A Local Projection Stabilization FEM for the linearized stationary MHD problem

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Abstract. We present a local projection stabilization (LPS) type finite element (FE) method for the linearized stationary magnetohydrodynamics (MHD) problem. In contrast to the residual-based stabilization in [1]-[2], we investigate a symmetric LPS method comparable to the term-by-term stabilization in [3].

1 Introduction

Following the time discretization and linearization approach in [1]-[2], we consider the stationary MHD model

$$-\nu\Delta\mathbf{u} + (\mathbf{a}\cdot\nabla)\mathbf{u} + \nabla p - (\nabla\times\mathbf{b})\times\mathbf{d} = \mathbf{f_u}\ ,\ \nabla\cdot\mathbf{u} = 0, \tag{1}$$

$$\lambda\nabla\times(\nabla\times\mathbf{b}) + \nabla r - \nabla\times(\mathbf{u}\times\mathbf{d}) = \mathbf{f_b}\ ,\ \nabla\cdot\mathbf{b} = 0, \tag{2}$$

in a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$ with $\nabla\cdot\mathbf{a}=0$. $\mathbf{a}$ and $\mathbf{d}$ are the vector-fields for the velocity and magnetic field at linearization. For the unknown velocity field $\mathbf{u}$, magnetic field $\mathbf{b}$, pressure p and magnetic pseudo-pressure r (vanishing in the continuous case), we introduce the function spaces

$$V = \left\{\mathbf{v}\in\left[H^1(\Omega)\right]^d\,:\,\mathbf{v}=0\ \text{on}\,\partial\Omega\right\}\quad,\quad Q = L_0^2(\Omega),$$
$$C = \{\mathbf{c}\in H(curl\,;\,\Omega)\,:\,\mathbf{n}\times\mathbf{c}=0\ \text{on}\,\partial\Omega\}\quad,\quad S = H_0^1(\Omega)$$

where $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are appropriate inner and dual products. The variational problem reads: Find $\mathbf{U}:=(\mathbf{u},\mathbf{b},p,r)\in V\times C\times Q\times S$ such that

$$\mathcal{A}_G(\mathbf{U},\mathbf{V}) = \mathcal{F}_G(\mathbf{V}),\quad \forall\mathbf{V}:=(\mathbf{v},\mathbf{c},q,s)\in V\times C\times Q\times S \tag{3}$$

with

$$\begin{aligned}\mathcal{A}_G(\mathbf{U},\mathbf{V}) = {}& \nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \langle\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v}\rangle - (p,\nabla\cdot\mathbf{v}) - \langle(\nabla\times\mathbf{b})\times\mathbf{d},\mathbf{v}\rangle\\ &+ (\nabla\cdot\mathbf{u},q) - (\mathbf{b},\nabla s)\\ &+\lambda(\nabla\times\mathbf{b},\nabla\times\mathbf{c}) + (\nabla r,\mathbf{c}) - \langle\nabla\times(\mathbf{u}\times\mathbf{d}),\mathbf{c}\rangle,\end{aligned} \tag{4}$$

$$\mathcal{F}_G(\mathbf{V}) = \langle\mathbf{f_u},\mathbf{v}\rangle + \langle\mathbf{f_b},\mathbf{c}\rangle. \tag{5}$$

Let $\mathcal{T}_h$ be the primal grid with FE spaces of Taylor-Hood type

$$V_h\times Q_h/C_h\times S_h = \mathbb{P}^k_{\mathcal{T}_h}\times\mathbb{P}^{k-1}_{\mathcal{T}_h}\ \text{or}\ \mathbb{Q}^k_{\mathcal{T}_h}\times\mathbb{Q}^{k-1}_{\mathcal{T}_h},\quad k\in\mathbb{N}\setminus\{1\}. \tag{6}$$

The pair $V_h \times Q_h$ is discretely-divergence-free, thus

$$V_h^{div} := \{\mathbf{v_h} \in V_h : \ (\nabla \cdot \mathbf{v}_h, q_h) = 0 \ \forall q_h \in Q_h\} \neq \{0\}.$$

