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A PROJECTION-BASED VARIATIONAL MULTISCALE METHOD FOR LARGE-EDDY SIMULATION WITH APPLICATION TO NON-ISOTHERMAL FREE CONVECTION PROBLEMS

JOHANNES LÖWE

Departm. of Mathematics and Computer Science, NAM, University of Göttingen,
Lotzestrasse 17-18, D-37083, Göttingen, Germany
loewe@math.uni-goettingen.de

GERT LUBE

Departm. of Mathematics and Computer Science, NAM, University of Göttingen,
Lotzestrasse 17-18, D-37083, Göttingen, Germany
lube@math.uni-goettingen.de

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We consider a projection-based variational multiscale method for large-eddy simulation of the Navier-Stokes/Fourier model of incompressible, non-isothermal flows. For the semidiscrete problem, an a priori error estimate is given for rather general nonlinear, piecewise constant coefficients of the subgrid models for the unresolved scales of velocity, pressure, and temperature. Then we address aspects of the discretization in time. Finally, the design of the subgrid scale models is specified for the case of free convection problems and studied for the standard benchmark problem of free convection in a closed cavity.

Keywords: Navier-Stokes/Fourier model. Large-Eddy simulation, Variational multiscale problem. Free convection problem.

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1. Introduction

Incompressible non-isothermal viscous flows of a Newtonian fluid can be modeled by the Navier-Stokes/ Fourier equations which read: Given a bounded domain $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$, the simulation time T, and force fields $\mathbf{f} : (0,T]\times\Omega \to \mathbb{R}^3$ and $Q : (0,T] \times \Omega \to \mathbb{R}^3$, find a velocity field $\mathbf{u} : (0,T] \times \Omega \to \mathbb{R}^3$, a pressure field

$p : (0, T] \times \Omega \to \mathbb{R}$, and a temperature field $\theta : (0, T] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \mathbf{u} - 2\nu \nabla \cdot \mathbb{D}\mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \nabla p + \alpha \mathbf{g}\theta &= \mathbf{f} + \alpha \mathbf{g}\theta_0 && \text{in } (0, T] \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } [0, T] \times \Omega, \\ \partial_t \theta - \varkappa \Delta \theta + \mathbf{u} \cdot \nabla \theta &= Q && \text{in } (0, T] \times \Omega, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 && \text{in } \Omega, \\ \theta|_{t=0} &= \theta^0 && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity coefficient, $\varkappa > 0$ is the thermal diffusivity coefficient, and $\mathbb{D}\mathbf{u} := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ denotes the velocity deformation tensor. In system (1.1), the Boussinesq approximation is applied with the thermal expansion coefficient α, the gravitational acceleration vector $\mathbf{g}$, and the reference temperature θ_0. (Throughout this paper, we incorporate, for simplicity, the term $\alpha \mathbf{g}\theta_0$ into the pressure gradient.) Some boundary conditions have to be imposed on $\partial\Omega$ to obtain a closed set of equations. In the analysis below, we impose homogeneous Dirichlet conditions for velocity and temperature for simplicity, but see Remark 2.1. A global-in-time existence result for a more general Navier-Stokes/ Fourier model can be found in [10], Theorem 3.1.

Relevant dimensionless numbers are the Reynolds number $Re := UL/\nu$, the Peclet number $Pe := UL/\varkappa$, the Prandtl number $Pr := \nu/\varkappa$, and the Rayleigh number $Ra := \alpha|\mathbf{g}|L^3\delta\theta/(\nu\varkappa)$ with given characteristic length L, velocity scale U and a characteristic temperature difference $\delta\theta$. In many industrial applications, simulations of turbulent flows are of major interest. In mixed convection problems, such flows are typically characterized by large Rayleigh numbers.

The finite-element (FE) method is one of the most popular and mathematically sound variants of numerical approximation. The standard Galerkin method aims to simulate all persistent scales, but this is not feasible even in next futures for very large numbers of Ra. Standard residual-based stabilization techniques, like the streamline-upwind (or SUPG) method and/or the pressure stabilization (or PSPG) technique, add numerical viscosity acting at all scales (see [37] for a representative overview). The classical way of large-eddy simulation (LES) to simulate only the behavior of large scales accurately has several drawbacks like commutation errors and the open question of appropriate boundary conditions for the large scales. For a critical review, we refer to [5].

Alternatively and following ideas in [17,14,18], the idea of variational multiscale (VMS) methods is to define the large scales by projections into appropriate function spaces. Based on a three-scale decomposition of the flow field into large, resolved small, and unresolved scales, the influence of the unresolved small scales is described by a subgrid model acting directly only on the resolved small scales. A series of numerical studies reports good experience with VMS methods for standard benchmark problems. Meanwhile, different variants of VMS methods have been considered. For a review and comparison of different variants for the incompressible and isothermal case see [13,22].

The numerical analysis of VMS methods for turbulent flows is still in its infancy. Let us comment on the isothermal case first. The case of equal-order interpolation has been thoroughly considered by R. Codina and his co-workers, see, e.g., [32]. If inf-sup stable FE pairs

are applied (as in the present paper), the analysis differs in some aspects from the equal-order case. Here we refer to the contributions by V. John and his co-workers to projection-based variants of the VMS method with inf-sup stable FE pairs. A globally constant turbulent viscosity ν_T together with an elliptic projection for the definition of the large scales is considered in [19] and analyzed in [20], but the approach leads to some open problems of the elliptic projection. As a remedy, an L^2-projection together with a Smagorinsky-type subgrid model is analyzed in [21]. The subgrid modelling of the unresolved pressure scales based on grad-div stabilization is discussed in [31]. Moreover, we refer to the paper [36] where a modified projection-based FE VMS method is discussed which had been presented in [27,19]. Here, the subgrid model for the unresolved velocity scales is based on the L^2-projection Π_H for the definition of the large scales of the velocity deformation tensor. Contrary to the approach in [21], the so-called fluctuation operator $I - \Pi_H$ is applied to the velocity deformation tensor whereas the velocity deformation tensor is applied to the fluctuation operator in [21] first. These operators do not commute in the general case [30].
Unfortunately, there are only very few papers on the numerical analysis of thermally coupled flows. Early papers are on the stationary case in [6,4] and for the time-dependent cases in [7]. An extension of VMS methods to non-isothermal flows based on equal-order interpolation of all unknowns has been considered in [9]. To the best of our knowledge, there are no papers dealing with the numerical analysis of thermally coupled turbulent flows.
In the present paper, we extend the analysis of our former paper [36] to thermally coupled incompressible flows. We derive an a priori error estimate for the spatially semidiscretized problem where the definition of the piecewise subgrid models for the unresolved velocity, pressure, and temperature scales remains rather general. Then we address aspects of the time discretization of the semidiscrete problem. For the case of free convection flow, we then specify the velocity subgrid model. Finally, the parametrization of the three subgrid models is checked for the case of free convection in a closed cavity.
The paper is organized as follows: In Section 2, we introduce the projection-based VMS method under consideration. Then, in Section 3, we provide the error analysis for the model after spatial semidiscretization based on inf-sup stable finite element pairs for velocity and pressure. In Section 4 we discuss aspects of the time discretization. Section 5 provides a specification of the subgrid models for the case of free convection flows. Then, Section 6 is devoted to the application of the approach to the standard benchmark of natural convection in a closed cavity. Finally, we summarize the results in Section 7 and give some conclusions.

2. A modified projection-based finite-element variational multiscale method

2.1. *Preliminaries*

For a bounded domain $\Omega \subset \mathbb{R}^3$, we apply standard notations for Lebesgues spaces $L^p(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$, together with the corresponding norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$ with $m \in \mathbb{N}, 1 \le p \le \infty$. The inner product in $[L^2(\Omega)]^3$ will be denoted by $(\cdot,\cdot)$. A similar notation will be used on subdomains $D \subseteq \Omega$. For clarity we write $\|\cdot\|_0$ for the L^2 norm $\|\cdot\|_{L^2(\Omega)}$ of the whole domain Ω.

For a normed space X with functions defined on Ω, let $L^p(0,t;X)$ be the space of all functions defined on $(0,t)\times X$ with

$$\|\mathbf{u}\|_{L^p(0,t;X)} := \Big(\int_0^t \|\mathbf{u}\|_X^p \, ds\Big)^{1/p} < \infty, \quad 1 \le p < \infty$$

and with the obvious modification for $p=\infty$.
Setting

$$V = [H_0^1(\Omega)]^3, \qquad Q = L^2_*(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}, \qquad \Psi := H_0^1(\Omega),$$

we consider the variational formulation of the Navier-Stokes/ Fourier model: find $\mathbf{u}\colon [0,T] \to V$, $p\colon (0,T] \to Q$, and $\theta\colon [0,T]\to\Psi$ satisfying

$$\begin{aligned}(\partial_t \mathbf{u}, \mathbf{v}) + (2\nu \mathbb{D}\mathbf{u}, \mathbb{D}\mathbf{v}) + b_S(\mathbf{u},\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + \alpha(\mathbf{g}\theta, \mathbf{v}) &= (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v}\in V,\\ (q, \nabla\cdot\mathbf{u}) &= \ 0 \qquad \forall q \in Q,\\ (\partial_t\theta, \psi) + (\varkappa\nabla\theta, \nabla\psi) + c_S(\mathbf{u},\theta,\psi) &= (Q,\psi) \quad \forall\psi\in\Psi.\end{aligned} \tag{2.1}$$

Here, the skew-symmetric trilinear forms

$$\begin{aligned} b_S(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \frac{1}{2}[((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})]\\ c_S(\mathbf{u},\theta,\psi) &:= \frac{1}{2}\big[(\mathbf{u}\cdot\nabla\theta,\psi) - (\mathbf{u}\cdot\nabla\psi,\theta)\big]\end{aligned}$$

have the important properties $b_S(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$ for all $\mathbf{u},\mathbf{v}\in V$ and $c_S(\mathbf{u},\psi,\psi)=0$ for all $(\mathbf{u},\psi)\in V\times\Psi$.
For the present analysis, we will use Korn's inequality with constant C_{Ko} and the Poincaré-Friedrichs inequality with constant C_F such that

$$\|\nabla\mathbf{v}\|_0 \le C_{\text{Ko}}\|\mathbb{D}\mathbf{v}\|_0 \quad \text{and} \quad \|\mathbf{v}\|_0 \le C_F\|\nabla\mathbf{v}\|_0 \qquad \forall\mathbf{v}\in V. \tag{2.2}$$

Remark 2.1. *The analysis of this paper can be applied in the case of periodic boundary conditions for the velocity as well. The proof for Korn's inequality under such conditions is very similar to the case of no-slip boundary conditions. It is possible to extend the analysis to cases where no-slip boundary conditions and periodic boundary conditions appear simultaneously, e.g. in channel flows.*

2.2. *Variational multiscale method*

Let $\mathcal{T}_h$ be an admissible triangulation of domain Ω following [11], with maximal diameter $h>0$ of the mesh cells $K\in\mathcal{T}_h$. The FE spaces $V_h\times Q_h\subset V\times Q$ of the basic Galerkin FE method for the Navier-Stokes model will be standard inf-sup stable velocity-pressure spaces, i.e. with

$$\inf_{q_h\in Q_h\setminus\{0\}} \sup_{\mathbf{v}_h\in V_h\setminus\{0\}} \frac{(q_h, \nabla\cdot\mathbf{v_h})}{\|q_h\|_0\|\nabla\mathbf{v}_h\|_0} \ge \beta > 0 \tag{2.3}$$

where β is h-independent. Moreover, let $\Psi_h \subset \Psi$ be a standard FE space for the Fourier model. Then the Galerkin FE method reads: find $\mathbf{u}_h\colon [0,T] \to V_h$, $p_h\colon (0,T] \to Q_h$, and $\theta_h\colon (0,T] \to \Psi_h$ such that

$$\begin{aligned} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + (2\nu \mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) & \\ + \alpha(\mathbf{g}\theta_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in V_h, \\ (q_h, \nabla \cdot \mathbf{u}_h) &= 0 \qquad \forall q_h \in Q_h, \\ (\partial_t \theta_h, \psi_h) + (\varkappa \nabla \theta_h, \nabla \psi_h) + c_S(\mathbf{u}_h, \theta_h, \psi_h) &= (Q, \psi_h) \qquad \forall \psi_h \in \Psi_h. \end{aligned}$$

For turbulent flows, let a three-scale decomposition of the flow and pressure fields be given

$$\mathbf{v} = \overline{\mathbf{v}}_h + \tilde{\mathbf{v}}_h + \hat{\mathbf{v}} \quad \forall \mathbf{v} \in V; \quad q = \overline{q}_h + \tilde{q}_h + \hat{q} \quad \forall q \in Q, \quad \psi = \overline{\psi}_h + \tilde{\psi}_h + \hat{\psi} \quad \forall \psi \in \Psi.$$

We search for the resolved scales

$$(\mathbf{v}_h, q_h, \psi_h) := (\overline{\mathbf{v}}_h + \tilde{\mathbf{v}}_h, \overline{q}_h + \tilde{q}_h, \overline{\psi}_h + \tilde{\psi}_h) \in V_h \times Q_h \times \Psi_h \subset V \times Q \times \Psi.$$

Following a variant of the VMS approach of [27], Section 3, we model the influence of the unresolved small scales on the resolved small scales starting from the following notation: Let $\mathcal{T}_H$ be the triangulation of a coarser grid, i.e. $H \geq h$. Throughout this paper, $\mathcal{T}_h$ is a conforming refinement of $\mathcal{T}_H$.

