in \mathcal{T}_h$, there exists an affine mapping $F_T : \hat{T} \to T$ which maps $\hat{T}$ onto T. This quite restrictive assumption for quadrilaterals/hexahedra can be weakened to asymptotically affine linear mappings.[3]

Let us denote by $\mathcal{E}_h$ the set of element faces (for $d = 3$) and element edges (for $d = 2$) induced by the finite element mesh $\mathcal{T}_h$ on $\partial\Omega$. Moreover, we assume that the Robin part Γ_R of the boundary is exactly triangulated by elements of $\mathcal{E}_h$.

Set $P_{k,\mathcal{T}_h} := \{v_h \in L^2(\Omega); v_h \circ F_T \in P_k(\hat{T}), T \in \mathcal{T}_h\}$ with the space $P_k(\hat{T})$ of complete polynomials of degree $k \in \{0, 1\}$ defined on $\hat{T}$ and $Q_{k,\mathcal{T}_h} := \{v_h \in L^2(\Omega); v_h \circ F_T \in Q_k(\hat{T}),\, T \in \mathcal{T}_h\}$ with the space $Q_k(\hat{T})$ of all polynomials on $\hat{T}$ with maximal degree k in each coordinate direction. We shall approximate the space V by a finite element space $V_h \subset V$ such that

$$V_h \supset P_{1,\mathcal{T}_h} \cap V \quad \text{or} \quad V_h \supset Q_{1,\mathcal{T}_h} \cap V.$$

Similarly, let $Q_{h,\Omega} \subset H^1(\Omega)$ be a finite element space for the control variable and $Q_{h,\Gamma} = Q_{h,\Omega}|_{\Gamma_R}$ its restriction to Γ_R.

3.2. *Local projection stabilization (LPS) for the state problems*

The basic Galerkin discretization of the state problem (2.1) reads: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h), \quad \forall\, v_h \in V_h. \tag{3.1}$$

For $0 < \varepsilon \ll 1$, the solution u_h of (3.1) may suffer from spurious oscillations. As in Ref. 5 we consider the local projection stabilization (LPS) approach which results in a symmetric discrete optimality system. The idea of LPS methods is to split the discrete function spaces into small and large scales and to add stabilization terms of diffusion-type acting only on the small scales. There are two obvious choices of the space of large scales:

The *two-level variant* starts from the given space $V_h \supset P_{1,\mathcal{T}_h} \cap V$ or $V_h \supset Q_{1,\mathcal{T}_h} \cap V$ for simplicial or hexahedral elements. The large scales are determined with the help of a coarse mesh. This mesh $\mathcal{M}_h$ is constructed by coarsening the basic mesh $\mathcal{T}_h$ such that each macro-element $M \in \mathcal{M}_h$ is the union of one or more neighboring cells $T \in \mathcal{T}_h$. The diameter of $M \in \mathcal{M}_h$ is denoted by h_M. We assume that the decomposition $\mathcal{M}_h$ of Ω is non-overlapping and shape-regular. In addition, the interior cells are supposed to be of the same size as the corresponding macro-cell:

$$\exists\, C > 0: \quad h_M \le C h_T, \quad \forall\, T \in \mathcal{T}_h, M \in \mathcal{M}_h \quad \text{with } T \subset M. \tag{3.2}$$

The discrete space $D_h \subset L^2(\Omega)$ is the discontinuous finite element space of piecewise constant functions defined on the macro-partition $\mathcal{M}_h$. The restriction of D_h on $M \in \mathcal{M}_h$ is denoted by $D_h(M) := \{v_h|_M; v_h \in D_h\}$. The next ingredient is a local projection $\pi_M : L^2(M) \to D_h(M)$ which defines the global projection $\pi_h : L^2(\Omega) \to D_h$ by $(\pi_h v)|_M := \pi_M(v|_M)$ for all $M \in \mathcal{M}_h$. A standard variant is the local orthogonal L^2 projection. Denoting the identity on $L^2(\Omega)$ by id, the fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ is defined by $\kappa_h := \mathrm{id} - \pi_h$.

The second approach, the *one-level variant*, consists in choosing the discontinuous finite element space D_h of piecewise constant functions on the original mesh $\mathcal{T}_h$ and constructing a proper enriched space V_h. The same abstract framework as in the first approach can be used by setting $\mathcal{M}_h = \mathcal{T}_h$.

For both variants, the stabilized discrete formulation reads: find $u_h \in V_h$ such that

$$a_{lps}(u_h, v_h) := a(u_h, v_h) + s_h(u_h, v_h) = f(v_h), \quad \forall\, v_h \in V_h, \tag{3.3}$$

where the additional stabilization term is given by

$$s_h(u_h, v_h) := \sum_{M \in \mathcal{M}_h} \tau_M (\kappa_h(\mathbf{b} \cdot \nabla u_h), \kappa_h(\mathbf{b} \cdot \nabla v_h))_M. \tag{3.4}$$

The stabilization s_h acts solely on the small scales. The constants τ_M will be determined later based on *a priori* error analysis.

3.3. *Some variants of one- and two-level variant*

Different variants for the choice of the discrete spaces V_h and D_h are given in Ref. 13. Here we describe some details.

The *one-level approach* with $\mathcal{M}_h = \mathcal{T}_h$ starts from a given discontinuous space D_h and uses an enrichment of the spaces $P_{1,\mathcal{T}_h} \cap V$ or $Q_{1,\mathcal{T}_h} \cap V$. For simplicial elements, we set

$$D_h := P_{0,\mathcal{T}_h}, \quad V_h := \{v \in V; v|_T \circ F_T \in P_1^{bub}(\hat{T})\ \forall T \in \mathcal{T}_h\},$$

where

$$P_1^{bub}(\hat{T}) := P_1(\hat{T}) + \hat{b} \cdot P_0(\hat{T}), \quad \hat{b}(\hat{x}) := (d+1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i(\hat{x})$$

with the barycentric coordinates $\hat{\lambda}_i, i = 1, \ldots, d+1$. For quadrilateral/hexahedral elements, we can use either $D_h = P_{0,\mathcal{T}_h}$ or $D_h = Q_{0,\mathcal{T}_h}$. Setting $\hat{D} = P_0(\hat{T})$ or $\hat{D} = Q_0(\hat{T})$, respectively, the spaces V_h are constructed analogously as for simplices with

$$Q_1^{bub}(\hat{T}) := Q_1(\hat{T}) + \hat{b} \cdot \hat{D}, \quad \hat{b}(\hat{x}) := \prod_{i=1}^{d} (1 - \hat{x}_i^2), \quad \hat{T} = (-1,1)^d.$$

Now consider the *two-level approach* (cf. Fig. 1 for $d = 2$). For quadrilateral/hexahedral elements, each $M \in \mathcal{M}_h$ is uniformly refined into 2^d sub-elements. In the simplicial case, each $M \in \mathcal{M}_h$ is divided into $d+1$ simplices by connecting the barycenter of M with the vertices of M. For simplices and for quadrilaterals/hexahedra, respectively, set

$$V_h := P_{1,\mathcal{T}_h} \cap V, \quad D_h := P_{0,\mathcal{M}_h} \quad \text{and} \quad V_h := Q_{1,\mathcal{T}_h} \cap V, \quad D_h := Q_{0,\mathcal{M}_h}.$$

Note that, for the two-level approach based on simplicial finite elements, the space V_h can be written in the form:

$$V_h = \{v \in V : v|_M \circ F_M \in P_1(\hat{T}) \oplus \hat{B}_1 \forall M \in \mathcal{M}_h\},$$

where $\hat{B}_1 \subset H_0^1(\hat{T})$ is a finite-dimensional space consisting of continuous piecewise polynomial functions of degree 1. Therefore, the simplicial two-level approach can be treated as a one-level approach with respect to the mesh $\mathcal{M}_h$.