Let $\mathcal{M}_h = \mathcal{T}_h$ or $\mathcal{M}_h = \mathcal{T}_{2h}$ be the macro grid with discontinuous FE spaces $D_h^{u/b} \subset [L^2(\Omega)]^d$. The local orthogonal L^2-projectors are denoted as $\pi_M^{u/b} : [L^2(M)]^d \to D_h^{u/b}|_M$. The global projections $\pi_h^{u/b} : [L^2(\Omega)]^d \to D_h^{u/b}$ are given as $(\pi_h^{u/b}\mathbf{w})|_M := \pi_M^{u/b}(\mathbf{w}|_M)$. The fluctuation operator $\kappa_h^{u/b} : [L^2(\Omega)]^d \to [L^2(\Omega)]^d$ with $\kappa_h^{u/b}\mathbf{w} := \left((id - \pi_h^{u/b})w_i\right)_{i=1}^d$ is assumed to have the approximation property

$$\|\kappa_h^{u/b}\mathbf{v}\|_{0,M} \leq Ch_M^l\|\mathbf{v}\|_{l,M} \quad \forall \mathbf{v} \in [W^{l,2}(M)]^d, l = 0, \ldots, s, \ s \in \{0, \ldots, k\}. \tag{7}$$

Let $\mathbf{U_h} = (\mathbf{u_h}, \mathbf{b_h}, p_h, r_h)\,, \mathbf{V_h} = (\mathbf{v_h}, \mathbf{c_h}, q_h, s_h) \in V_h \times C_h \times Q_h \times S_h \subset V \times C \times Q \times S$. Then the LPS terms read

$$\begin{aligned}
\mathcal{S}_{lps}(\mathbf{U_h}, \mathbf{V_h}) = \sum_M \{&\tau_1 \left(\kappa_h^u((\mathbf{a_M} \cdot \nabla)\mathbf{u_h}), \kappa_h^u((\mathbf{a_M} \cdot \nabla)\mathbf{v_h})\right)_M + \tau_2 \left(\nabla \cdot \mathbf{u_h}, \nabla \cdot \mathbf{v_h}\right)_M \\
&+\tau_3 \left(\kappa_h^b((\nabla \times \mathbf{b_h}) \times \mathbf{d_M}), \kappa_h^b((\nabla \times \mathbf{c_h}) \times \mathbf{d_M})\right)_M \\
&+\tau_4 \left(\kappa_h^b((\nabla \times (\mathbf{u_h} \times \mathbf{d_M})), \kappa_h^b(\nabla \times (\mathbf{v_h} \times \mathbf{d_M}))\right)_M \\
&+ \tau_5 \left(\nabla r_h, \nabla s_h\right)_M + \tau_6 \left(\nabla \cdot \mathbf{b_h}, \nabla \cdot \mathbf{c_h}\right)_M\}
\end{aligned}$$

where $(\cdot, \cdot)_M$ is the L^2 scalar product on cell M. Here, $\mathbf{a_M}$ and $\mathbf{d_M}$ are elementwise constant approximations of $\mathbf{a}|_\mathbf{M}$ and $\mathbf{d}|_\mathbf{M}$. The LPS problem consists of finding $\mathbf{U_h} \in V_h \times C_h \times Q_h \times S_h$ such that for all $\mathbf{V_h} \in V_h \times C_h \times Q_h \times S_h$:

$$\mathcal{A}_{stab}(\mathbf{U_h}, \mathbf{V_h}) = \mathcal{A}_G(\mathbf{U_h}, \mathbf{V_h}) + \mathcal{S}_{lps}(\mathbf{U_h}, \mathbf{V_h}) = \mathcal{F}_G\left(\mathbf{V_h}\right). \tag{8}$$

2 Stability of the proposed method

For $k \in \mathbb{N}_0$ and $D \subseteq \Omega$, we use the notation $|\cdot|_{k,D} := |\cdot|_{H^k(D)}$ and $\|\cdot\|_{p,D} := \|\cdot\|_{L^p(D)}$ with $1 \leq p \leq \infty$. In case of $D = \Omega$, we omit index D.