We start with the subgrid model for the small velocity scales. The FE space L_H of coarse scales of the deformation tensor is given by

$$\{0\} \subseteq L_H \subseteq \mathbb{D}V_h \subseteq L := \left\{ \mathbf{L} = (l_{ij}) \in [L^2(\Omega)]^{3\times 3} \mid l_{ij} = l_{ji} \right\}.$$

Let $\Pi_H^u\colon L \to L_H \subset L$ be the L^2-orthogonal projection. The *fluctuation operator* is

$$\kappa_u \colon = Id - \Pi_H^u.$$

The operator κ_u takes the resolved small-scale fluctuations of the deformation. Following our approach to the isothermal case in [36], we assume a cellwise constant turbulent viscosity coefficient $\nu_T(\mathbf{u}_h, \theta_h)$ with $\nu_T(\mathbf{u}_h, \theta_h)|_K =: \nu_T^K(\mathbf{u}_h, \theta_h)$ per cell $K \subset \Omega$ and introduce as symmetric subgrid viscosity term for the velocities

$$\begin{aligned} (\nu_T(\mathbf{u}_h, \theta_h)\kappa_u(\mathbb{D}\mathbf{u}_h), \mathbb{D}\mathbf{v}_h) &= \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h, \theta_h)(\kappa_u(\mathbb{D}\mathbf{u}_h), \mathbb{D}\mathbf{v}_h)_K \\ &= \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h, \theta_h)(\kappa_u(\mathbb{D}\mathbf{u}_h), \kappa_u(\mathbb{D}\mathbf{v}_h))_K \\ &= (\nu_T(\mathbf{u}_h, \theta_h)\kappa_u(\mathbb{D}\mathbf{u}_h), \kappa_u(\mathbb{D}\mathbf{v}_h)). \end{aligned} \tag{2.4}$$

This is, in the context of stabilization techniques based on local projection, strongly related to the so-called *gradient-based* local projection stabilization [25]. In the first part of the paper, we consider a rather general subgrid model for $\nu_T(\mathbf{u}_h, \theta_h)$. Later on, we consider a Smagorinsky/Eidson-type model. Another possibility is Vreman's subgrid model [39].

In a second step, we incorporate a pressure subgrid scale model using the so-called grad-div stabilization:

$$(\gamma(\mathbf{u}_h, p_h)(\nabla \cdot \mathbf{u}_h), \nabla \cdot \mathbf{v}_h) := \sum_{K \in \mathcal{T}_h} \gamma_K(\mathbf{u}_h, p_h)(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_K$$

where $\gamma(\mathbf{u}_h, \theta_h)$ denotes a non-negative user-chosen parameter function being cellwise constant on $K \in \mathcal{T}_h$. (For simplicity, we introduce no coarse grid space for the pressure.)
Finally, to give a subgrid model for the small temperature scales, we introduce a FE space M_H of coarse scales of the temperature gradient such that

$$\{0\} \subseteq M_H \subseteq \nabla\Psi_h \subseteq M := \left\{\mathbf{M} = (m_i) \in [L^2(\Omega)]^3\right\}.$$

Let $\Pi_H^\theta \colon L \to M_H \subset M$ be the L^2-orthogonal projection. The fluctuation operator is

$$\kappa_\theta \colon = Id - \Pi_H^\theta$$

which takes the resolved small-scale fluctuations of the temperature gradient. We assume a cellwise constant turbulent viscosity coefficient $\varkappa_T(\mathbf{u}_h, \theta_h)$ with $\varkappa_T(\mathbf{u}_h, \theta_h)|_K =: \varkappa_T^K(\mathbf{u}_h, \theta_h)$ on each cell $K \subset \Omega$ and introduce the symmetric subgrid viscosity term for the temperature

$$\begin{aligned}(\varkappa_T(\mathbf{u}_h, \theta_h)\kappa_\theta(\nabla\theta_h), \nabla\psi_h) &= \sum_{K\in\mathcal{T}_h} \varkappa_T^K(\mathbf{u}_h, \theta_h)(\kappa_\theta(\nabla\theta_h), \nabla\psi_h)_K \\ &= (\varkappa_T(\mathbf{u}_h, \theta_h)\kappa_\theta(\nabla\theta_h), \kappa_\theta(\nabla\psi_h)).\end{aligned} \tag{2.5}$$

Now the modified VMS method reads: find $\mathbf{u}_h \colon (0, T] \to V_h$, $p_h \colon (0, T] \to Q_h$, $\theta_h \colon (0, T] \to \Psi_h$ such that

$$\begin{aligned}(\partial_t\mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (\nabla\cdot\mathbf{v}_h, p_h) + \alpha(\mathbf{g}\theta_h, \mathbf{v}_h)& \\ +(\gamma(\mathbf{u}_h, p_h)(\nabla\cdot\mathbf{u}_h), \nabla\cdot\mathbf{v}_h) + (\nu_T(\mathbf{u}_h, \theta_h)\kappa_u(\mathbb{D}\mathbf{u}_h), \kappa_u(\mathbb{D}\mathbf{v}_h)) &= (\mathbf{f}, \mathbf{v}_h) \\ (q_h, \nabla\cdot\mathbf{u}_h) &= 0 \\ (\partial_t\theta_h, \psi_h) + (\varkappa\nabla\theta_h, \nabla\psi_h) + c_S(\mathbf{u}_h, \theta_h, \psi_h)& \\ +(\varkappa_T(\mathbf{u}_h, \theta_h)\kappa_\theta(\nabla\theta_h), \kappa_\theta(\nabla\psi_h)) &= (Q, \psi_h)\end{aligned} \tag{2.6}$$

for all $(\mathbf{v}_h, q_h, \psi_h) \in V_h \times Q_h \times \Psi_h$. A possible choice of the spaces L_H and M_h with $H = h$ will be discussed in Section 5. In particular, an adaptive choice of L_H following [22] is possible.

3. A Priori Error Analysis of a General VMS Model

In this section, we will consider the a priori analysis of the VMS-scheme (2.6) where the (nonlinear) coefficients of the subgrid models are not specified yet. Thus we extend our approach in [36] from the isothermal case to the coupled Navier-Stokes/Fourier model. To keep the analysis as short as possible, we cite technical results from [36].
In the subsequent analysis, we take advantage of the space of discretly divergence-free functions

$$V_{h,\text{div}} \colon = \{\mathbf{v}_h \in V_h \mid (\nabla\cdot\mathbf{v}_h, q_h) = 0\ \forall q_h \in Q_h\}.$$

The space $V_{h,\text{div}}$ is not empty since we will use only FE spaces V_h and $\mathbb{Q}_h$ which fulfill the discrete Ladyžhenskaya-Babuška-Brezzi condition (2.3). Then problem (2.6) is equivalent to: find $\mathbf{u}_h \colon [0,T] \longrightarrow V_{h,\text{div}}$ and $\theta_h \colon (0,T] \to \Psi_h$ satisfying

$$
\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \alpha(\mathbf{g}\theta_h, \mathbf{v}_h)& \\
+(\gamma(\mathbf{u}_h, p_h)(\nabla\cdot\mathbf{u}_h), \nabla\cdot\mathbf{v}_h) + (\nu_T(\mathbf{u}_h,\theta_h)\kappa_u(\mathbb{D}\mathbf{u}_h), \kappa_u(\mathbb{D}\mathbf{v}_h)) &= (\mathbf{f}, \mathbf{v}_h) \\
(\partial_t \theta_h, \psi_h) + (\varkappa\nabla\theta_h, \nabla\psi_h) + c_S(\mathbf{u}_h, \theta_h, \psi_h)& \\
+(\varkappa_T(\mathbf{u}_h,\theta_h)\kappa_\theta(\nabla\theta_h), \kappa_\theta(\nabla\psi_h)) &= (Q, \psi_h)
\end{aligned}
\tag{3.1}
$$

for all $(\mathbf{v}_h, \psi_h) \in V_{h,\text{div}} \times \Psi_h$.

3.1. *Stability*

We derive a semidiscrete a priori error estimate for the problem (3.1). Therefore one has to prove the stability of the continuous and the discrete solutions $\mathbf{u}$ and $\mathbf{u}_h$. It turns out that the coupling between the Navier-Stokes model and the Fourier model is rather weak.

Lemma 3.1. *Let $\mathbf{u}_h$ be the solution of* (3.1)*. Assume $\mathbf{f} \in [L^1(0,T;L^2(\Omega))]^3$, $Q \in L^1(0,T;L^2(\Omega))$, $\mathbf{u}_0 \in [L^2(\Omega)]^3$, and $\theta_0 \in L^2(\Omega)$. Then we obtain $\mathbf{u}_h \in [L^\infty(0,T;L^2(\Omega))]^3$, $\mathbb{D}\mathbf{u}_h \in [L^2(0,T;L^2(\Omega))]^{3\times3}$, $\theta_h \in L^\infty(0,T;L^2(\Omega)$, and $\nabla\theta_h \in [L^2(0,T;L^2(\Omega)]^3$. For all $t \in (0,T]$, there is control of thermal and kinetic energies*

$$
\begin{aligned}
\|\theta_h\|_{L^\infty(0,t;L^2(\Omega))} &\le K_1(Q,\theta_0) \qquad := \|\theta_0\|_0 + \|Q\|_{L^1(0,t;L^2(\Omega))} \\
\|\mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega))} &\le K_2(\mathbf{f},\mathbf{u}_0,Q,\theta_0) := \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))} + C\alpha\|\mathbf{g}\|_\infty K_1(Q,\theta_0),
\end{aligned}
$$

and control of dissipation and subgrid terms

$$
\varkappa\|\nabla\theta_h\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \sum_{K\in\mathcal{T}_h} \varkappa_T^K(\mathbf{u}_h,\theta_h)\|\kappa_\theta(\nabla\theta_h)\|^2_{L^2(K)} \le \frac{3}{2}K_1^2(Q,\theta_0),
$$

$$
\begin{aligned}
\nu\,\|\mathbb{D}\mathbf{u}_h\|^2_{L^2(0,t;L^2(\Omega))} &+ \frac{1}{2}\int_0^t \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u(\mathbb{D}\mathbf{u}_h)\|^2_{L^2(K)}\,dt \\
&+\frac{1}{2}\int_0^t \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h,p_h)\,\|\nabla\cdot\mathbf{u}_h\|^2_{L^2(K)}\,dt \le 3K_2^2(\mathbf{f},\mathbf{u}_0,Q,\theta_0).
\end{aligned}
$$

Proof. See Appendix A.

Remark 3.2. *One can prove a similar stability result for the solution $(\mathbf{u},\theta)$ of the continuous problem* (2.1)*. For this purpose it is easy to adapt the proof in* [26].