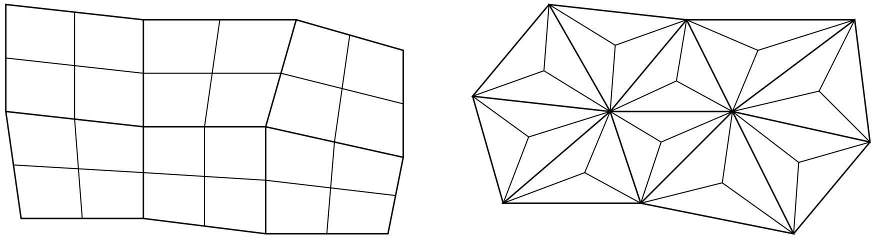


Fig. 1. Two-level approach with meshes $\mathcal{M}_h$ (bold lines) and $\mathcal{T}_h$ (fine lines).

3.4. *Discrete optimality system*

The discretized control problem to (1.1)–(1.2) is formulated in $V_h \times Q_{h,\Omega} \times Q_{h,\Gamma}$ as follows:

$$\min J(u_h, q_{h,\Omega}, q_{h,\Gamma}), \quad u_h \in V_h, \quad q_{h,\Omega} \in Q_{h,\Omega}, \quad q_{h,\Gamma} \in Q_{h,\Gamma}, \tag{3.5}$$

subject to

$$a_{lps}(u_h, v_h) = (f + q_{h,\Omega}, v_h) + (g + q_{h,\Gamma}, v_h)_{\Gamma_R}, \quad \forall\, v_h \in V_h. \tag{3.6}$$

This discrete optimal control problem admits a unique solution $(\bar{u}_h, \bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})$. Now we introduce a discrete solution operator $S_h : Q_\Omega \times Q_\Gamma \to V_h$ by

$$a_{lps}(S_h(q_{h,\Omega}, q_{h,\Gamma}), v_h) = (f + q_{h,\Omega}, v_h)_\Omega + (g + q_{h,\Gamma}, v_h)_{\Gamma_R} \quad \forall\, v_h \in V_h.$$

Moreover, the discrete reduced cost functional is formulated as

$$j_h(q_{h,\Omega}, q_{h,\Gamma}) = J(S_h(q_{h,\Omega}, q_{h,\Gamma}), q_{h,\Omega}, q_{h,\Gamma}).$$

For all $k_{h,\Omega} \in Q_{h,\Omega}, k_{h,\Gamma} \in Q_{h,\Gamma}$, the necessary (and sufficient) optimality conditions read

$$D_{q_\Omega} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma}) \cdot (k_{h,\Omega} - \bar{q}_{h,\Omega}) = (\alpha_\Omega \bar{q}_{h,\Omega} + \bar{p}_h, k_{h,\Omega} - \bar{q}_{h,\Omega})_\Omega = 0, \tag{3.7}$$

$$D_{q_\Gamma} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma}) \cdot (k_{h,\Gamma} - \bar{q}_{h,\Gamma}) = (\alpha_\Gamma \bar{q}_{h,\Gamma} + \bar{p}_h, k_{h,\Gamma} - \bar{q}_{h,\Gamma})_{\Gamma_R} = 0, \tag{3.8}$$

hence

$$\alpha_\Omega \bar{q}_{h,\Omega} + \bar{p}_h = 0, \qquad \alpha_\Gamma \bar{q}_{h,\Gamma} + \bar{p}_h = 0.$$

Here the discrete adjoint state $p_h \in V_h$ is the solution of the discrete adjoint state problem

$$a_{lps}(v_h, p_h) = \lambda_\Omega(u_h - u_\Omega, v_h)_\Omega + \lambda_\Gamma(u_h - u_\Gamma, v_h)_{\Gamma_R}, \tag{3.9}$$

where $u_h = S_h(q_\Omega, q_\Gamma)$ is the associated discrete state to (q_Ω, q_Γ).

Remark 3.1. The symmetry of the LPS term implies that the operations "optimize" and "discretize" commute, see Ref. 5.

Finally, the second-order derivatives of $j_h(q_\Omega, q_\Gamma)$ do not depend on (q_Ω, q_Γ) and admit the estimates

$$D_{q_\Omega q_\Omega} j_h(q_\Omega, q_\Gamma) \cdot (k_{h,\Omega}, k_{h,\Omega}) \geq \alpha_\Omega \|k_{h,\Omega}\|^2_{0,\Omega}, \quad \forall\, k_{h,\Omega} \in Q_{h,\Omega}, \tag{3.10}$$

$$D_{q_\Gamma q_\Gamma} j_h(q_\Omega, q_\Gamma) \cdot (k_{h,\Gamma}, k_{h,\Gamma}) \geq \alpha_\Gamma \|k_{h,\Gamma}\|^2_{0,\Gamma_R}, \quad \forall\, k_{h,\Gamma} \in Q_{h,\Gamma}. \tag{3.11}$$

4. *A Priori* Error Analysis for the Optimal Control Problem

In this section, we provide the error analysis for the optimal control problem (1.1)–(1.2).

4.1. *Some auxiliary results*

It turns out that additional assumptions for the LPS method are required. In order to control the consistency error of the stabilization term, the discontinuous space D_h on the coarse mesh $\mathcal{M}_h$ has to be large enough; more precisely:

Assumption 4.1. The fluctuation operator $\kappa_h = \mathrm{id} - \pi_h$, see Sec. 3.2, satisfies for $s \in [0,1]$ the following approximation property:

$$\exists\, C_\kappa > 0: \quad \|\kappa_h q\|_{0,M} \leq C_\kappa h_M^s |q|_{s,M}, \quad \forall\, q \in W^{s,2}(M), \quad \forall\, M \in \mathcal{M}_h. \tag{4.1}$$

Remark 4.1. (i) Assumption 4.1 is valid if the local L^2-projection operator π_h is chosen in the definition of the fluctuation operator $\kappa_h = \mathrm{id} - \pi_h$.

(ii) The original version of (4.1) in Ref. 13 only considers $s \in \{0, 1\}$.

Now we construct a special interpolation $j_0 : V \to V_h$ such that the error $v - j_0 v$ is L^2-orthogonal to D_h for all $v \in V$. In order to conserve the standard approximation properties, we additionally assume

Assumption 4.2. There exists a constant $\beta > 0$ such that, for any $M \in \mathcal{M}_h$,

$$\inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M} \|q_h\|_{0,M}} \geq \beta > 0, \tag{4.2}$$

where $Y_h(M) := \{v_h|_M : v_h \in V_h, v_h = 0 \text{ on } \Omega \backslash M\}$.