Lemma 1. *For $\mathbf{U}, \mathbf{V} \in V \times C \times Q \times S$, it holds for the symmetric LPS terms*

$$(i) \quad \mathcal{S}_{lps}\left(\mathbf{U}, \mathbf{U}\right) \geq 0, \quad (ii) \quad |\mathcal{S}_{lps}\left(\mathbf{U}, \mathbf{V}\right)| \leq \left(\mathcal{S}_{lps}\left(\mathbf{U}, \mathbf{U}\right)\right)^{\frac{1}{2}} \left(\mathcal{S}_{lps}\left(\mathbf{V}, \mathbf{V}\right)\right)^{\frac{1}{2}}.$$

Let $\mathbf{V} = (\mathbf{v}, \mathbf{c}, q, s) \in V \times C \times Q \times S$. Integration by parts yields $\langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{v}\rangle = 0$ and $\langle (\nabla \times \mathbf{c}) \times \mathbf{d}, \mathbf{v}\rangle = -\langle \nabla \times (\mathbf{v} \times \mathbf{d}), \mathbf{c}\rangle$, hence

$$\mathcal{A}_G\left(\mathbf{V}, \mathbf{V}\right) = \nu\|\nabla \mathbf{v}\|_0^2 + \lambda\|\nabla \times \mathbf{c}\|_0^2.$$

We define the following expressions

$$\|\mathbf{V}\|_{\mathrm{G}}^2 = \nu\|\nabla \mathbf{v}\|_0^2 + \lambda\|\nabla \times \mathbf{c}\|_0^2, \qquad \|\mathbf{V}\|_{\mathrm{lps}}^2 = \mathcal{S}_{lps}\left(\mathbf{V}, \mathbf{V}\right). \tag{9}$$

Symmetric testing $\mathbf{V_h} = \mathbf{U_h}$ yields

$$(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{V_h}, \mathbf{V_h}) = \|\mathbf{V_h}\|_{\mathrm{G}}^2 + \|\mathbf{V_h}\|_{\mathrm{lps}}^2. \tag{10}$$

Using Young's inequality and the definition

$$\|(\mathbf{f_u}, \mathbf{f_b})\|_{G,\star} = \sup_{(v_h, c_h)\in V_h\times C_h} \mathcal{F}_G(\mathbf{V_h}) / \|\mathbf{V_h}\|_G,$$

we get the unique existence of the velocity field via

$$\|\mathbf{U_h}\|_{\mathrm{G}}^2 + 2\|\mathbf{U_h}\|_{\mathrm{lps}}^2 \leq \|(\mathbf{f_u}, \mathbf{f_b})\|_{G,\star}^2. \tag{11}$$

By the discrete Babuška-Brezzi-condition, we have for all $p_h \in Q_h$ the unique existence of $\mathbf{v_h} \in V_h$ with

$$\nabla\cdot\mathbf{v_h} = -p_h\ ,\ |\mathbf{v_h}|_1 \leq \beta_u^{-1}\|p_h\|_0. \tag{12}$$

Examining the term $(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U_h}, (\mathbf{v_h}, \mathbf{0}, 0, 0))$, we end up with

$$\begin{aligned}\|p_h\|_0^2 \leq \|\mathbf{f_u}\|_{-1}|\mathbf{v_h}|_1 + \nu\|\nabla\mathbf{u_h}\|_0|\mathbf{v_h}|_1 + |\mathcal{S}_{lps}((\mathbf{u_h}, \mathbf{0}, 0, 0), (\mathbf{v_h}, \mathbf{0}, 0, 0))| \\ -(\mathbf{a}\cdot\nabla\mathbf{u_h}, \mathbf{v_h}) + ((\nabla\times\mathbf{b_h})\times\mathbf{d}, \mathbf{v_h})\end{aligned}$$

by using that $\mathbf{u_h} \in V_h^{div}$. Based on the inequalities

$$-(\mathbf{a}\cdot\nabla\mathbf{u_h}, \mathbf{v_h}) = (\mathbf{a}\cdot\nabla\mathbf{v_h}, \mathbf{u_h}) \leq C_p\|\mathbf{a}\|_\infty|\mathbf{u_h}|_1|\mathbf{v_h}|_1, \tag{13}$$