3.2. *A Priori Error Estimate*

In a next step, we prove an a priori error estimate for the semidiscretized problem (3.1). To this goal we have to take into account the regularity of the continuous and discrete solutions. For the proof of Theorem 3.4, we need that for the continuous solution $(\mathbf{u},\theta)$ of (2.1) and the discrete solution $(\mathbf{u}_h,\theta_h)$ of (3.1)

$$
\begin{aligned}
\partial_t\theta \in L^2(0,t;H^{-1}(\Omega)), \quad \partial_t\mathbf{u}, \partial_t\mathbf{u}_h \in L^2(0,t;H^{-1}(\Omega)),& \\
\nabla\theta \in L^4(0,t;L^2(\Omega)), \quad \nabla\mathbf{u} \in L^4(0,t;L^2(\Omega))&
\end{aligned}
\tag{3.2}
$$

holds true. Suppose $(\mathbf{u}_i, \theta_i) \in V_{div} \times \Psi, i = 1, 2$ with $V_{div} := \{\mathbf{v} \in V : (q, \nabla \cdot \mathbf{v}) = 0, \forall q \in Q\}$ are solutions of the continuous problem (2.1). Then a straightforward application of the Gronwall Lemma to the error equations for $\phi := \mathbf{u}_1 - \mathbf{u}_2$ ad $\chi := \theta_1 - \theta_2$ provides $\phi = 0, \chi = 0$ and therefore uniqueness of the solution $(\mathbf{u}, \theta) \in V_{div} \times \Psi$.
For better readability, we introduce some notation where we will omit the dependence of the parameters ν_T^K, $\varkappa_T^K$, and γ_K on $\mathbf{u}_h, \theta_h, p_h$. Remind that $\Pi_H^u : L \to L_H \subset L$ and $\Pi_H^\theta : M \to M_H \subset M$ are the L^2-orthogonal projections of Subsection 2.2. Using (2.4) and (2.5), we introduce the elementwise *multiscale viscosities*

$$\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h) \left\|\kappa_u \mathbb{D}\mathbf{v}_h\right\|^2_{L^2(K)} = \sum_{K\in\mathcal{T}_h} \underbrace{\nu_T^K(\mathbf{u}_h,\theta_h)\left(1 - \frac{\left\|\Pi_H^u \mathbb{D}\mathbf{v}_h\right\|^2_{L^2(K)}}{\left\|\mathbb{D}\mathbf{v}_h\right\|^2_{L^2(K)}}\right)}_{=:\nu^K_{\mathrm{VMS}}(\mathbf{v}_h)\geq 0} \left\|\mathbb{D}\mathbf{v}_h\right\|^2_{L^2(K)},$$

$$\sum_{K\in\mathcal{T}_h} \varkappa_T^K(\mathbf{u}_h,\theta_h) \left\|\kappa_\theta \nabla\psi_h\right\|^2_{L^2(K)} = \sum_{K\in\mathcal{T}_h} \underbrace{\varkappa_T^K(\mathbf{u}_h,\theta_h)\left(1 - \frac{\left\|\Pi_H^\theta \nabla\psi_h\right\|^2_{L^2(K)}}{\left\|\nabla\psi_h\right\|^2_{L^2(K)}}\right)}_{=:\varkappa^K_{\mathrm{VMS}}(\psi_h)\geq 0} \left\|\nabla\psi_h\right\|^2_{L^2(K)}$$

where we take advantage of the projector properties of the fluctuation operators. Then we define the modified elementwise viscosities

$$\nu^K_{\mathrm{mod}}(\mathbf{v}_h) := 2\nu + \nu^K_{\mathrm{VMS}}(\mathbf{v}_h), \quad \varkappa^K_{\mathrm{mod}}(\psi_h) := \varkappa + \varkappa^K_{\mathrm{VMS}}(\psi_h)$$

which contain the sums of the model and subgrid viscosities.

Remark 3.3. *The subgrid models only formally act on the small resolved scales, but, due to the properties of the L^2-projections Π_H^u and Π_H^θ, the modified viscosities measure the influence on all resolved scales. In case of $\|\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)} = 0$, we set $\nu^K_{\mathrm{VMS}}(\mathbf{v}_h) = 0$. For this reason one could demand $\nu_T^K(\mathbf{u}_h,\theta_h) = 0$ if $\|\mathbb{D}\mathbf{u}_h\|^2_{L^2(K)} = 0$ as in Smagorinsky-type subgrid models. A similar argument is valid for the temperature-dependent subgrid viscosity term.*

In the analysis below, we will apply mesh-dependent norms according to

$$|||\mathbf{u}(t)|||^2 := \|\mathbf{u}(t)\|_0^2 + \sum_{K\in\mathcal{T}_h} \int_0^t \left(\frac{\nu^K_{\mathrm{mod}}(\mathbf{u})}{2} \|\mathbb{D}\mathbf{u}\|^2_{L^2(K)} + \gamma_K \|\nabla\cdot\mathbf{u}\|^2_{L^2(K)} \right) dt, \quad (3.3)$$

$$|[\theta(t)]|^2 := \|\theta(t)\|_0^2 + \sum_{K\in\mathcal{T}_h} \int_0^t \frac{1}{2} \varkappa^K_{\mathrm{mod}}(\theta) \|\nabla\theta\|^2_{L^2(K)}\, dt. \quad (3.4)$$

Theorem 3.4. *Let $(\mathbf{u},\theta)$ and $(\mathbf{u}_h,\theta_h)$ be the solutions of* (2.1) *and of* (3.1)*, respectively. Let $\mathbf{f} \in [L^1(0,T;L^2(\Omega))]^3$, $\mathbf{u}_0 \in [L^2(\Omega)]^3$, let $\mathbf{I}_h$, J_h, and I_h be interpolation operators onto $L^4(0,T;V_{h,div})$, $L^2(0,T;Q_h)$, and $L^2(0,T;\Psi_h)$, respectively. Suppose that the* *regularity assumption* (3.2) *is true and let*

$$\sum_{K\in\mathcal{T}_h} \nu_{\mathrm{VMS}}^K(\mathbf{u}-\mathbf{I}_h\mathbf{u})\,\|\mathbb{D}(\mathbf{u}-\mathbf{I}_h\mathbf{u})\|_{L^2(K)}^2 \in L^1(0,T), \tag{3.5}$$

$$\sum_{K\in\mathcal{T}_h} \nu_{\mathrm{VMS}}^K(\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|_{L^2(K)}^2 \in L^1(0,T); \tag{3.6}$$

then

$$|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 + |[(\theta-\theta_h)(t)]|^2 \le 2 \inf_{\substack{\tilde{\mathbf{u}}_h \,\in\, L^2(0,\,t;\,V_h^{div}) \\ \tilde{\theta}_h \,\in\, L^2(0,\,t;\,\Psi_h)}} |||(\mathbf{u}-\tilde{\mathbf{u}}_h)(t)|||^2 + |[(\theta-\tilde{\theta}_h)(t)]|^2 \tag{3.7}$$

$$+e^{\int_0^t g_3(s)ds} \inf_{\substack{\tilde{\mathbf{u}}_h \,\in\, L^4(0,\,t;\,V_h^{div}) \\ \tilde{p}_h \,\in\, L^2(0,\,t;\,Q_h) \\ \tilde{\theta}_h \,\in\, L^2(0,\,t;\,\Psi_h)}} \left(\|(\mathbf{u}_h-\tilde{\mathbf{u}}_h)(0)\|_0^2 + \|(\theta_h-\tilde{\theta}_h)(0)\|_0^2 + \int_0^t g_2(s)ds \right)$$

with

$$\begin{aligned} g_3(t) :=& \frac{27C_{LT}^4}{2\inf_{t\in(0,T)}\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h-\tilde{\mathbf{u}}_h)^3}\,\|\mathbb{D}\mathbf{u}\|_0^4 + 2\alpha\,\|\mathbf{g}\|_\infty \\ &+ \frac{8C_1^4}{\inf_{t\in(0,T)}\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h-\tilde{\mathbf{u}}_h)\ \varkappa_{\mathrm{mod}}^{\min}(\theta_h-\tilde{\theta}_h)^2}\,\|\nabla\theta\|_0^4, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} g_2(t) :=& 2\sum_{K\in\mathcal{T}_h}\Bigg[\min\left(\frac{9C_{Ko}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h-\tilde{\mathbf{u}}_h)},\frac{1}{\gamma_K}\right)\left(\|p-\tilde{p}_h\|_{L^2(K)}^2 + \gamma_K^2\,\|\nabla\cdot(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_{L^2(K)}^2\right) \\ &+6\Big(\nu+\nu_{\mathrm{VMS}}^K(\mathbf{u}-\tilde{\mathbf{u}}_h)\Big)\,\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_{L^2(K)}^2 + \Big(2\varkappa+4\varkappa_{\mathrm{VMS}}^K(\theta-\tilde{\theta}_h)\Big)\,\Big\|\nabla(\theta-\tilde{\theta}_h)\Big\|_{L^2(K)}^2 \\ &+6\nu_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u(\mathbb{D}\mathbf{u})\|_{L^2(K)}^2 + 4\varkappa_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_\theta(\nabla\theta)\|_{L^2(K)}^2\Bigg] + \alpha|\mathbf{g}|_\infty\,\Big\|\theta-\tilde{\theta}_h\Big\|_0^2 \\ &+\frac{6C_{Ko}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h-\tilde{\mathbf{u}}_h)}\,\|\partial_t(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_{-1,\Omega}^2 + \frac{4}{\varkappa_{\mathrm{mod}}^{\min}(\theta-\tilde{\theta}_h)}\,\Big\|\partial_t(\theta-\tilde{\theta}_h)\Big\|_{-1,\Omega}^2 \\ &+\frac{6C_{LT}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h-\tilde{\mathbf{u}}_h)}\Big(C_FC_{Ko}\,\|\mathbb{D}\mathbf{u}\|_0^2 + \|\mathbf{u}_h\|_0\,\|\mathbb{D}\mathbf{u}_h\|_0\Big)\,\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_0^2 \\ &+\frac{4C_1^2C_{Ko}}{\varkappa_{\mathrm{mod}}^{\min}(\theta_h-\tilde{\theta}_h)}\left(C_FC_{Ko}\,\|\nabla\theta\|_0^2\,\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}_h)\|_0^2 + \|\mathbf{u}_h\|_0\,\|\mathbb{D}\mathbf{u}_h\|_0\,\Big\|\nabla(\theta-\tilde{\theta}_h)\Big\|_0^2\right) \end{aligned} \tag{3.9}$$

where

$$\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h-\tilde{\mathbf{u}}_h) := \min_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h-\tilde{\mathbf{u}}_h), \qquad \varkappa_{\mathrm{mod}}^{\min}(\theta_h-\tilde{\theta}_h) := \min_{K\in\mathcal{T}_h}\varkappa_{\mathrm{mod}}^K(\theta_h-\tilde{\theta}_h).$$

C_F and C_{Ko} are the constants of the inequalities of Friedrichs and Korn. C_{LT} and C_1 are related to upper bounds of the advective terms.

Proof. See Appendix B.

Finally we are interested in an L^2-error estimate for the pressure.

Corollary 3.5. *Let* $(\mathbf{u},p) \in V \times Q$ *and* $(\mathbf{u}_h,p_h) \in V_h \times Q_h$ *be solutions of* (2.1) *and* (2.6)*, respectively; then*

$$\|p-p_h\|^2_{L^2(0,t;L^2(\Omega))} \leq 2\Big(1+\frac{\sqrt{3}}{\beta}\Big)^2 \|p-J_hp\|^2_{L^2(0,t;L^2(\Omega))} + C_u \,|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2$$
$$+\frac{14}{\beta^2}\|\partial_t(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(0,t;H^{-1}(\Omega))} + 2C_F^2\alpha^2\|\mathbf{g}\|^2_\infty \|\theta-\theta_h\|^2_{L^2(0,t;L^2(\Omega))}$$
$$+\frac{14}{\beta^2}\int_0^t \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)^2 \|\kappa_u \mathbb{D}\mathbf{u}\|^2_{L^2(K)}\,dt, \quad (3.10)$$

where

$$C_u = \frac{14}{\beta^2}\Bigg(4\nu + \max_{\tilde{t}\in(0,t),K\in\mathcal{T}_h} 3\gamma_K(\mathbf{u}_h) + \max_{\tilde{t}\in(0,t),K\in\mathcal{T}_h} 2\nu_T^K(\mathbf{u}_h,\theta_h)$$
$$+\max_{\tilde{t}\in(0,t)} \frac{2C_{LT2}^2\left(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0\right)^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}-\mathbf{u}_h)}\Bigg). \quad (3.11)$$

Proof. See Appendix C.