Remark 4.2. The inf−sup condition (4.2) implies that the space D_h must not be too rich. On the other hand, D_h must be rich enough to fulfil the approximation property (4.1). Assumption 4.2 is valid for the discrete spaces discussed in Sec. 3.3, cf. Ref. 9, Sec. 4.

Lemma 4.1. *Let Assumption 4.2 be satisfied. Then there is an interpolation operator $j_0 : V \to V_h$ such that*

$$(v - j_0 v, q_h)_\Omega = 0, \quad \forall\, q_h \in D_h, \quad \forall\, v \in V \tag{4.3}$$

and

$$\|v - j_0 v\|_{0,M} + h_M |v - j_0 v|_{1,M} + h_M^{\frac{1}{2}} \|v - j_0 v\|_{0,E} \lesssim h_M^{1+\lambda} \|v\|_{1+\lambda,2,M} \tag{4.4}$$

for all $M \in \mathcal{M}_h$ and for $v \in V \cap W^{1+\lambda,2}(\Omega)$ with $1 + \lambda > \frac{d}{2}$.

Proof. This is a simple extension of the proof with $\lambda \in \{0,1\}$ in Ref. 13. In particular, the modified analysis takes advantage of the Lagrangian interpolation properties of the space V_h

$$\exists\, C > 0: \quad \|v - I_h v\|_{m,T} \leq C h_T^{1+\lambda-m} \|v\|_{1+\lambda,2,T}, \quad m \in \{0,1\} \tag{4.5}$$

for $v \in W^{1+\lambda,2}(T)$, $\forall\, T \in \mathcal{T}_h$ with $\lambda \in [0,1)$ such that $1 + \lambda > \frac{d}{2}$, see Ref. 7, Theorem 2.25 and Remark 2.1. Moreover, for $E \subseteq \partial T$ one obtains

$$\exists\, C > 0: \quad \|v - I_h v\|_{0,E} \leq C h_T^{\lambda + \frac{1}{2}} \|v\|_{1+\lambda,2,T}. \tag{4.6}$$

□

4.2. *Analysis of the state problems*

The next goal is to derive error estimates for the state problems (3.6) and (3.9). First, the stability of the scheme will be given in the mesh-dependent norm

$$|||v||| := (\varepsilon|v|^2_{1,\Omega} + \sigma\|v\|^2_{0,\Omega} + \|\tilde{\beta}^{\frac{1}{2}}v\|^2_{0,\Gamma_R} + s_h(v,v))^{\frac{1}{2}}, \quad \forall\, v \in V.$$

Lemma 4.2. *The LPS schemes* (3.6) *and* (3.9) *for the discrete state and the adjoint states admit unique solutions.*

Proof. For any $v \in V$, integration by parts yields $(\mathbf{b}\cdot\nabla v, v)_\Omega = \frac{1}{2}((\mathbf{b}\cdot\mathbf{n})v, v)_{\Gamma_R}$, hence

$$a_{lps}(v,v) = \varepsilon|v|^2_{1,\Omega} + \sigma\|v\|^2_{0,\Omega} + \|\tilde{\beta}^{\frac{1}{2}}v\|^2_{0,\Gamma_R} + s_h(v,v) = |||v|||^2, \quad \forall\, v \in V \tag{4.7}$$

with $\tilde{\beta} = \beta + \frac{1}{2}\mathbf{b}\cdot\mathbf{n}$. This implies $|||u_h|||^2 \le (\tilde{f}, u_h)_\Omega + (\tilde{g}, u_h)_{\Gamma_R}$, hence existence and uniqueness of $u_h \in V_h$ in the scheme (3.6). The result for (3.9) follows similarly. □

The following *a priori* estimate can be proven using the standard technique of combining stability and consistency results based on the auxiliary results of the last subsection. Here, and in the following lemma, we fix some controls $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$ which will be later on, in the proof of the main theorem, chosen as the Lagrangian interpolants of the optimal controls $(\bar{q}_\Omega, \bar{q}_\Gamma)$.

Lemma 4.3. *Let* $(q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma$, $u = S(q_\Omega, q_\Gamma) \in V$. *For some* $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$, *let* $w_h = S_h(p_\Omega, p_\Gamma) \in V_h$ *be the solution of*

$$a_{lps}(w_h, v_h) = (f + p_\Omega, v_h)_\Omega + (g + p_\Gamma, v_h)_{\Gamma_R} \quad \forall\, v_h \in V_h. \tag{4.8}$$

Let the stabilization parameters be chosen as

$$\tau_M \sim \frac{h_M}{\varepsilon h_M^{-1} + \|\mathbf{b}\|_{[L^\infty(M)]^d}}. \tag{4.9}$$

Then, under the assumptions of Lemma 2.1, *there holds the following a priori error estimate*

$$\begin{aligned} |||u - w_h||| &\le C_\Omega\|q_\Omega - p_\Omega\|_{0,\Omega} + C_\Gamma\|q_\Gamma - p_\Gamma\|_{0,\Gamma_R} \\ &\quad + C\left(\sum_{M\in\mathcal{M}_h} h_M^{2\lambda+1}\left\{\frac{|\mathbf{b}\cdot\nabla u|^2_{\lambda,2,M}}{\varepsilon h_M^{-1} + \|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M\|u\|^2_{1+\lambda,2,M}\right\}\right)^{\frac{1}{2}} \end{aligned} \tag{4.10}$$

with

$$C_M := \frac{\varepsilon}{h_M} + \sigma h_M + \|\mathbf{b}\|_{[L^\infty(M)]^d} + \|\beta\|_{L^\infty(\partial M\cap\Gamma_R)} + \|\mathbf{b}\cdot\mathbf{n}\|_{L^\infty(\partial M\cap\Gamma_R)},$$

$$C_\Omega := \min\left\{\frac{1}{\sqrt{\sigma}}; \frac{C_P}{\sqrt{\varepsilon}}\right\}; \quad C_\Gamma := \min\left\{\frac{1}{\sqrt{\beta_0}}; \frac{C_P}{\sqrt{\varepsilon}}\right\}.$$

Proof. The error is split into $u - w_h = (u - j_0 u) + (j_0 u - w_h)$. For the approximation error $u - j_0 u$, Lemma 4.1 and Assumption 4.1 with $s = 0$ yield

$$|||u - j_0 u||| \lesssim \left(\sum_{M \in \mathcal{M}_h} \left[\varepsilon + \sigma h_M^2 + \tau_M \|\mathbf{b}\|^2_{[L^\infty(M)]^d} + \|\tilde{\beta}\|_{L^\infty(\partial M \cap \Gamma_R)} h_M \right] h_M^{2\lambda} \|u\|^2_{1+\lambda,2,M} \right)^{\frac{1}{2}}. \tag{4.11}$$