$$((\nabla\times\mathbf{b_h})\times\mathbf{d}, \mathbf{v_h}) \leq C_p\|\mathbf{d}\|_\infty\|\nabla\times\mathbf{b_h}\|_0|\mathbf{v_h}|_1, \tag{14}$$

$$\|(\mathbf{v_h}, \mathbf{0}, 0, 0)\|_{\mathrm{lps}}^2 \leq \max_M\left(\tau_1|\mathbf{a_M}|^2 + \tau_2 d + \tau_4|\mathbf{d}_M|^2\right)|\mathbf{v_h}|_1^2 \tag{15}$$

together with (12), we obtain after some calculation the unique existence of the fluid pressure thanks to

$$\begin{aligned}\beta_u\|p_h\|_0 \leq \|\mathbf{f_u}\|_{-1} + \left(\sqrt{\nu} + \frac{C_p\|\mathbf{a}\|_\infty}{\sqrt{\nu}} + \frac{C_p\|\mathbf{d}\|_\infty}{\sqrt{\lambda}}\right)\|\mathbf{U_h}\|_G \\ + \left(\max_M(\sqrt{\tau_1}|\mathbf{a_M}|) + \max_M\sqrt{\tau_2 d} + \sqrt{\tau_4}|\mathbf{d}_M|\right)\|\mathbf{U_h}\|_{\mathrm{lps}}.\end{aligned}$$

We define the norms

$$\|\mathbf{c}\|_C = \sqrt{\lambda}\left(L_0^{-1}\|\mathbf{c}\|_0 + \|\nabla\times\mathbf{c}\|_0\right)\ ,\ \|s\|_S = (\|s\|_0 + L_0\|\nabla s\|_0)/\sqrt{\lambda} \tag{16}$$

on the spaces C and S with a length-scale $L_0 \sim \mathrm{diam}(\Omega)$. Using integration by parts, we define the bilinear form of the Maxwell problem

$$\mathcal{C}_{Max}((\mathbf{b}, r), (\mathbf{c}, s)) = \mathcal{A}_G((\mathbf{0}, \mathbf{b}, 0, r), (\mathbf{0}, \mathbf{c}, 0, s)).$$

For this problem, the continuous Babuška-Brezzi-condition

$$\inf_{(\mathbf{b},r)\in C\times S}\sup_{(\mathbf{c},s)\in C\times S}\frac{\mathcal{C}_{Max}((\mathbf{b}, r), (\mathbf{c}, s))}{(\|\mathbf{b}\|_C + \|r\|_S)(\|\mathbf{c}\|_C + \|s\|_S)} \geq \beta_m \tag{17}$$

holds. Let $(\mathbf{b_h}, 0) \in C_h \times S_h \subset C \times S$. By (17) there exists a unique $(\overline{\mathbf{c}}, \overline{s}) \in C \times S$ with $\|\overline{\mathbf{c}}\|_C + \|\overline{s}\|_S = 1$ such that

$$\begin{aligned}\beta_m \|\mathbf{b_h}\|_C &\leq \mathcal{C}_{Max}\left((\mathbf{b_h}, 0), (\overline{\mathbf{c}}, \overline{s})\right) \\ &= \mathcal{C}_{Max}\left((\mathbf{b_h}, 0), (\overline{\mathbf{c}}, 0)\right) + \mathcal{C}_{Max}\left((\mathbf{b_h}, 0), (0, \overline{s})\right) =: I + II\end{aligned} \tag{18}$$

holds. We get with $\|\overline{s}\|_S \leq 1$ and $\|\overline{\mathbf{c}}\|_C \leq 1$ the estimates

$$I = \lambda \left(\nabla \times \mathbf{b_h}, \nabla \times \overline{\mathbf{c}}\right) \leq \sqrt{\lambda}\|\nabla \times \mathbf{b_h}\|_0 \sqrt{\lambda}\|\nabla \times \overline{\mathbf{c}}\|_0 \leq \|\mathbf{U_h}\|_G, \tag{19}$$