Remark 3.6. *The control of the* L^2*-error of pressure in Corollary 3.5 is not complete so far as only the error terms* $|||(\mathbf{u}-\mathbf{u}_h)(t)|||$ *and* $\|\theta-\theta_h\|_{L^2(0,t;L^2(\Omega))}$ *are bounded by* (3.7)*. For laminar flows, i.e. without any subgrid model, an estimate of the term* $\|\partial_t(\mathbf{u}-\mathbf{u}_h)\|_{L^2(0,t;H^{-1}(\Omega))}$ *can be found in* [15]*, Section 7. The possibility to modify the analysis of* [15] *for a projection-based VMS method with globally constant* ν_T *is mentioned in* [20]*, Remark 4.3. In the case of piecewise constant, but possibly nonlinear subgrid coefficients* ν_T*, the corresponding analysis remains open.*

3.3. *Discussion of the result*

Let us briefly discuss the quasi-optimal a-priori estimates of Theorem 3.4 and of Corollary 3.5. First we consider the case of *isotropic* meshes $\mathcal{T}_h$: All right hand side terms in (3.8)-(3.9) depend on the *approximation* properties of a standard (quasi)-interpolation operator in the discrete pressure space Q_h, of the fluctuation operators $\kappa_{u/\theta} = \mathrm{I}d - \Pi_h^{u/\theta}$, and of a divergence-preserving interpolation operator I_h in the space $V_{h,div}$. The existence of the operator I_h onto $V_{h,div}$ and corresponding interpolation estimates were recently shown in [12].

Let $(\mathbf{u},p,\theta) \in [W^{k+1,2}(\Omega)]^d \times W^{k,2}(\Omega) \times W^{k+1,2}(\Omega)$ for $t \in (0,T]$ with $k \in \mathbb{N}$, and let the FE-spaces $V_h \times Q_h \times \Psi_h$ of velocity/pressure/temperature be of piecewise order k, $k-1$ and k, respectively. Then, for fixed viscosities ν und $\varkappa$, the convergence order of the corresponding left hand side terms in the a priori error estimate (3.7) is $\mathcal{O}(h^k)$. The (potentially large) exponential factor in (3.7)-(3.8) reflects a worst-case scenario of solution of the Navier-Stokes model. For a discussion of the problem and improved estimates for exponentially stable solutions, we refer to [15].

An open problem is the existence of appropriate approximation properties of the divergence-preserving interpolation operator I_h on *anisotropic* meshes. Numerical experience with Taylor-Hood elements $V_h \times Q_h$ on hexahedral meshes show that at least moderately large aspect ratios of such elements are possible, see Section 6. A preliminary analysis indicates the dependence of the interpolation estimates on the aspect ratio of the elements. A potential remedy is to imply on isotropic meshes Dirichlet boundary conditions weakly in a DG-type penalty formulation, cf. [2,3].
A potential drawback of the estimates in Theorem 3.4 and in Corollary 3.5 is that not all right-hand side terms in (3.8)-(3.9) are in elementwise fashion. This might be desirable as the behavior of the continuous solution is usually very different, e.g., in boundary layers, wakes, and away from layers. Moreover, it would be desirable to apply different turbulence models in different parts of the domain as is meanwhile standard practice in CFD.

4. Aspects of time discretization

The semidiscrete system (2.6) can be written as a system of differential-algebraic equations (DAE) of the form

$$\begin{pmatrix} M_u & 0 & 0 \\ 0 & M_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h'(t) \\ \theta_h'(t) \\ \mathbf{p}_h'(t) \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h(t) \\ \mathbf{q}_h(t) \\ 0 \end{pmatrix} - \begin{pmatrix} A_u(\mathbf{u}_h, \theta_h) & C & B \\ 0 & A_\theta(\mathbf{u}_h, \theta_h) & 0 \\ B^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h(t) \\ \theta_h(t) \\ \mathbf{p}_h(t) \end{pmatrix} \tag{4.1}$$

with symmetric positive definite mass matrices M_u, M_θ and stiffness matrices $A_u(\mathbf{u}_h, \theta_h), A_\theta(\mathbf{u}_h, \theta_h)$. Matrix C stems from the discretized Boussinesq term whereas B and B^T represent the discretized gradient and divergence operators, respectively. All these matrices posess full rank and the DAE could be made semi-explicit upon multiplying the first two equations with M_u^{-1} and M_θ^{-1} at the cost of loosing the sparsity of the coefficient matrices. Using that $B^T M_u^{-1} B$ has a bounded inverse one can show that the DAE-system has differentiation index 2 [8] and perturbation index 2 [35].
The time discretization of this DAE-system requires some care. A critical comparison of different schemes can be rarely found in the literature. The study [23] considered the generalized trapezoidal rule and Rosenbrock schemes. One conclusion was that only second-order methods are sufficient for obtaining accurate results. The paper [24] is focussed on implicit and linearly implicit schemes; in particular, diagonal-implicit Runge-Kutta (DIRK) and Rosenbrock-Wanner (ROW) methods were considered in combination with adaptive time step control.
In the numerical experiments we employ the second order BDF multi-step method, where the time-derivatives in (4.1) are replaced by a backwards differentiation formula

$$\mathbf{u}_h'(t_{n+1}) \approx \frac{3\mathbf{u}_h(t_{n+1}) - 4\mathbf{u}_h(t_n) + \mathbf{u}_h(t_{n-1})}{2}$$

and analogously for $\theta_h'(t_{n+1})$. This scheme possesses nice stability properties and does not suffer from order reduction for the algebraic variables. In order to resolve the nonlinearity one can replace all occurrences of $\mathbf{u}_h(t_{n+1})$ in the coefficient matrices with the linear extrapolation $\mathbf{w}_h(t_{n+1}) = 2\mathbf{u}_h(t_n) - \mathbf{u}_h(t_{n-1})$. For the non-linear implicit scheme we use this extrapolation as a starting value for a fixed-point iteration.

5. Specification of the subgrid models for free convection flows

We consider $\mathbf{u}_h, \theta_h$ to be approximations with adequate resolution (filter width $\Delta \geq 2h$) to the spatially filtered quantities $\langle \mathbf{u} \rangle$ and $\langle \theta \rangle$, where $\langle \cdot \rangle$ is usually a Gaussian filter. Applying the filter to the full set of equations (1.1) one obtains the equations for the filtered quantities with two additional unclosed residual terms. These are the residual stress tensor τ^R and the residual temperature flux $\mathbf{h}$:

$$\tau^R = \langle \mathbf{u} \otimes \mathbf{u} \rangle - \mathbf{u}_h \otimes \mathbf{u}_h, \qquad \mathbf{h} = \langle \mathbf{u}\, \theta \rangle - \mathbf{u}_h \theta_h. \tag{5.1}$$

We define the residual kinetic energy $k_r := \frac{1}{2} \sum_{i=1}^{d} \tau_{ii}^R$ and substract the normal stresses $\frac{2}{d} k_r \mathbb{I}$ (acting like pressure forces) from τ^R to get the anisotropic residual stress tensor $\tau^r := \tau^R - \frac{2}{d} k_r \mathbb{I}$. These terms are modeled with the linear eddy-viscosity asumption by Boussinesq and the gradient-diffusion hypothesis

$$\tau^r = -2\nu_t\, \mathbb{D}\mathbf{u}_h, \qquad \mathbf{h} = -a_t\, \nabla \theta_h. \tag{5.2}$$

It remains to parametrize the model parameters ν_t and a_t. One very simple and often used choice is a model based on the work of Lilly and Eidson summarized in [34]

$$\nu_t(\mathbb{D}\mathbf{u}_h, \nabla\theta_h) = (C_E \Delta)^2 \max \left(0\,, \|\mathbb{D}\mathbf{u}_h\|_F^2 + \frac{\beta}{Pr_t} \mathbf{g} \cdot \nabla\theta_h \right)^{1/2}, \quad a_t = Pr_t^{-1} \nu_t. \tag{5.3}$$

with $C_E = 0.21$ and $Pr_t = 0.4$. These expressions are reasonable approximations remote from the wall and are multiplied by a van Driest-type damping function $f(y^+) = [1 - \exp(-y^+/A)]^2$ for reasonable near wall behavior, where y^+ is a suitably scaled wall distance. This model was studied together with a variant with dynamic parameters in [34]. For $\mathbf{g} \cdot \nabla\theta_h = \mathbf{0}$ the model reduces to the classical Smagorinsky model. In the numerical experiments below, we will denote this variant as *full Smagorinsky-Eidson model*.

An often made observation for this type of models is the fact, that they are too dissipative. One opinion is, that eddy-viscosity should be limited to modeling the influence of the fourth order sub-grid scale tensor on the mean velocity and other (non-diffusive) models should be used for the remaining second order large scale and cross scale terms [5]. In order to reduce the dissipation introduced by the model, we extend the model to

$$\tau^r = -2\nu_t\, \kappa_u(\mathbb{D}\mathbf{u}_h), \qquad \mathbf{h} = -a_t\, \kappa_\theta(\nabla\theta_h). \tag{5.4}$$

This restricts the effect of additional dissipation to the small resolved scales $\tilde{\mathbf{u}}_h$ and $\tilde{\theta}_h$ and corresponds to the subgrid viscosity terms present in (2.6). In the numerical experiments below, we denote this variant as *projection-based Smagorinsky-Eidson model*. The original behavior can be recovered by choosing the coarse spaces to be $\{0\}$ or equivalently setting $\kappa_{u/\theta} = I_{u/\theta}$.

Defining $\nu_t = \nu_t(\kappa_u(\mathbb{D}\mathbf{u}_h), \kappa_\theta(\nabla\theta_h))$ in terms of the resolved small scales is another way to further reduce the dissipation. This leads to two variants where either all resolved (*all-small*) or only the small resolved scales (*small-small*) are used for the computation of the model parameter ν_t.
For the parameter of the pressure subgrid model we restrict ourselves to the simple choice $\gamma(\mathbf{u}_h) = \gamma =$ constant. This term is a consistent term (for the continuous model) but serves the analysis quite well and improves the numerical accuracy of the model in some cases tremendously [31].

Remark 5.1. *It remains to validate the assumptions* (3.5) *of Theorem 3.4. This can be done following the approach in* [36]*, Corollary 4.1 for the projection-based Smagorinsky model in the isothermal case.*

6. Numerical experiments for two-dimensional closed cavity flow

Now we apply the method to natural convection in a differentially heated cavity as a standard benchmark problem for non-isothermal incompressible flows, see, e.g., [29,40,34]. Heating $\theta = \theta_{max}$ and cooling $\theta = \theta_{min}$ is performed at lateral boundaries, whereas the upper and lower boundaries are highly conducting. As suggested in [38] we use experimental data as boundary conditions on these walls. No-slip conditions $\mathbf{u} = \mathbf{0}$ for velocity are given at the whole boundary $\partial\Omega$.
Relevant information regarding the boundary layer behavior of (turbulent) natural convection flow can be found, e.g., in [16].
In the remainder of this paper, let $\Omega := (0,1)^2$. For the spatial discretization we apply quadrilateral meshes with FE spaces $\mathbb{Q}_2/\mathbb{Q}_1/\mathbb{Q}_2$ for velocity/pressure and temperature within the FE package `deal.II`, see [1].

6.1. *Laminar flows*

Let us start with results for laminar flows with Rayleigh numbers $Ra \leq 10^8$. In our implementation, we use a one-level approach with $H = h$. Results are given on isotropic meshes with $h = 2^{-l}$ up to level $l = 7$. For the following simulations we don't apply any turbulence model, i.e., $L_h = \{\mathbb{D}(V_h)\}$ and $M_h = \{\nabla(\Psi_h)\}$, but we comment on its influence below.
In Table 1 we present results for the maximal value and position of stream-function Φ_{max}, the maximal values and position of $u_1(0.5, x_2)$ and of $u_2(x_1, 0.5)$ for different Ra-numbers. In Table 2 we show results for Nusselt numbers for different Ra-numbers. Here we use the following definitions:

$$Nu_{avg} = \int_\Omega q(x)dx, \qquad Nu_{\frac{1}{2}} = \int_0^1 q(0.5, x_2)dx_2,$$
$$Nu_{min} = \min_{0\leq x_2\leq 1} q(0, x_2), \qquad Nu_{max} = \max_{0\leq x_2\leq 1} q(0, x_2)$$

with $q(x) = (u_1\theta - \theta_{x_1})(x)$. The results are in very good agreement with benchmark results of [29].