Now we estimate the remaining part $z_h := j_0 u - w_h$ using (4.7)

$$\begin{aligned} |||j_0 u - w_h||| &= \frac{(a + s_h)(j_0 u - w_h, z_h)}{|||z_h|||} \\ &= \frac{(a + s_h)(u - w_h, z_h)}{|||z_h|||} + \frac{(a + s_h)(j_0 u - u, z_h)}{|||z_h|||} =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We start with term I. Subtracting (4.8) from (2.1), one obtains the perturbed Galerkin orthogonality relation

$$(a + s_h)(u - w_h, v_h) = s_h(u, v_h) + (q_\Omega - p_\Omega, v_h)_\Omega + (q_\Gamma - p_\Gamma, v_h)_{\Gamma_R}, \quad \forall\, v_h \in V_h. \tag{4.12}$$

Assumption 4.1 yields

$$|s_h(u, v_h)| \le s_h^{\frac{1}{2}}(u, u) s_h^{\frac{1}{2}}(v_h, v_h) \le C \left(\sum_{M \in \mathcal{M}_h} \tau_M h_M^{2\lambda} |\mathbf{b} \cdot \nabla u|^2_{\lambda,2,M} \right)^{\frac{1}{2}} |||v_h||| \quad \forall\, v_h \in V_h.$$

Moreover, under assumption of Lemma 2.1 there holds

$$\begin{aligned} (q_\Omega - p_\Omega, v_h)_\Omega &\le C_\Omega \|q_\Omega - p_\Omega\|_{0,\Omega} |||v_h|||, \quad C_\Omega := \min\left\{ \frac{1}{\sqrt{\sigma}}; \frac{C_P}{\sqrt{\varepsilon}} \right\}, \\ (q_\Gamma - p_\Gamma, v_h)_{\Gamma_R} &\le C_\Gamma \|q_\Gamma - p_\Gamma\|_{0,\Gamma_R} |||v_h|||, \quad C_\Gamma := \min\left\{ \frac{1}{\sqrt{\beta_0}}; \frac{C_P}{\sqrt{\varepsilon}} \right\}, \end{aligned}$$

where C_P denotes the Poincaré constant. Setting $v_h = z_h$, we obtain

$$I \le C \left(\sum_{M \in \mathcal{M}_h} \tau_M h_M^{2\lambda} |\mathbf{b} \cdot \nabla u|^2_{\lambda,2,M} \right)^{\frac{1}{2}} + C_\Omega \|q_\Omega - p_\Omega\|_{0,\Omega} + C_\Gamma \|q_\Gamma - p_\Gamma\|_{0,\Gamma_R}.$$

Now we consider the terms of II separately. Integration by parts and the orthogonality property (4.3) and the estimate (4.4) yield for $w_h \in V_h$ that

$$\begin{aligned} \frac{a(j_0 u - u, w_h)}{|||w_h|||} &= \frac{1}{|||w_h|||} (\varepsilon(\nabla(j_0 u - u), \nabla w_h)_\Omega - (\kappa_h(\mathbf{b} \cdot \nabla w_h), j_0 u - u)_\Omega \\ &\quad + \sigma(j_0 u - u, w_h)_\Omega + (\tilde{\beta}(j_0 u - u), w_h)_{\Gamma_R}) \\ &\le C \left(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda} \left[\varepsilon + (\sigma + \tau_M^{-1}) h_M^2 + \|\tilde{\beta}\|_{L^\infty(\partial M \cap \Gamma_R)} h_M \right] \|u\|^2_{1+\lambda,2,M} \right)^{\frac{1}{2}}. \end{aligned}$$

The estimate of the stabilization term follows using (4.1) with $s = 0$ and (4.4)

$$\frac{s_h(j_0 u - u, w_h)}{|||w_h|||} \le C \left(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda} \tau_M \|\mathbf{b}\|^2_{[L^\infty(M)]^d} \|u\|^2_{1+\lambda,2,M} \right)^{\frac{1}{2}}. \tag{4.13}$$

Summing up all inequalities in this proof gives the assertion

$$|||j_0 u - w_h||| \le C \left(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda} \left\{ \tau_M |\mathbf{b} \cdot \nabla u|^2_{\lambda,2,M} + \tilde{C}_M \|u\|^2_{1+\lambda,2,M} \right\} \right)^{\frac{1}{2}} + C_\Omega \|q_\Omega - p_\Omega\|_{0,\Omega} + C_\Gamma \|q_\Gamma - p_\Gamma\|_{0,\Gamma_R} \tag{4.14}$$

with

$$\tilde{C}_M := \varepsilon + \sigma h_M^2 + \tau_M^{-1} h_M^2 + \tau_M \|\mathbf{b}\|^2_{[L^\infty(M)]^d} + (\|\beta\|_{L^\infty(\partial M \cap \Gamma_R)} + \|\mathbf{b} \cdot \mathbf{n}\|_{L^\infty(\partial M \cap \Gamma_R)}) h_M.$$

In the advection-dominated case, the parameters τ_M are determined by balancing the terms $\tau_M^{-1} h_M^2 \sim \tau_M \|\mathbf{b}\|^2_{[L^\infty(M)]^d}$, hence $\tau_M \sim \frac{h_M}{\|\mathbf{b}\|_{[L^\infty(M)]^d}}$. In the diffusion-dominated case, we balance the terms $\varepsilon \sim \tau_M^{-1} h_M^2$. The combination of both cases leads to the choice (4.9). Note that a deterioration of the denominator of τ_M in case of $\mathbf{b} = 0$ is avoided. Finally, the triangle inequality concludes the proof. □

Remark 4.3. The constants C_Ω and C_Γ are critical in the case of $0 < \varepsilon \ll 1$ together with $0 \le \sigma, \beta_0 \ll 1$. Let us discuss some relevant situations:

- For singularly perturbed diffusion-reaction problems, i.e. with $\mathbf{b} \equiv 0$, it is reasonable to assume that $\sigma > 0$ is independent of ε.
- For singularly perturbed advection-diffusion problems, there occurs the case that all subcharacteristics of the first-order operator $\mathbf{b} \cdot \nabla$ leave the domain $\bar{\Omega}$ in finite time. This excludes periodic subcharacteristics and stagnation points of $\mathbf{b}$ in $\bar{\Omega}$. Then it is possible to transform the elliptic operator to the form $L = -\varepsilon L_2 + \mathbf{b} \cdot \nabla + \tilde{c}$ with $\tilde{c}(x) \ge \sigma$ with arbitrary large σ.
- For Robin boundary control (or regularized Dirichlet control), it is reasonable to assume $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \ge \beta_0 > 0$ with $\beta_0 = \mathcal{O}(1)$.

Remark 4.4. In the limit case $\lambda = 1$, i.e. for $u \in H^2(\Omega)$, we obtain the well-known optimal convergence rate $\mathcal{O}(h_M^{\frac{3}{2}})$ with respect to h_M.

Remark 4.5. The LPS method is still a matter of ongoing research. Recent results provide improved stability and convergence results of the LPS method.

(i) In Lemma 4.1 of Ref. 10, it is shown for the one-level method that the LPS-norm $|||\cdot|||$ gives control of the weighted streamline derivative $(\sum_K \tau_K \|\mathbf{b} \cdot \nabla(\cdot)\|^2_{0,K})^{\frac{1}{2}}$.
(ii) Theorem 2 of Ref. 9 states that, for simplicial elements, the one- and the two-level approach are algebraically equivalent to a residual-based stabilization scheme, to the unusual Galerkin/least-squares stabilization or algebraic subscale method.[6]

Similarly, we obtain the following *a priori* error estimate for the adjoint problem (3.9).