Table 1. Maximal value and position of stream-function Φ_{max}, maximal values and position of $u_1(0.5, x_2)$ and of $u_2(x_1, 0.5)$ for different Ra-numbers

Ra	Level	Φ_{max}	$x_{1,\Phi}$	$x_{2,\Phi}$	$u_{1,\max}$	$x_{1,max}$	$u_{2,max}$	$x_{2,max}$
	5	16.7971	0.1263	0.5469	64.8193	0.8496	218.7909	0.0381
10^6	6	16.8100	0.1484	0.5469	64.8344	0.8496	220.4804	0.0381
	7	16.8100	0.1523	0.5469	64.8342	0.8499	220.5943	0.0378
Ref.[29]		16.8110	0.1500	0.5470	64.8300	0.8500	220.6000	0.0380
	5	9.4164	0.0937	0.5469	147.4576	0.8789	720.9979	0.0225
10^7	6	9.5361	0.0859	0.5547	148.5186	0.8794	696.8134	0.0220
	7	9.5386	0,0859	0.5547	148.5846	0.8794	699.8374	0.0212
Ref.[29]		9.5390	0.0860	0.5560	148.5954	0.8790	699.1796	0.0210
	6	5.3392	0,0469	0.5547	322.9088	0.9297	2279.0403	0.0122
10^8	7	5.3815	0.0469	0.5547	321.6551	0.9277	2224.6928	0.0122
Ref.[29]		5.3850	0.0480	0.5530	321.9000	0.9280	2222.0000	0.0120

Table 2. Results for Nusselt numbers for different Ra-numbers

Ra	Level	Nu_{min}	x_{min}	Nu_{max}	x_{max}	Nu_{avg}	$Nu_{\frac{1}{2}}$
	5	0.992502	0.9990	18.39393	0.0313	8.814861	9.113911
10^6	6	0.983896	0.9995	17.73365	0.0391	8.824718	8.877341
	7	0.980687	1.0000	17.56368	0.0391	8.825174	8.832769
Ref.[29]		0.979500	1.0000	17.53600	0.0390	8.822500	8.825000
	5	1.433298	1.0000	43.56854	0.0225	16.176103	17.970533
10^7	6	1.380943	0.9995	41.71878	0.0156	16.509660	16.998765
	7	1.371551	0.9998	39.94763	0.0178	16.522482	16.608537
Ref.[29]		1.366000	0.9990	38.94000	0.0200	16.523000	16.523000
	6	1.944536	1.0000	96.35414	0.0112	29.799044	32.687167
10^8	7	1.934311	0.9998	93.17395	0.0078	30.207906	30.964267
Ref.[29]		1.919000	1.0000	87.24000	0.0080	30.223000	30.225000

Additionally, we considered the influence of the Smagorinsky-Eidson model for laminar flows on anisotropic meshes (see Subsec. 6.2). For $Ra = 10^6$, we compare in Fig. 1 the velocity and temperature profiles $u_1(x_1, 0.5), u_2(0.5, x_2)$ and $\theta(x_1, 0.5), \theta(0.5, x_2)$ on moderate isotropic meshes. The solutions with (blue) and without (black) the projection-based VMS model are obviously grid-converged, thus indicating a negligible model error. The solutions with the full Smagorinsky-Eidson model with $\Delta = 2\text{diam}(K)$ (green) and anisotropic choice of Δ (red) deviate from the solution without turbulence model, thus

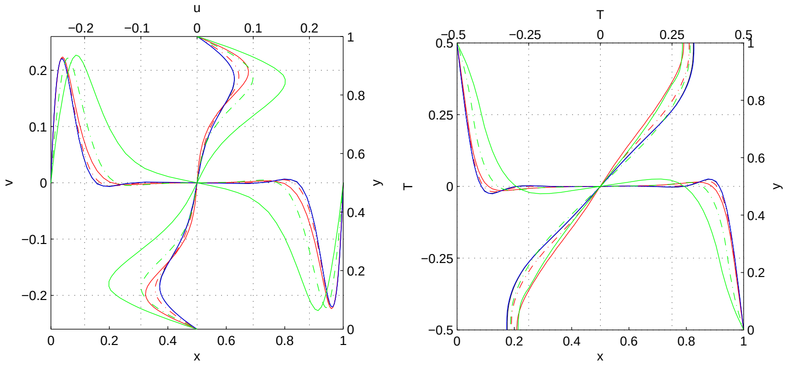

Fig. 1. Velocity profile $u_1(x_1, 0.5)$ and $u_2(0.5, x_2)$ (left) and temperature profiles $\theta(x_1, 0.5)$ and $\theta(0.5, x_2)$ (right) with full and projection-based Smagorinsky-Eidson model for $Ra = 10^6$ and anisotropic meshes with 32×32 and nodes (full lines) and 64×64 nodes (dashed lines).

showing an over-diffusive behavior of the model error.

6.2. *Low-turbulence flow*

Now we present results for time-averaged quantities of a low-turbulence flow at $Ra = 1.58 \times 10^9$. In our implementation, we use a one-level approach with $H = h$. Computations were done on two meshes with 64 and 32 cells in each dimension. An anisotropic mesh refinement had been performed at all boundaries by transforming an equidistant reference mesh with

$$x = \hat{x} - \frac{19}{40\pi} \sin(2\pi\hat{x}), \quad y = \hat{y} - \frac{7}{16\pi} \sin(2\pi\hat{y}) \ .$$

The maximum aspect-ratio of cells at the vertical walls was about 36:1. In the Smagorinsky-Eidson subgrid model we apply for the filter width Δ an anisotropic scaling matrix that takes local mesh anisotropy and orientation into account. This approach gave better results than taking an isotropic filter width (e.g. length of shortest edge).

On both meshes we compare the results for velocity and temperature profiles (see Fig. 2-4) and of wall shear stress (see Fig. 5). In particular we used the proposed VMS with

- the *full Smagorinsky-Eidson parametrization*, i.e. $L_h = \{0\}$, $M_h = \{0\}$, with van-Driest damping (left column) and
- the *projection-based Smagorinsky-Eidson model* $L_h = \mathbb{Q}_0^{d\times d}$, $M_h = \mathbb{Q}_0^d$ without van Driest damping (right column).

Moreover, we used grad-div stabilization with constant $\gamma_K = 0.3$ to improve the mass conservation properties of the scheme.

In general, the results are in good agreement to experimental data of [38], but the full and the projection-based variants behave in a different way. The solutions for the full Smagorinsky-Eidson model are obviously not grid-converged, but the solution on the fine mesh compares

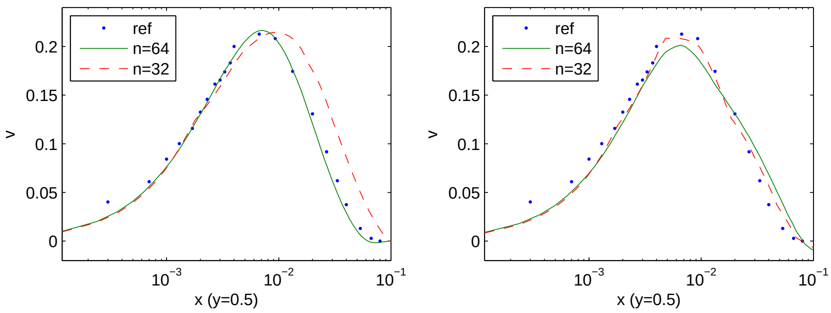

Fig. 2. Boundary layer profiles for horizontal velocity profile $v(x, 0.5)$ with full (left) and projection-based (right) Smagorinsky-Eidson model and experimental data [38]

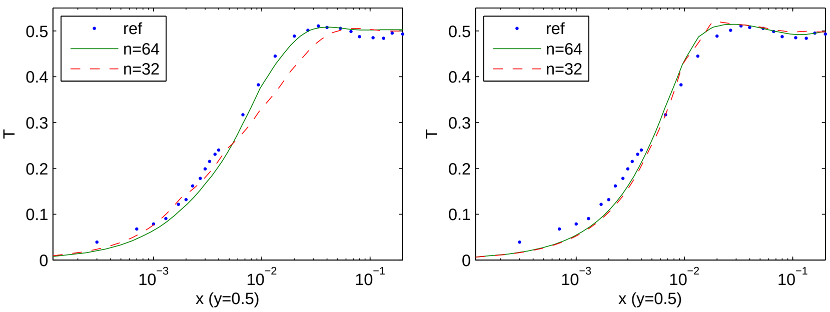

Fig. 3. Temperature profile on vertical centerline $T(0.5, y)$ with full (left) and projection-based (right) Smagorinsky-Eidson model and experimental data of [38]

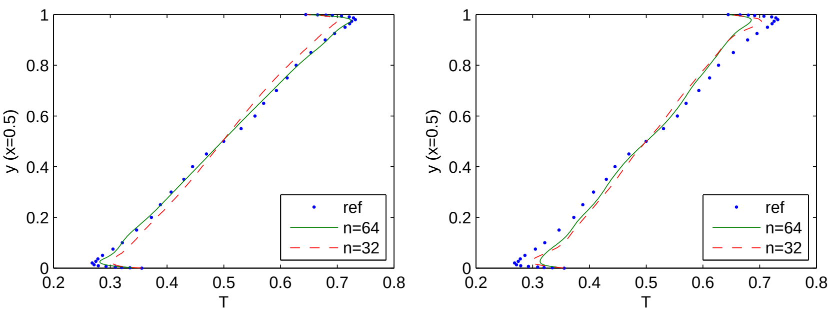

Fig. 4. Temperature profile on vertical centerline $T(0.5, y)$ with full (left) and projection-based (right) Smagorinsky-Eidson model and experimental data of [38]

very well with the experimental data. On the coarse mesh the boundary layers are too thick. This reflects that this turbulence model is too dissipative. Interestingly, for the projection-based variant, we observe already almost grid-convergence of the solutions. On the other hand, it deviates more from the experimental data.

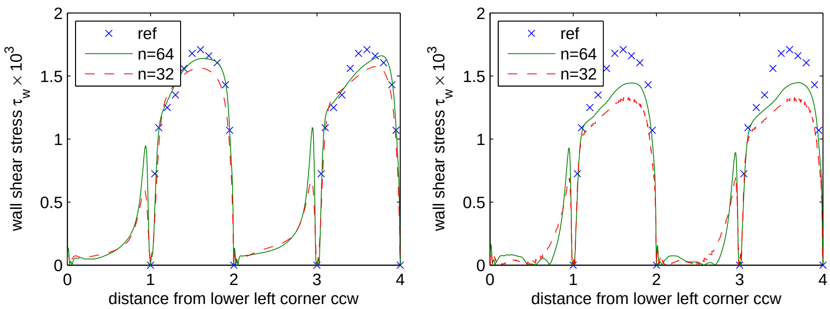

Fig. 5. Wall-shear stress with full (left) and projection-based (right) Smagorinsky-Eidson model and experimental data [38]

The results for the wall-shear stress in Fig. 5 – and similar results for the Nusselt number (not shown) – show grid-dependence of both variants together with better behavior for the full Smagorinsky-Eidson model.

One critical point of the simulation is the separation of the flow at the vertical walls and its reattachment at the horizontal walls. Experiments show small counter-rotating vortices in these corners, which we also found in our simulations on the fine mesh. On the coarse mesh these vortices are missing and the wall shear stress at reattachment is too small.

7. Summary

In this paper, we applied a variational multiscale model to the time-dependent Navier-Stokes/Fourier model of incompressible and non-isothermal flows. For the case of piece-wise nonlinear subgrid models for the unresolved velocity, temperature, and pressure fluctuations, an a priori analysis of the nonlinear semidiscrete problem was given. Then we specified the subgrid model for natural convection flow based on the classical Smagorinsky-Eidson model. Finally, we applied the approach to the standard benchmark problem of natural convection problem in a two-dimensional differentially heated two-dimensional cavity. For laminar flows we observed already for the method without stabilization (or turbulence model) very good agreement with benchmark results. For a low-turbulence flow we found good agreement of the VMS-based turbulence models with experimental data.

Some open problems are the extension to the three-dimensional case of the closed cavity, to Rayleigh-Benard convection and to mixed convection problems in indoor air-flow simulation. This will be considered in future research.