Lemma 4.4. *For* $(q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma$, *let* $p \in V$ *be the solution of the adjoint state problem* (2.7) *and for some* $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$, *let* $y_h \in V_h$ *be the adjoint discrete solution. Let the stabilization parameters be chosen as in* (4.9). *Then, there holds the a priori error estimate*

$$\begin{aligned} |||p - y_h||| &\leq (C_\Omega^2 \lambda_\Omega + C_\Gamma^2 \lambda_\Gamma) |||u - w_h||| \\ &\quad + C \left(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda+1} \left\{ \frac{|\mathbf{b} \cdot \nabla p|^2_{\lambda,2,M}}{\varepsilon h_M^{-1} + \|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|p\|^2_{1+\lambda,2,M} \right\} \right)^{\frac{1}{2}} \end{aligned} \tag{4.15}$$

with C_M, C_Ω *and* C_Γ *as in the previous lemma.*

Proof. The equations for $p \in V$ and $y_h \in V_h$

$$\begin{aligned} a(v, p) &= \lambda_\Omega (u - u_\Omega, v)_\Omega + \lambda_\Gamma (u - u_\Gamma, v)_{\Gamma_R} \qquad \forall v \in V \\ a(v_h, y_h) + s_h(y_h, v_h) &= \lambda_\Omega (w_h - u_\Omega, v_h)_\Omega + \lambda_\Gamma (w_h - u_\Gamma, v_h)_{\Gamma_R}, \quad \forall v_h \in V_h \end{aligned}$$

lead to the error equation

$$\begin{aligned} a(v_h, p - y_h) + s_h(p - y_h, v_h) = s_h(p, v_h) &+ \lambda_\Omega (u - w_h, v_h)_\Omega \\ &+ \lambda_\Gamma (u - w_h, v_h)_{\Gamma_R} \quad \forall v_h \in V_h. \end{aligned}$$

The remaining part of the proof follows the lines of the previous proof. □

Remark 4.6. The term $|||u - w_h|||$ in (4.10) can be further estimated via Lemma 4.3.

4.3. *Main result for unconstrained case*

We are now in a position to prove the main result for the unconstrained optimal control problem.

Theorem 4.1. *Let the assumptions of Lemma* 2.1 *and Assumption* 2.1 *be valid. Moreover, let* $(\bar{u}, \bar{q}_\Omega, \bar{q}_\Gamma)$ *be the solution of the optimal control problem* (1.1)–(1.2) *and* $(\bar{u}_h, \bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})$ *the solution of the discretized problem* (3.5)–(3.6). *Finally, let* $\alpha_\Omega, \alpha_\Gamma > 0$. *Then there exists a constant* $C > 0$ *depending on* $\lambda_\Omega, \lambda_\Gamma, \alpha_\Omega, \alpha_\Gamma, C_\Omega, C_\Gamma$ *such that the following error estimate holds*:

$$\begin{aligned} &\|\bar{q}_\Omega - \bar{q}_{h,\Omega}\|_{0;\Omega} + \|\bar{q}_\Gamma - \bar{q}_{h,\Gamma}\|_{0;\Gamma_R} \\ &\quad \leq C \Bigg\{ \left(\sum_{M \in \mathcal{M}_h} h_M^{1+2\lambda} |\bar{q}_\Omega|^2_{1+\lambda;2,M} \right)^{\frac{1}{2}} + \left(\sum_{E \in \mathcal{E}_h \cap \Gamma_R} h_E^{1+2\lambda} |\bar{q}_\Gamma|^2_{1+\lambda;2,E} \right)^{\frac{1}{2}} \\ &\qquad + \left(\sum_M h_M^{1+2\lambda} \left(\frac{|\mathbf{b} \cdot \nabla \bar{u}|^2_{\lambda;2,M}}{\varepsilon h_M^{-1} + \|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|\bar{u}\|^2_{1+\lambda,2,M} \right) \right)^{\frac{1}{2}} \\ &\qquad + \left(\sum_M \left(h_M^{1+2\lambda} \frac{|\mathbf{b} \cdot \nabla \bar{p}|^2_{\lambda;2,M}}{\varepsilon h_M^{-1} + \|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|\bar{p}\|^2_{1+\lambda,2,M} \right) \right)^{\frac{1}{2}} \Bigg\} \end{aligned}$$

with C_M, C_Ω *and* C_Γ *as in Lemma* 4.3.

Proof. Let $(z_{h,\Omega}, z_{h,\Gamma}) \in Q_{h,\Omega} \times Q_{h,\Gamma}$ be arbitrary so far. A straightforward calculation gives together with (2.11) and (2.12)

$$\begin{aligned}
&D_{q_\Omega} j_h(z_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Omega} - \bar{q}_{h,\Omega}) - D_{q_\Omega} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Omega} - \bar{q}_{h,\Omega}) \\
&\qquad = D_{q_\Omega, q_\Omega} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Omega} - \bar{q}_{h,\Omega}, z_{h,\Omega} - \bar{q}_{h,\Omega}) \\
&\qquad \geq \alpha_\Omega \|z_{h,\Omega} - \bar{q}_{h,\Omega}\|_{0;\Omega}^2, \\
&D_{q_\Gamma} j_h(z_{h,\Omega}, z_{h,\Gamma})(w_{h,\Gamma} - \bar{q}_{h,\Gamma}) - D_{q_\Gamma} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Gamma} - \bar{q}_{h,\Gamma}) \\
&\qquad = D_{q_\Gamma, q_\Gamma} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Gamma} - \bar{q}_{h,\Gamma}, z_{h,\Gamma} - \bar{q}_{h,\Gamma}) \\
&\qquad \geq \alpha_\Gamma \|z_{h,\Gamma} - \bar{q}_{h,\Gamma}\|_{0,\Gamma_R}^2.
\end{aligned}$$

As the gradient vanishes at the optimal point for the unconstrained case, there holds

$$D_{q_\Omega} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Omega} - \bar{q}_{h,\Omega}) = 0 = D_{q_\Omega} j(\bar{q}_\Omega, \bar{q}_\Gamma)(z_{h,\Omega} - \bar{q}_{h,\Omega}), \tag{4.16}$$

$$D_{q_\Gamma} j_h(\bar{q}_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Gamma} - \bar{q}_{h,\Gamma}) = 0 = D_{q_\Gamma} j(\bar{q}_\Omega, \bar{q}_\Gamma)(z_{h,\Gamma} - \bar{q}_{h,\Gamma}) \tag{4.17}$$

which leads in the previous inequalities to

$$\begin{aligned}
\alpha_\Omega \|z_{h,\Omega} - \bar{q}_{h,\Omega}\|_{0;\Omega}^2 &\leq D_{q_\Omega} j_h(z_{h,\Omega}, \bar{q}_{h,\Gamma})(z_{h,\Omega} - \bar{q}_{h,\Omega}) - D_{q_\Omega} j(\bar{q}_\Omega, \bar{q}_\Gamma)(z_{h,\Omega} - \bar{q}_{h,\Omega}), \\
\alpha_\Gamma \|z_{h,\Gamma} - \bar{q}_{h,\Gamma}\|_{0;\Gamma_R}^2 &\leq D_{q_\Gamma} j_h(\bar{q}_{h,\Omega}, z_{h,\Gamma})(z_{h,\Gamma} - \bar{q}_{h,\Gamma}) - D_{q_\Gamma} j(\bar{q}_\Omega, \bar{q}_\Gamma)(z_{h,\Gamma} - \bar{q}_{h,\Gamma}).
\end{aligned}$$

Now the discrete analogue of Lemma 2.2 gives

$$\begin{aligned}
\alpha_\Omega \|z_{h,\Omega} - \bar{q}_{h,\Omega}\|_{0;\Omega}^2 &\leq (\alpha_\Omega z_{h,\Omega} + y_{h,\Omega}, z_{h,\Omega} - \bar{q}_{h,\Omega})_\Omega - (\alpha_\Omega \bar{q}_\Omega + \bar{p}, z_{h,\Omega} - \bar{q}_{h,\Omega})_\Omega \\
&= (\alpha_\Omega(z_{h,\Omega} - \bar{q}_\Omega) + (y_{h,\Omega} - \bar{p}), z_{h,\Omega} - \bar{q}_{h,\Omega})_\Omega,
\end{aligned}$$

where $y_{h,\Omega}$ denotes the associated discrete adjoint state to $z_{h,\Omega}$. This implies

$$\|z_{h,\Omega} - \bar{q}_{h,\Omega}\|_{0;\Omega} \leq \|z_{h,\Omega} - \bar{q}_\Omega\|_{0;\Omega} + \frac{C_\Omega}{\alpha_\Omega} |||y_h - \bar{p}|||$$

and via triangle inequality

$$\|\bar{q}_\Omega - \bar{q}_{h,\Omega}\|_{0;\Omega} \leq 2\|z_{h,\Omega} - \bar{q}_\Omega\|_{0;\Omega} + \frac{C_\Omega}{\alpha_\Omega} |||y_h - \bar{p}|||. \tag{4.18}$$

Similarly we obtain with the associated discrete adjoint state $y_{h,\Gamma}$ to $w_{h,\Gamma}$ that

$$\alpha_\Gamma \|z_{h,\Gamma} - \bar{q}_{h,\Gamma}\|_{0;\Gamma_R}^2 \leq (\alpha_\Gamma(z_{h,\Gamma} - \bar{q}_\Gamma) + (y_{h,\Gamma} - \bar{p}), z_{h,\Gamma} - \bar{q}_{h,\Gamma})_{\Gamma_R}$$

and

$$\|\bar{q}_\Gamma - \bar{q}_{h,\Gamma}\|_{0;\Gamma_R} \leq 2\|z_{h,\Gamma} - \bar{q}_\Gamma\|_{0;\Gamma_R} + \frac{C_\Gamma}{\alpha_\Gamma} |||y_h - \bar{p}|||. \tag{4.19}$$

The continuous optimality system (1.2), (2.7)–(2.9) gives $\bar{q}_\Omega = -\frac{1}{\alpha_\Omega}\bar{p}$ and $\bar{q}_\Gamma = -\frac{1}{\alpha_\Gamma}\bar{p}|_\Gamma$. Consequently, the regularity of the adjoint state $\bar{p}$ implies $(\bar{q}_\Omega, \bar{q}_\Gamma) \in W^{1+\lambda,2}(\Omega) \times W^{\frac{1}{2}+\lambda,2}(\Gamma_R)$ with $1 + \lambda > \frac{d}{2}$. This allows one to select $z_{h,\Omega}$ and $z_{h,\Gamma}$ as the

Lagrangian interpolants of $\bar{q}_\Omega$ and $\bar{q}_\Gamma$, respectively; hence

$$\|z_{h,\Omega} - \bar{q}_\Omega\|_{0;\Omega} \leq C\left(\sum_{M\in\mathcal{M}_h} h_M^{1+2\lambda}|\bar{q}_\Omega|^2_{1+\lambda;2,M}\right)^{\frac{1}{2}}, \tag{4.20}$$

$$\|z_{h,\Gamma} - \bar{q}_\Gamma\|_{0;\Gamma_R} \leq C\left(\sum_{E\in\mathcal{E}_h\cap\Gamma_R} h_E^{1+2\lambda}|\bar{q}_\Gamma|^2_{\frac{1}{2}+\lambda;2,E}\right)^{\frac{1}{2}}. \tag{4.21}$$

The estimates (4.18), (4.19) together with the latter interpolation estimates, Lemmas 4.4 and 4.3 with $p_\Omega := z_{h,\Omega}$ and $p_\Gamma := z_{h,\Gamma}$ prove the assertion. □

5. Regularized Dirichlet Control

In applications, a Dirichlet boundary control

$$u = q$$

might be desirable. A review of different variants is given in Ref. 12. One possibility is to approximate the Dirichlet boundary control by a Robin boundary control of the form

$$\delta\nabla u\cdot\mathbf{n} + \beta(u-q) = 0, \quad \beta = \mathcal{O}(1)$$

for $\delta \to +0$, but the choice of the regularization parameter δ is delicate. For the case of the singularly perturbed problem (1.2), a rather natural choice seems to be $\delta = \varepsilon$. This would allow one to interpret the Robin boundary control within this paper as a regularization of Dirichlet boundary control. Nevertheless, some care is necessary.

In order to describe potential problems, define the subsets Γ_-, Γ_0 and Γ_+ of the boundary $\partial\Omega$, depending on the sign of $(\mathbf{b}\cdot\mathbf{n})(x)$, as the inflow, characteristic and outflow part for the flow field $\mathbf{b}$. Typically, the solution u of problem (1.2) has boundary layers at the outflow part Γ_+ with steep gradient $|\varepsilon\nabla u\cdot\mathbf{n}| \sim 1$ and at characteristic boundaries Γ_0 with (at most) $|\varepsilon\nabla u\cdot\mathbf{n}| \sim \sqrt{\varepsilon}$. Clearly, at the inflow part Γ_-, one only has $|\varepsilon\nabla u\cdot\mathbf{n}| \sim \epsilon$. This observation motivates one to exclude a Dirichlet control at the outflow boundary Γ_+ whereas the Robin regularization

$$\varepsilon\nabla u\cdot\mathbf{n} + \beta(u-q) = 0 \tag{5.1}$$

with $\beta + \frac{1}{2}\mathbf{b}\cdot\mathbf{n} \geq \beta_0 > 0$ is a potential approximation of a Dirichlet condition $u = q$ at $\Sigma \subseteq \partial\Omega\backslash\Gamma_+$.