Appendix A: Proof of the stability estimate (Lemma 3.1)

The proof is a variant of a similar proof in [26], Section 1. Setting $\psi_h = \theta_h$ and $\mathbf{v}_h = \mathbf{u}_h$ in (3.1) and using $b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h) = 0$, $c_S(\mathbf{u}_h, \theta_h, \theta_h) = 0$, one gets global balance of kinetic energy and of heat energy:

$$\frac{1}{2}\partial_t \|\theta_h\|_0^2 + \varkappa\|\nabla\theta_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \varkappa_T^K(\mathbf{u}_h,\theta_h)\|\kappa_\theta(\nabla\theta_h)\|^2_{L^2(K)} = (Q,\theta_h) \tag{7.1}$$

$$\frac{1}{2}\partial_t \|\mathbf{u}_h\|_0^2 + 2\nu\,\|\mathbb{D}\mathbf{u}_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \big(\gamma_K(\mathbf{u}_h,p_h)\,\|\nabla\cdot\mathbf{u}_h\|^2_{L^2(K)} + \nu_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u(\mathbb{D}\mathbf{u}_h)\|^2_{L^2(K)}\big) = (\mathbf{f}-\alpha\mathbf{g}\theta_h,\mathbf{u}_h). \tag{7.2}$$

The non-negativity of the parameters $\nu, \varkappa, \nu_T^K(\mathbf{u}_h,\theta_h), \varkappa_T^K(\mathbf{u}_h,\theta_h), \gamma_K(\mathbf{u}_h,p_h)$ implies via Cauchy-Schwarz inequality that $\partial_t\|\theta_h\|_0 \le \|Q\|_0$ and $\partial_t\|\mathbf{u}_h\|_0 \le \|\mathbf{f}\|_0 + \alpha\|\mathbf{g}\|_\infty\|\theta_h\|_0$ and, after integrating

$$\|\theta_h\|_{L^\infty(0,t;L^2(\Omega))} \le \|\theta_0\|_0 + \|Q\|_{L^1(0,t;L^2(\Omega))} \equiv K_1$$
$$\|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))} \le \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))} + \alpha\|\mathbf{g}\|_\infty\|\theta_h\|_{L^1(0,t;L^2(\Omega))}$$

for all $t\in(0,T]$. Then, the inequality $\|\theta_h\|_{L^1(0,t;L^2(\Omega))} \le C\|\theta_h\|_{L^\infty(0,t;L^2(\Omega))}$ implies the first part.
For the second part, we first use (7.1) and integrate on $(0,t)$ to obtain

$$\varkappa\|\nabla\theta(t)\|^2_{|L^2(0,t;L^2(\Omega))} + \int_0^t \sum_{K\in\mathcal{T}_h} \varkappa_T(\mathbf{u}_h,\theta_h)\|\kappa_\theta(\nabla\theta_h)\|^2_{0,K} \le \frac{1}{2}\|\theta_0\|_0^2 + \int_0^t (Q,\theta_h)\,dt.$$

As in [36] one observes via the first part of the Lemma together with the Cauchy-Schwarz and Young inequalities

$$\int_0^t (Q,\theta_h)\,dt' \le \frac{1}{2}\|\theta_0\|_0^2 + \frac{3}{2}\|Q\|^2_{L^1(0,t;L^2(\Omega))}.$$

thus

$$\begin{aligned}\varkappa\|\nabla\theta_h\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \sum_{K\in\mathcal{T}_h} \varkappa_T(\mathbf{u}_h,\theta_h)\|\kappa_\theta(\nabla\theta_h)\|^2_{0,K}) &\le \frac{1}{2}\|\theta_0\|_0^2 + \frac{3}{2}\|Q\|^2_{L^1(0,t;L^2(\Omega))}\\ &\le \frac{3}{2}K_1^2(Q,\theta_0).\end{aligned}$$

Similarly we obtain

$$\int_0^t (\mathbf{f}-\alpha\mathbf{g}\theta_h,\mathbf{u}_h)\,dt' \le 3K_2^2(\mathbf{f},\mathbf{u}_0,Q,\theta_0)$$

via (7.2), hence

$$\begin{aligned}&\nu\,\|\mathbb{D}\mathbf{u}_h\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2}\int_0^t \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u(\mathbb{D}\mathbf{u}_h)\|^2_{L^2(K)}\,dt\\ &+\frac{1}{2}\int_0^t \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h,p_h)\,\|\nabla\cdot\mathbf{u}_h\|^2_{L^2(K)}\,dt \le 3K_2^2(\mathbf{f},\mathbf{u}_0,Q,\theta_0).\end{aligned}$$

Appendix B: Proof of the a priori error velocity estimate (Theorem 3.4)

We split the errors into model errors $\mathbf{e}_h^u, e_h^\theta$ and approximation error $\epsilon^u, \epsilon^\theta$

$$\mathbf{u}_h - \mathbf{u} = (\mathbf{u}_h - \tilde{\mathbf{u}}_h) - (\mathbf{u} - \tilde{\mathbf{u}}_h) =: \mathbf{e}_h^u - \epsilon^u,$$
$$\theta_h - \theta = (\theta_h - \tilde{\theta}_h) - (\theta - \tilde{\theta}_h) =: e_h^\theta - \epsilon^\theta,$$

Now one can subtract (2.1) from (3.1), use $\mathbf{e}_h^u \in V_{h,\mathrm{div}}$ and $e_h^\theta \in \Psi_h$, respectively, as test functions and obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t \|\mathbf{e}_h^u\|_0^2 + \sum_{K\in\mathcal{T}_h} \nu_{\mathrm{mod}}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \gamma_K\,\|\nabla\cdot\mathbf{e}_h^u\|_{L^2(K)}^2 \\
&\quad= (\partial_t \epsilon^u, \mathbf{e}_h^u) + (2\nu\mathbb{D}\epsilon, \mathbb{D}\mathbf{e}_h^u) + b_S(\mathbf{u},\mathbf{u},\mathbf{e}_h^u) - b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{e}_h^u) - (p-\lambda_h, \nabla\cdot\mathbf{e}_h^u) \\
&\qquad+ \sum_{K\in\mathcal{T}_h} \gamma_K\,(\nabla\cdot\epsilon^u, \nabla\cdot\mathbf{e}_h^u)_K + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,(\kappa_u(\mathbb{D}\epsilon^u), \kappa_u(\mathbb{D}\mathbf{e}_h^u))_K \\
&\qquad\quad- \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,(\kappa_u(\mathbb{D}\mathbf{u}), \kappa_u(\mathbb{D}\mathbf{e}_h^u))_K - \alpha(\mathbf{g}e_h^\theta, \mathbf{e}_h^u) + \alpha(\mathbf{g}\epsilon^\theta, \mathbf{e}_h^u) \qquad (7.3)
\end{aligned}$$

for all $\lambda_h \in Q_h$ and

$$\begin{aligned}
&\frac{1}{2}\partial_t \left\|e_h^\theta\right\|_0^2 + \sum_{K\in\mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta)\,\left\|\nabla e_h^\theta\right\|_{L^2(K)}^2 \\
&\qquad\qquad= \left(\partial_t\epsilon^\theta, e_h^\theta\right) + (\varkappa\nabla\epsilon^\theta, \nabla e_h^\theta) + c_S(\mathbf{u},\theta,e_h^\theta) - c_S(\mathbf{u}_h,\theta_h,e_h^\theta) \\
&+\sum_{K\in\mathcal{T}_h} \varkappa_T^K(\mathbf{u}_h,\theta_h)\,(\kappa_u(\mathbb{D}\epsilon^u), \kappa_u(\mathbb{D}\mathbf{e}_h^u))_K - \sum_{K\in\mathcal{T}_h} \varkappa_T^K(\mathbf{u}_h,\theta_h)\,(\kappa_u(\mathbb{D}\mathbf{u}), \kappa_u(\mathbb{D}\mathbf{e}_h^u))_K\,.
\end{aligned} \qquad (7.4)$$

Next one has to estimate all the terms on the right hand side of (7.3) and (7.4). For convenience, we first summarize from [36] the corresponding linear terms for the error equation of the Navier-Stokes model:

$$(\partial_t\epsilon^u, \mathbf{e}_h^u) \le \frac{3C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)}\,\|\partial_t\epsilon^u\|_{H^{-1}(\Omega)}^2 + \frac{1}{12}\sum_{K\in\mathcal{T}_h} \nu_{\mathrm{mod}}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2$$

$$(2\nu\mathbb{D}\epsilon^u, \mathbb{D}\mathbf{e}_h^u) \le 6\nu\,\|\mathbb{D}\epsilon^u\|_0^2 + \frac{2\nu}{12}\,\|\mathbb{D}\mathbf{e}_h^u\|_0^2$$

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,(\kappa_u(\mathbb{D}\epsilon^u), \kappa_u(\mathbb{D}\mathbf{e}_h^u))_K &\le \sum_{K\in\mathcal{T}_h} 6\nu_{\mathrm{VMS}}^K(\epsilon^u)\,\|\mathbb{D}\epsilon^u\|_{L^2(K)}^2 \\
&\quad+ \sum_{K\in\mathcal{T}_h} \frac{\nu_{\mathrm{VMS}}^K(\mathbf{e}_h^u)}{24}\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2 \\
\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,(\kappa_u(\mathbb{D}\mathbf{u}), \kappa_u(\mathbb{D}\mathbf{e}_h^u))_K &\le \sum_{K\in\mathcal{T}_h} 6\nu_{\mathrm{VMS}}^K(\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|_{L^2(K)}^2 \\
&\quad+ \sum_{K\in\mathcal{T}_h} \frac{\nu_{\mathrm{VMS}}^K(\mathbf{e}_h^u)}{24}\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2\,.
\end{aligned}$$

$$(p-\lambda_h, \nabla\cdot \mathbf{e}_h^u) \le \sum_{K\in\mathcal{T}_h} \min\left(\frac{9C_{\rm Ko}^2}{\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)}, \frac{1}{\gamma_K}\right) \|p-\lambda_h\|_{L^2(K)}^2 + \frac{1}{12}\sum_{K\in\mathcal{T}_h} \nu_{\rm mod}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h}\frac{\gamma_K}{4}\,\|\nabla\cdot\mathbf{e}_h^u\|_{L^2(K)}^2\,,$$

$$\sum_{K\in\mathcal{T}_h}\gamma_K(\mathbf{u}_h)\,(\nabla\cdot\epsilon^u, \nabla\cdot\mathbf{e}_h^u)_K \le \sum_{K\in\mathcal{T}_h} \min\left(\frac{9C_{\rm Ko}^2}{\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)}, \frac{1}{\gamma_K}\right)\gamma_K^2\,\|\nabla\cdot\epsilon^u\|_{L^2(K)}^2 + \frac{1}{12}\sum_{K\in\mathcal{T}_h} \nu_{\rm mod}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h}\frac{\gamma_K}{4}\,\|\nabla\cdot\mathbf{e}_h^u\|_{L^2(K)}^2\,.$$

For the nonlinear convective term we start in (7.3) as in [36] from

$$b_S(\mathbf{u},\mathbf{u},\mathbf{e}_h^u) - b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{e}_h^u) = b_S(\epsilon^u,\mathbf{u},\mathbf{e}_h^u) - b_S(\mathbf{e}_h^u,\mathbf{u},\mathbf{e}_h^u) + b_S(\mathbf{u}_h,\epsilon^u,\mathbf{e}_h^u).$$

Using an estimate of the skew-symmetric trilinear form

$$b_S(\mathbf{u},\mathbf{v},\mathbf{w}) \le C_{LT}\,\|\mathbf{u}\|_0^{\frac12}\,\|\mathbb{D}\mathbf{u}\|_0^{\frac12}\,\|\mathbb{D}\mathbf{v}\|_0\,\|\mathbb{D}\mathbf{w}\|_0\,. \tag{7.5}$$

in [28], Lemma 2.2 (f), we obtained [36]

$$b_S(\mathbf{u},\mathbf{u},\mathbf{e}_h^u) - b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{e}_h^u) \le \frac{27C_{LT}^4}{4\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)^3}\,\|\mathbb{D}\mathbf{u}\|_0^4\,\|\mathbf{e}_h^u\|_0^2 + \frac{3C_{LT}^2}{\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)}\Big(C_F C_{\rm Ko}\,\|\mathbb{D}\mathbf{u}\|_0^2 + \|\mathbf{u}_h\|_0\,\|\mathbb{D}\mathbf{u}_h\|_0\Big)\,\|\mathbb{D}\epsilon^u\|_0^2 + \frac{5}{12}\sum_{K\in\mathcal{T}_h}\nu_{\rm mod}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2\,.$$