A typical situation is the flow in a domain $\Omega = (0, L)\times(-\frac{H}{2}, \frac{H}{2})$ of channel type with the flow field

$$b(x) = \left(\left(\frac{H}{2} - |x_2|\right)^\kappa, 0\right)^T \quad \text{with } \kappa \geq 0.$$

The case $\kappa > 0$ corresponds to a no-slip condition of the flow field $\mathbf{b}$ whereas $\kappa = 0$ represents a slip-condition of $\mathbf{b}$. The solution u of (1.2) can be seen as a temperature field or as the density of some chemical reactant. Let us describe two potential applications of Dirichlet control:

(i) *Regularization of wall Dirichlet control*:
A Dirichlet condition $u = q$ is given at a part $\Sigma \subset \Gamma_0 = (0, L) \times \{-\frac{H}{2}, \frac{H}{2}\}$ of the channel walls whereas an insulation condition is given on $\Gamma_0 \backslash \Sigma$. An inflow condition $\varepsilon \frac{\partial u}{\partial x_1} + \beta(u - g) = 0$ with $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \geq \beta_0 > 0$ is prescribed on $\Gamma_- = \{0\} \times (-\frac{H}{2}, \frac{H}{2})$. A "do-nothing" condition $\varepsilon \frac{\partial u}{\partial x_1} = 0$ might be prescribed on $\Gamma_+ = \{1\} \times (-\frac{H}{2}, \frac{H}{2})$.

(ii) *Regularization of inflow Dirichlet control*:
A Dirichlet condition $u = g$ is given at a part $\Sigma \subset \Gamma_-$ whereas a Robin boundary condition $\varepsilon \frac{\partial u}{\partial x_1} + \beta(u - g) = 0$ with $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \geq \beta_0 > 0$ is prescribed on $\Gamma_- \backslash \Sigma$. A "do-nothing" condition $\varepsilon \frac{\partial u}{\partial x_1} = 0$ might be prescribed on Γ_+. An "insulation" condition $\varepsilon \frac{\partial u}{\partial x_2} = 0$ is given at the channel walls Γ_0.

Replacing the Dirichlet control on $\Sigma \subseteq \Gamma_- \cup \Gamma_0$ by the Robin boundary control (5.1), one can take advantage of the results of this paper. We will discuss an example for case (i) in the next section. An analysis of this approach and numerical experiments for case (ii) will be reported elsewhere.

6. Numerical Experiments

Meanwhile, several authors contributed to the theoretical and practical investigations of LPS methods. A detailed discussion of pro's and con's of the one- and two-level variant can be found in Ref. 9. As a result of the latter studies, no significant preference of one of the methods was observed. For the following numerical experiments with the two-level variant of the LPS method, the C++ libray `deal.II`[4] is used.

The goal of the first example is to show the effect of stabilization and the convergence of the method for vanishing regularization parameter α_Ω.

Example 6.1. Consider the unconstrained optimization problem

$$\min J(q_\Omega, q_\Gamma, u) := \frac{1}{2} \|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Omega}{2} \|q_\Omega\|^2_{L^2(\Omega)}$$

such that

$$\begin{aligned} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + \sigma u &= q_\Omega \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6.1}$$

with $\varepsilon = 10^{-3}, \mathbf{b} = (-1, -2)^t, \sigma = 1$. In order to obtain results on the convergence of the control in the sense of $q_\Omega \to q_{\Omega,\text{ref}}$ for $\alpha_\Omega \to 0$, we prescribe the control as

$$q_{\Omega,\text{ref}}(x) = (\sin(\pi x_1))^{0.3} (\sin(\pi x_2))^{0.3}.$$

Then we compute the solution of (6.1) with given source term q_Ω and prescribe the solution as desired state u_Ω.

Table 1. Different error measures for the unstabilized scheme with mesh width $h = 2^{-5}$.

	Control			State		
α_Ω	L^∞	L^2	H^1	L^∞	L^2	H^1
1e+0	9.47E-01	6.97E-01	5.98E+00	4.01E-01	1.54E-01	3.45E+00
1e-1	6.92E-01	5.16E-01	9.54E+00	2.54E-01	1.02E-01	2.73E+00
1e-2	7.23E-01	2.63E-01	1.68E+01	1.51E-01	3.48E-02	4.32E+00
1e-3	2.43E+00	3.41E-01	4.48E+01	1.24E-01	2.07E-02	4.43E+00
1e-4	1.04E+01	1.11E+00	1.97E+02	7.67E-02	1.11E-02	2.35E+00
1e-5	2.23E+01	2.07E+00	3.87E+02	2.38E-02	2.84E-03	5.68E-01
1e-6	2.64E+01	2.43E+00	4.55E+02	3.19E-03	3.66E-04	7.18E-02

Table 2. Different error measures for LPS-stabilization with $\tau = 0.034h$ and mesh width $h = 2^{-5}$.

	Control			State		
α_Ω	L^∞	L^2	H^1	L^∞	L^2	H^1
1e+0	9.46E-01	6.97E-01	5.89E+00	4.09E-01	1.54E-01	3.55E+00
1e-1	6.87E-01	5.12E-01	5.31E+00	2.79E-01	1.03E-01	2.60E+00
1e-2	5.57E-01	2.23E-01	6.74E+00	8.54E-02	2.77E-02	9.67E-01
1e-3	2.96E-01	8.04E-02	5.29E+00	1.94E-02	4.37E-03	2.35E-01
1e-4	1.64E-01	2.74E-02	2.85E+00	3.57E-03	5.81E-04	4.77E-02
1e-5	4.95E-02	6.79E-03	9.53E-01	4.81E-04	7.06E-05	7.77E-03
1e-6	7.08E-03	9.81E-04	1.56E-01	5.12E-05	7.64E-06	9.45E-04

If problem (6.1) is solved without stabilization, then the control tries, in the case of small values of α_Ω, to reduce the existing oscillations in order to reach the (smooth) desired state. The convergence of the state is obtained as well for the unstabilized as for the stabilized case, see Tables 1 and 2. Nevertheless, in the unstabilized case, the control is subject to spurious oscillations whereas in the case of stabilization the convergence of the control is observed.

In the following example we revisit a problem which had been considered in Ref. 5 for the case of box-constraints for the control. Here we consider the case without constraints. The numerical solution in Ref. 5 for $\varepsilon = 10^{-3}$ with the two-level variant of the LPS method gave strong oscillations in the boundary layer regions. Here, a significantly smaller value $\varepsilon = 10^{-5}$ of the singular perturbation parameter is chosen.

Example 6.2. We consider the optimization problem

$$\min J(q_\Omega, q_\Gamma, u) := \frac{1}{2}\|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Omega}{2}\|q_\Omega\|^2_{L^2(\Omega)},$$

such that

$$\begin{aligned} -\varepsilon\Delta u + (\mathbf{b}\cdot\nabla)u + \sigma u &= f + q_\Omega \quad \text{in } \Omega = (0,1)^2 \\ u &= 0 \quad\quad\quad\ \ \text{on } \partial\Omega \end{aligned}$$

with $q_\Omega \in L^2(\Omega)$ and $\varepsilon = 10^{-5}, \mathbf{b} = (-1,-2)^t, \sigma = 1, f = 1, u_\Omega = 1$ and $\alpha_\Omega = 0.1$.