For the Boussinesq term in (7.3), we obtain

$$-\alpha(\mathbf{g}e_h^\theta,\mathbf{e}_h^u) + \alpha(\mathbf{g}\epsilon^\theta,\mathbf{e}_h^u) \le \alpha\,\|\mathbf{g}\|_\infty\left(\left\|e_h^\theta\right\| + \left\|\epsilon^\theta\right\|\right)\|\mathbf{e}_h^u\|_0 \le \alpha\,\|\mathbf{g}\|_\infty\,\|\mathbf{e}_h^u\|_0^2 + \frac{\alpha\,\|\mathbf{g}\|_\infty}{2}(\left\|e_h^\theta\right\|_0^2 + \left\|\epsilon^\theta\right\|_0^2)$$

Now each term in (7.3) is estimated and we can summarize

$$\begin{aligned}
&\frac12\partial_t\,\|\mathbf{e}_h^u\|_0^2 + \frac14\sum_{K\in\mathcal{T}_h}\nu_{\rm mod}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{L^2(K)}^2 + \frac12\sum_{K\in\mathcal{T}_h}\gamma_K\,\|\nabla\cdot\mathbf{e}_h^u\|_{L^2(K)}^2\\
&\le \sum_{K\in\mathcal{T}_h}\Bigg[\min\left(\frac{9C_{\rm Ko}^2}{\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)}, \frac{1}{\gamma_K}\right)\Big(\|p-\lambda_h\|_{L^2(K)}^2 + \gamma_K^2\,\|\nabla\cdot\epsilon^u\|_{L^2(K)}^2\Big)\\
&\qquad + 6\big(\nu+\nu_{\rm VMS}^K(\epsilon^u)\big)\,\|\mathbb{D}\epsilon^u\|_{L^2(K)}^2 + 6\nu_{\rm VMS}^K(\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|_{L^2(K)}^2\Bigg]\\
&\qquad + \frac{3C_{\rm Ko}^2}{\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)}\,\|\partial_t\epsilon^u\|_{H^{-1}(\Omega)}^2 + \frac{\alpha\|\mathbf{g}\|_\infty}{2}\,\left\|\epsilon^\theta\right\|_0^2\\
&\qquad + \frac{3C_{LT}^2}{\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)}\Big(C_F C_{\rm Ko}\,\|\mathbb{D}\mathbf{u}\|_0^2 + \|\mathbf{u}_h\|_0\,\|\mathbb{D}\mathbf{u}_h\|_0\Big)\,\|\mathbb{D}\epsilon^u\|_0^2\\
&\quad + \left(\frac{27C_{LT}^4}{4\nu_{\rm mod}^{\rm min}(\mathbf{e}_h^u)^3}\,\|\mathbb{D}\mathbf{u}\|_0^4 + \alpha\,\|\mathbf{g}\|_\infty\right)\|\mathbf{e}_h^u\|_0^2 + \frac{\alpha\,\|\mathbf{g}\|_\infty}{2}\,\left\|e_h^\theta\right\|_0^2\,.
\end{aligned} \tag{7.6}$$

Similar estimates of the terms in (7.4) lead to

$$\left(\partial_t \epsilon^\theta, e_h^\theta\right) \le \frac{2}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)} \left\|\partial_t \epsilon^\theta\right\|_{H^{-1}(\Omega)}^2 + \frac{1}{8} \sum_{K \in \mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta) \left\|\nabla e_h^\theta\right\|_{L^2(K)}^2$$

$$(\varkappa \nabla \epsilon^\theta, \nabla e_h^\theta) \le 2\varkappa \left\|\nabla \epsilon^\theta\right\|_0^2 + \frac{1}{8} \varkappa \left\|\nabla e_h^\theta\right\|_0^2$$

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \varkappa_T^K \left(\kappa_\theta(\nabla \epsilon^\theta), \kappa_\theta(\nabla e_h^\theta)\right)_K &\le \sum_{K \in \mathcal{T}_h} 4\varkappa_{\mathrm{VMS}}^K(\epsilon^\theta) \left\|\nabla \epsilon^\theta\right\|_{L^2(K)}^2 \\ &\quad + \frac{1}{16} \sum_{K \in \mathcal{T}_h} \varkappa_{\mathrm{VMS}}^K(e_h^\theta) \left\|\nabla e_h^\theta\right\|_{L^2(K)}^2 \\ - \sum_{K \in \mathcal{T}_h} \varkappa_T^K \left(\kappa_\theta(\nabla \theta), \kappa_\theta(\nabla e_h^\theta)\right)_K &\le \sum_{K \in \mathcal{T}_h} 4\varkappa_{\mathrm{VMS}}^K(\theta) \left\|\nabla \theta\right\|_{L^2(K)}^2 \\ &\quad + \frac{1}{16} \sum_{K \in \mathcal{T}_h} \varkappa_{\mathrm{VMS}}^K(e_h^\theta) \left\|\nabla e_h^\theta\right\|_{L^2(K)}^2 \end{aligned}$$

For the advective terms we obtain

$$\begin{aligned} & c_S(\mathbf{u}, \theta, e_h^\theta) - c_S(\mathbf{u}_h, \theta_h, e_h^\theta) \\ & = c_S(\mathbf{u} - \tilde{\mathbf{u}}_h + \tilde{\mathbf{u}}_h - \mathbf{u}_h + \mathbf{u}_h, \theta, e_h^\theta) - c_S(\mathbf{u}_h, \theta_h - e_h^\theta, e_h^\theta) \\ & = c_S(\epsilon^u, \theta, e_h^\theta) - c_S(\mathbf{e}_h^u, \theta, e_h^\theta) + c_S(\mathbf{u}_h, \theta, e_h^\theta) - c_S(\mathbf{u}_h, \tilde{\theta}_h, e_h^\theta) \\ & = c_S(\epsilon^u, \theta, e_h^\theta) - c_S(\mathbf{e}_h^u, \theta, e_h^\theta) + c_S(\mathbf{u}_h, \epsilon^\theta, e_h^\theta). \end{aligned}$$

Starting from

$$c_S(\mathbf{u}, \theta, \vartheta) \le C_1 \left\|\mathbf{u}\right\|_0^{\frac{1}{2}} \left\|\nabla \mathbf{u}\right\|_0^{\frac{1}{2}} \left\|\nabla \theta\right\|_0 \left\|\nabla \vartheta\right\|_0,$$

we obtain

$$\begin{aligned} c_S(\mathbf{e}_h^u, \theta, e_h^\theta) &\le C_1 \left\|\mathbf{e}_h^u\right\|_0^{\frac{1}{2}} \left\|\nabla \mathbf{e}_h^u\right\|_0^{\frac{1}{2}} \left\|\nabla \theta\right\|_0 \left\|\nabla e_h^\theta\right\|_0 \le \frac{2C_1^4 C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)\, \varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)^2} \left\|\nabla \theta\right\|_0^4 \left\|\mathbf{e}_h^u\right\|_0^2 \\ &\quad + \frac{1}{8} \sum_{K \in \mathcal{T}_h} \nu_{\mathrm{mod}}^K(\mathbf{e}_h^u) \left\|\mathbb{D}\mathbf{e}_h^u\right\|_{L^2(K)}^2 + \frac{1}{4} \sum_{K \in \mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta) \left\|\nabla e_h^\theta\right\|_{L^2(K)}^2, \end{aligned}$$

$$\begin{aligned} c_S(\epsilon^u, \theta, e_h^\theta) &\le C_1 \left\|\epsilon^u\right\|_0^{\frac{1}{2}} \left\|\nabla \epsilon^u\right\|_0^{\frac{1}{2}} \left\|\nabla \theta\right\|_0 \left\|\nabla e_h^\theta\right\|_0 \\ &\le \frac{2C_1^2 C_F C_{\mathrm{Ko}}^2}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)} \left\|\nabla \theta\right\|_0^2 \left\|\mathbb{D}\epsilon^u\right\|_0^2 + \frac{1}{8} \sum_{K \in \mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta) \left\|\nabla e_h^\theta\right\|_{L^2(K)}^2, \end{aligned}$$

$$\begin{aligned} c_S(\mathbf{u}_h, \epsilon^\theta, e_h^\theta) &\le C_1 \left\|\mathbf{u}_h\right\|_0^{\frac{1}{2}} \left\|\nabla \mathbf{u}_h\right\|_0^{\frac{1}{2}} \left\|\nabla \epsilon^\theta\right\|_0 \left\|\nabla e_h^\theta\right\|_0 \\ &\le \frac{2C_1^2 C_{\mathrm{Ko}}}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)} \left\|\mathbf{u}_h\right\|_0 \left\|\mathbb{D}\mathbf{u}_h\right\|_0 \left\|\nabla \epsilon^\theta\right\|_0^2 + \frac{1}{8} \sum_{K \in \mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta) \left\|\nabla e_h^\theta\right\|_{L^2(K)}^2. \end{aligned}$$

Now we can summarize the estimates of the terms in (7.4):

$$
\begin{aligned}
&\frac{1}{2}\partial_t \left\|e_h^\theta\right\|_0^2 - \frac{1}{8}\sum_{K\in\mathcal{T}_h} \nu_{\mathrm{mod}}^K(\mathbf{e}_h^u)\left\|\mathbb{D}\mathbf{e}_h^u\right\|_{L^2(K)}^2 + \frac{1}{4}\sum_{K\in\mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta)\left\|\nabla e_h^\theta\right\|_{L^2(K)}^2 \\
&\quad \le \sum_{K\in\mathcal{T}_h}\left[\left(2\varkappa + 4\varkappa_{\mathrm{VMS}}^K(\epsilon^\theta)\right)\left\|\nabla\epsilon^\theta\right\|_{L^2(K)}^2 + 4\varkappa_{\mathrm{VMS}}^K(\theta)\left\|\nabla\theta\right\|_{L^2(K)}^2\right] \\
&\qquad + \frac{2C_1^2 C_{\mathrm{Ko}}}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)}\left(C_F C_{\mathrm{Ko}}\left\|\nabla\theta\right\|_0^2\left\|\mathbb{D}\epsilon^u\right\|_0^2 + \left\|\mathbf{u}_h\right\|_0\left\|\mathbb{D}\mathbf{u}_h\right\|_0\left\|\nabla\epsilon^\theta\right\|_0^2\right) \\
&\qquad + \frac{2}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)}\left\|\partial_t\epsilon^\theta\right\|_{H^{-1}(\Omega)}^2 + \frac{2C_1^4 C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)\,\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)^2}\left\|\nabla\theta\right\|_0^4\left\|\mathbf{e}_h^u\right\|_0^2 \qquad (7.7)
\end{aligned}
$$