Figure 2 shows the stabilized control and state for the problem. We present the discrete solution on the coarse grid for the two-level approach with Q_1-elements and

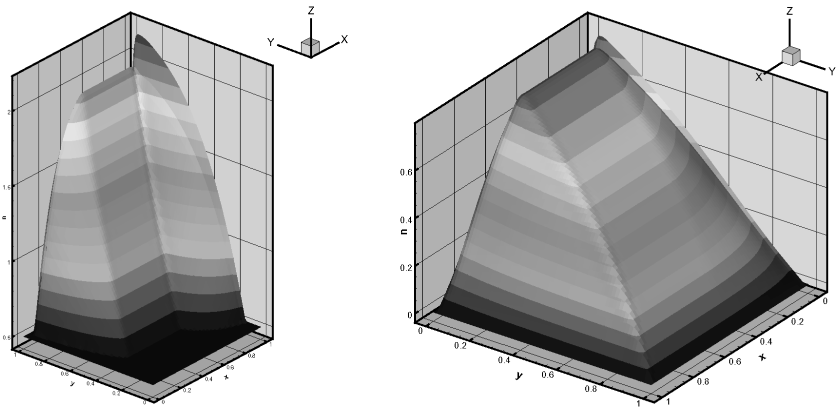


Fig. 2. Optimal discrete control and state for example with $\varepsilon = 10^{-5}$ and LPS parameters $\tau = 0.1h$.

Table 3. Example 6.2 h-convergence of the cost functional.

$h = 2^{-l}$	$J(\bar{q}_h, \bar{u}_h)$	$J(\bar{q}_h, \bar{u}_h) - J(\bar{q}_{2h}, \bar{u}_{2h})$	Num. conv. rate
2	3.08191E-01	—	—
3	2.76675E-01	3.15159E-02	—
4	2.63904E-01	1.27704E-02	1.30
5	2.60156E-01	3.74789E-03	1.77
6	2.59242E-01	9.13856E-04	2.04
7	2.59068E-01	1.74289E-04	2.39
8	2.59057E-01	1.07450E-05	4.01

$h = \frac{1}{128}$. Notice that the spurious oscillations for the discrete control and state in the boundary layer regions are significantly reduced as compared to the results given in Ref. 5.

Table 3 gives the convergence history of the cost functional J. Moreover, the numerical convergence rate is computed. The averaged rate is $r \approx 2.30$.

Finally, we present a numerical experiment for a regularized Dirichlet control according to case (i) in Sec. 5.

Example 6.3. We consider the optimization problem

$$\min J(u, q_\Omega, q_\Gamma) = \frac{\lambda_\Omega}{2} \|u - \tilde{u}_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2} \|u - \tilde{u}_\Gamma\|^2_{L^2(\Gamma_R)} + \frac{\alpha_\Omega}{2} \|q_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Gamma}{2} \|q_\Gamma\|^2_{L^2(\Gamma_R)}$$

such that

$$-\varepsilon \Delta u + (\mathbf{b} \cdot \nabla) u = q_\Omega \quad \text{in } \Omega = (0,2) \times \left(\frac{1}{2}, \frac{1}{2}\right)$$

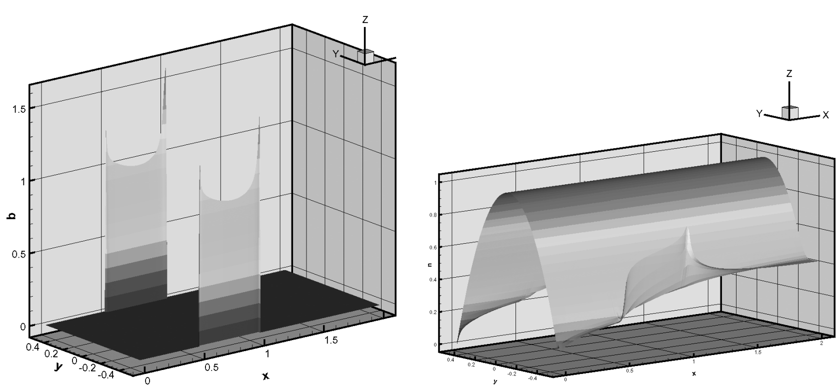
with $\varepsilon = 10^{-4}$, $\mathbf{b}(x) = ((\frac{1}{2} - |x_2|), 0)^T$ and $\tilde{u}_\Omega = 1$, $\tilde{u}_\Gamma = 0$ and $\alpha_\Omega = 10^5$, $\alpha_\Gamma = 10^{-2}$, $\lambda_\Omega = 1$ and $\lambda_\Gamma = 10^{-3}$. The boundary conditions are chosen as

$$\begin{aligned}
\varepsilon \nabla u \cdot \mathbf{n} + u &= g \quad \text{on } \Gamma_- = \{0\} \times \left(-\frac{1}{2}, \frac{1}{2}\right) \\
\varepsilon \nabla u \cdot \mathbf{n} + u &= q_\Gamma \quad \text{on } \Sigma \subset \Gamma_0 = (0, 2) \times \left\{-\frac{1}{2}, \frac{1}{2}\right\} \\
\varepsilon \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_0 \backslash \Sigma \\
\varepsilon \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_+ = \{2\} \times \left(-\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}$$

with the boundary part $\Sigma = (\frac{1}{2}, 1) \times \{-\frac{1}{2}\} \cup (\frac{1}{2}, 1) \times \{\frac{1}{2}\}$ with regularized Dirichlet control and with $g(x) = 1 - 4x_2^2$.

The two-level variant of LPS-stabilization with $\tau = 0.5h$ and mesh width $h = \frac{1}{64}$ is applied. Figure 3 shows (a) the boundary control q_Γ and (b) the state u. The interesting result is that, for this singularly perturbed problem, the influence of the boundary control is strongly restricted to the downstream boundary layer regions, i.e. the desired influence of the boundary control on the global behavior of the state u in Ω fails. The explanation is that, for this channel type flows, perturbations of the boundary data at $\Gamma_0 \cup \Gamma_+$ decay exponentially fast perpendicular to the boundary. This allows the conclusion that any kind of boundary control at characteristic and outflow boundaries is not useful here.

The situation is different for channel type flows with boundary control at the inflow part Γ_-, see case (ii) in Sec. 5. Another situation is given in the case $\sigma = 0$ and $f \equiv 0$ for cavity-type flows, where no inflow and outflow parts of the boundary exist. Then the effect of boundary control at characteristic parts Γ_0 of the boundary is relevant. We will report on results and arising problems elsewhere.



(a) (b)

Fig. 3. Robin control on part of wall as regularized Dirichlet control: (a) boundary control q_Γ and (b) state u.

7. Summary and Outlook

In this paper we considered the numerical analysis of discretized optimal control problems governed by a linear advection-diffusion-reaction equation without pointwise control constraints. The standard Galerkin discretization is stabilized via the local projection approach which leads to a symmetric optimality system at the discrete level. The optimal control problem simultaneously covers distributed and Robin boundary control. In contrast to Ref. 5, we allow the application of shape-regular, locally quasi-uniform meshes.

In the singularly perturbed case, the boundary control at certain parts of the boundary can formally be seen as regularization of a Dirichlet boundary control. Our first results show that boundary control at characteristic parts of the boundary has only influence on the boundary layer region. Further investigation will be concerned with boundary control at inflow boundaries and on boundary control for cavity-type flows (without in- and outflow). Moreover, it seems to be interesting to consider the less academic situation of coupled flow problems, e.g. with thermal coupling, as boundary control (e.g. of temperature) may influence the flow field.

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