The next step in the proof will be the application of Gronwall's Lemma, see [33], Lemma 1.4.1. It states that for $g_i(t) \in L^1(0,T)$ ($i \in \{1,2,3\}$), $\int_0^t g_1(s)ds$, $\int_0^t g_2(s)ds$ continuous and non-decreasing on $[0,T]$, $g_3(t)$ non-negative and

$$
\partial_t\left(\left\|\mathbf{e}_h^u\right\|_0^2 + \left\|e_h^\theta\right\|_0^2\right) + g_1(t) \le g_2(t) + g_3(t)\left(\left\|\mathbf{e}_h^u\right\|_0^2 + \left\|e_h^\theta\right\|_0^2\right) \qquad (7.8)
$$

there holds

$$
\begin{aligned}
&\left\|\mathbf{e}_h^u(t)\right\|_0^2 + \left\|e_h^\theta(t)\right\|_0^2 + \int_0^t g_1(t)dt \le \exp\left(\int_0^t g_3(s)ds\right)\times \qquad (7.9) \\
&\qquad\qquad \times\left(\left\|\mathbf{e}_h^u(0)\right\|_0^2 + \left\|\mathbf{e}_h^\theta(0)\right\|_0^2 + \int_0^t g_2(s)ds\right) \qquad (7.10)
\end{aligned}
$$

for all $t \in [0,T]$. At this point we have to find functions g_i which are contained in $L^1(0,T)$. We set

$$
\begin{aligned}
g_1(t) :=& \frac{1}{4}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{e}_h)\left\|\mathbb{D}\mathbf{e}_h^u\right\|_{L^2(K)}^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\varkappa_{\mathrm{mod}}^K(e_h^\theta)\left\|\nabla e_h^\theta\right\|_{L^2(K)}^2 \\
&+ \sum_{K\in\mathcal{T}_h}\gamma_K\left\|\nabla\cdot\mathbf{e}_h\right\|_{L^2(K)}^2,
\end{aligned}
$$

$$
\begin{aligned}
g_2(t) :=& 2\sum_{K\in\mathcal{T}_h}\left[\min\left(\frac{9C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)}, \frac{1}{\gamma_K}\right)\left(\left\|p-\lambda_h\right\|_{L^2(K)}^2 + \gamma_K^2\left\|\nabla\cdot\epsilon^u\right\|_{L^2(K)}^2\right)\right. \\
&+ 6\left(\nu + \nu_{\mathrm{VMS}}^K(\epsilon^u)\right)\left\|\mathbb{D}\epsilon^u\right\|_{L^2(K)}^2 + \left(2\varkappa + 4\varkappa_{\mathrm{VMS}}^K(\epsilon^\theta)\right)\left\|\nabla\epsilon^\theta\right\|_{L^2(K)}^2 \\
&\left. + 6\nu_{\mathrm{VMS}}^K(\mathbf{u})\left\|\mathbb{D}\mathbf{u}\right\|_{L^2(K)}^2 + 4\varkappa_{\mathrm{VMS}}^K(\theta)\left\|\nabla\theta\right\|_{L^2(K)}^2\right] \\
&+ \frac{6C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)}\left\|\partial_t\epsilon^u\right\|_{H^{-1}(\Omega)}^2 + \frac{4}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)}\left\|\partial_t\epsilon^\theta\right\|_{H^{-1}(\Omega)}^2 + \alpha\|\mathbf{g}\|_\infty\left\|\epsilon^\theta\right\|_0^2 \\
&+ \frac{6C_{LT}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)}\left(C_F C_{\mathrm{Ko}}\left\|\mathbb{D}\mathbf{u}\right\|_0^2 + \left\|\mathbf{u}_h\right\|_0\left\|\mathbb{D}\mathbf{u}_h\right\|_0\right)\left\|\mathbb{D}\epsilon^u\right\|_0^2 \\
&+ \frac{4C_1^2 C_{\mathrm{Ko}}}{\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)}\left(C_F C_{\mathrm{Ko}}\left\|\nabla\theta\right\|_0^2\left\|\mathbb{D}\epsilon^u\right\|_0^2 + \left\|\mathbf{u}_h\right\|_0\left\|\mathbb{D}\mathbf{u}_h\right\|_0\left\|\nabla\epsilon^\theta\right\|_0^2\right),
\end{aligned}
$$

$$g_3(t) := \frac{27C_{LT}^4}{2\inf_{t\in[0,T]}\left(\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)^3\right)}\|\mathbb{D}\mathbf{u}\|_0^4 + \frac{8C_1^4}{\inf_{t\in[0,T]}\left(\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)\ \varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)^2\right)}\|\nabla\theta\|_0^4 + 2\alpha\|\mathbf{g}\|_\infty .$$

With these definitions we have (7.8). By using (7.6) and if the $L^1(0,T)$ regularity is checked the setting will comply the requirements, because the other conditions are straightforward. To prove this we will use the stability result of Lemma 3.1 for the terms stemming from the nonlinearities in g_2, all the other terms are directly clear. We obtain

$$\begin{aligned}
&\int_0^t \|\mathbf{u}_h\|_0\|\mathbb{D}\mathbf{u}_h\|_0\|\mathbb{D}\epsilon^u\|_0^2\,dt \\
&\le \|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))}\|\mathbb{D}\mathbf{u}_h\|_{L^2(0,t;L^2(\Omega))}\|\mathbb{D}\epsilon^u\|_{L^4(0,t;L^2(\Omega))}^2 \\
&\le \sqrt{\frac{2}{\inf_{t\in[0,T]}\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h)}}\left(\|\mathbf{u}_0\|_0^2 + \frac{3}{2}\|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))}^2\right)\|\mathbb{D}\epsilon^u\|_{L^4(0,t;L^2(\Omega))}^2 < \infty,
\end{aligned}$$

$$\int_0^t \|\mathbb{D}\mathbf{u}\|_0^2\|\mathbb{D}\epsilon^u\|_0^2\,dt \le \|\mathbb{D}\mathbf{u}\|_{L^4(0,t;L^2(\Omega))}^2\|\mathbb{D}\epsilon^u\|_{L^4(0,t;L^2(\Omega))}^2 < \infty,$$

$$\int_0^t \|\mathbf{u}_h\|_0\|\mathbb{D}\mathbf{u}_h\|_0\left\|\nabla\epsilon^\theta\right\|_0^2 \le \|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))}\|\mathbb{D}\mathbf{u}_h\|_{L^2(0,t;L^2(\Omega))}\left\|\nabla\epsilon^\theta\right\|_{L^4(0,t;L^2(\Omega))}^2 < \infty,$$

and

$$\int_0^t \|\nabla\theta\|_0^2\|\mathbb{D}\epsilon^u\|_0^2 \le \|\nabla\theta\|_{L^4(0,t;L^2(\Omega))}^2\|\mathbb{D}\epsilon^u\|_{L^4(0,t;L^2(\Omega))}^2 < \infty$$

via Poincaré-Friedrichs inequality (2.2) and Hölder inequality. With the regularity assumptions of Theorem 3.4, in particular assumptions (3.2) and (3.5), we are now able to apply Gronwall's Lemma. This gives the estimate for the discretization error according to (7.9). Finally a careful consideration of the norm definition in (3.3)-(3.4) and the definition of $\nu_{\mathrm{VMS}}^K(\mathbf{e}_h), \varkappa_{\mathrm{VMS}}^K(e_h^\theta)$ shows

$$|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 + |[(\theta-\theta_h)(t)]|^2 \le 2\,|||\epsilon^u(t)|||^2 + 2\,|||\mathbf{e}_h^u(t)|||^2 + 2|[e_h^\theta(t)]|^2 + 2|[\epsilon^\theta(t)]|^2.$$

This concludes the proof.

Appendix C: Proof of the pressure estimate (Corollary 3.5)

The proof follows closely the proof of Corollary 3.6. in [36] with an additional term from the Boussinesq coupling. With the definitions $\mathbf{e}_u := \mathbf{u}_h - \mathbf{u}$ and $e_\theta := \theta_h - \theta$ we see that

$$\begin{aligned}
b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) - b_S(\mathbf{u},\mathbf{u},\mathbf{v}_h) &= b_S(\mathbf{e}_u,\mathbf{u}_h,\mathbf{v}_h) + b_S(\mathbf{u},\mathbf{e}_u,\mathbf{v}_h) \\
&\le C_{\mathrm{LT2}}\left(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0\right)\|\mathbb{D}\mathbf{e}_u\|_0\|\nabla\mathbf{v}_h\|_0
\end{aligned}$$

for all $\mathbf{v}_h \in V_h$, where the constant C_{LT2} comes from [28], Lemma 2.2 (e) together with Korn's inequality.

For the proof we subtract (2.1) from (2.6) with an arbitrary function $\mathbf{v}_h \in V_h$ and obtain

$$\begin{aligned}
&(\partial_t \mathbf{e}_u, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{e}_u, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b_S(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) \\
&+ \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)(\nabla\cdot\mathbf{e}_u, \nabla\cdot\mathbf{v}_h)_K + \alpha(\mathbf{g}e_\theta, \mathbf{v}_h) + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)(\kappa_u(\mathbb{D}\mathbf{e}_u), \mathbb{D}\mathbf{v}_h) \\
&\qquad + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)(\kappa_u(\mathbb{D}\mathbf{u}), \mathbb{D}\mathbf{v}_h) - (\nabla\cdot\mathbf{v}_h, (p_h - p)) = 0. \quad (7.11)
\end{aligned}$$

We split the pressure error as $p - p_h = (p - J_h p) + (J_h p - p_h)$. The discrete inf-sup condition (2.3) yields

$$\begin{aligned}
\beta\,\|p_h - p\|_0 &\le \beta\,\|p_h - J_h p\|_0 + \beta\,\|p - J_h p\|_0 \le \frac{(p_h - J_h p, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} + \beta\,\|p - J_h p\|_0 \\
&\le \frac{(p_h - p, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} + \frac{(p - J_h p, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} + \beta\,\|p - J_h p\|_0 \\
&\le \frac{(p_h - p, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} + (\beta + \sqrt{3})\,\|p - J_h p\|_0\,.
\end{aligned}$$

Furthermore, the remaining fraction can be estimated via (7.11) and the Cauchy-Schwarz inequality by

$$\begin{aligned}
\frac{(p_h - p, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} \le& \frac{1}{\|\nabla\mathbf{v}_h\|_0}\Bigg((\partial_t\mathbf{e}_u, \mathbf{v}_h) + 2\nu\,\|\mathbb{D}\mathbf{e}_u\|_0\,\|\mathbb{D}\mathbf{v}_h\|_0 + [b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b_S(\mathbf{u}, \mathbf{u}, \mathbf{v}_h)] \\
&+\alpha(\mathbf{g}e_\theta, \mathbf{v}_h) + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u\mathbb{D}\mathbf{e}_u\|_{L^2(K)}\,\|\mathbb{D}\mathbf{v}_h\|_{L^2(K)} \\
&+ \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u\mathbb{D}\mathbf{u}\|_{L^2(K)}\,\|\mathbb{D}\mathbf{v}_h\|_{L^2(K)} \\
&+ \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_u\|_{L^2(K)}\,\|\nabla\cdot\mathbf{v}_h\|_{L^2(K)} \Bigg) \\
\le& \frac{(\partial_t\mathbf{e}_u, \mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} + C_{\mathrm{LT2}}\,(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0)\,\|\mathbb{D}\mathbf{e}_u\|_0 + 2\nu\,\|\mathbb{D}\mathbf{e}_u\|_0 \\
&+C_F\alpha\,\|\mathbf{g}\|_\infty\,\|e_\theta\| + \sqrt{\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)\nu_{\mathrm{VMS}}^K(\mathbf{e_u})\,\|\mathbb{D}\mathbf{e}_u\|_{L^2(K)}^2} \\
&+ \sqrt{\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h,\theta_h)^2\,\|\kappa_u\mathbb{D}\mathbf{u}\|_{L^2(K)}^2} + \sqrt{3\sum_{K\in\mathcal{T}_h} \gamma_K^2(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_u\|_{L^2(K)}^2}.
\end{aligned}$$

That is why

$$\frac{(p_h - p, \nabla \cdot \mathbf{v}_h)^2}{7\,\|\nabla \mathbf{v}_h\|_0^2} \leq \frac{(\partial_t \mathbf{e}_u, \mathbf{v}_h)^2}{\|\nabla \mathbf{v}_h\|_0^2} + C_{\mathrm{LT2}}^2 \left(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0\right)^2 \|\mathbb{D}\mathbf{e}_u\|_0^2 + 4\nu^2 \|\mathbb{D}\mathbf{e}_u\|_0^2$$
$$+ C_F^2 \alpha^2 \|\mathbf{g}\|_\infty^2 \|e_\theta\|^2 + \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h, \theta_h) \nu_{\mathrm{VMS}}^K(\mathbf{e_u}) \|\mathbb{D}\mathbf{e}_u\|_{L^2(K)}^2$$
$$+ \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h, \theta_h)^2 \|\kappa_u \mathbb{D}\mathbf{u}\|_{L^2(K)}^2 + 3 \sum_{K \in \mathcal{T}_h} \gamma_K^2(\mathbf{u}_h) \|\nabla \cdot \mathbf{e}_u\|_{L^2(K)}^2 .$$

Integration over $(0, t)$ concludes the proof. In particular, the term stemming from the time derivative can be estimated by

$$\int_0^t \frac{(\partial_t \mathbf{e}_u, \mathbf{v}_h)^2}{\|\nabla \mathbf{v}_h^2\|_0} dt \leq \int_0^t \|\partial_t(\mathbf{u}_h - \mathbf{u})\|_{H^{-1}(\Omega)}^2 \, dt = \|\partial_t(\mathbf{u}_h - \mathbf{u})\|_{L^2(0,t;H^{-1}(\Omega))}^2$$

since $\mathbf{v}_h$ is not time-dependent.

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