$$\begin{aligned}II = |(\nabla \cdot \mathbf{b_h}, \overline{s})| &\leq \Big(\sum_M \tau_6 \|\nabla \cdot \mathbf{b_h}\|_{0,M}^2\Big)^{\frac{1}{2}} \Big(\sum_M \frac{1}{\tau_6} \|\overline{s}\|_{0,M}^2\Big)^{\frac{1}{2}} \\ &\leq \|\mathbf{U_h}\|_{lps} \cdot \Big(\max_M \frac{1}{\sqrt{\tau_6}}\Big) \cdot \sqrt{\lambda} \frac{\|\overline{s}\|_0}{\sqrt{\lambda}} = \max_M \sqrt{\frac{\lambda}{\tau_6}} \cdot \|\mathbf{U_h}\|_{\text{lps}}\end{aligned} \tag{20}$$

by the Cauchy-Schwarz inequality. Putting (19) and (20) into (18) leads to

$$\|\mathbf{b_h}\|_C \leq \frac{1}{\beta_m}\Big(\|\mathbf{U_h}\|_G + \max_M \sqrt{\frac{\lambda}{\tau_6}} \|\mathbf{U_h}\|_{\text{lps}}\Big) \tag{21}$$

and this gives the unique existence of the discrete magnetic field.

Finally, from the equation $(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U_h}, (\mathbf{0}, \mathbf{0}, 0, r_h)) = -(\mathbf{b_h}, \nabla r_h) + \sum_M \tau_5 \|\nabla r_h\|_{0,M}^2 = 0$, we conclude by using (16) that

$$\|\nabla r_h\|_0 \leq \Big(\min_M \sqrt{\tau_5}\Big)^{-1} \|\mathbf{b_h}\|_0 \leq L_0 \Big(\sqrt{\lambda} \min_M \sqrt{\tau_5}\Big)^{-1} \|\mathbf{b_h}\|_C. \tag{22}$$

This implies existence of the unique discrete magnetic pseudo-pressure. As full control of ∇r_h is essential to enforce condition $\nabla \cdot \mathbf{b}_h = 0$, (22) and (21) suggest the choices

$$\tau_5 \sim L_0^2/\lambda, \qquad \tau_6 \geq C\lambda. \tag{23}$$

3 Error analysis for smooth solutions

Subtracting (3) and (8) gives the approximate Galerkin orthogonality.

Lemma 2. *Let* $\mathbf{U}$ *and* $\mathbf{U_h}$ *be the solutions of (3) and (8). Then*

$$(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U} - \mathbf{U_h}, \mathbf{V_h}) = \mathcal{S}_{lps}(\mathbf{U}, \mathbf{V_h}), \quad \forall \mathbf{V_h} \in V_h \times C_h \times Q_h \times S_h. \tag{24}$$

Let $\mathbf{J} = (\mathbf{j^u}, \mathbf{j^b}, j^p, j^r)$ be appropriate interpolation operators. In particular, we assume that $\mathbf{j^u u} \in V_h^{div}$. We therefore decompose the error as

$$\mathbf{U} - \mathbf{U_h} = (\mathbf{U} - \mathbf{JU}) + (\mathbf{JU} - \mathbf{U_h}) = \varepsilon + \mathbf{E_h} \equiv (\varepsilon_{\mathbf{u}}, \varepsilon_{\mathbf{b}}, \varepsilon_p, \varepsilon_r) + (\mathbf{e_u}, \mathbf{e_b}, e_p, e_r).$$

Set now $\mathbf{V_h} = \mathbf{E_h}$ in (24), thus

$$\|\mathbf{E_h}\|_G^2 + \|\mathbf{E_h}\|_{lps}^2 = \underbrace{\mathcal{S}_{lps}(\mathbf{U}, \mathbf{E_h})}_{=I} \underbrace{-\mathcal{A}_G(\varepsilon, \mathbf{E_h})}_{=II} - \underbrace{\mathcal{S}_{lps}(\varepsilon, \mathbf{E_h})}_{=-III}. \tag{25}$$

We obtain

$$I \leq \left(\mathcal{S}_{lps}\left(\mathbf{U},\mathbf{U}\right)\right)^{\frac{1}{2}}\left(\mathcal{S}_{lps}\left(\mathbf{E_h},\mathbf{E_h}\right)\right)^{\frac{1}{2}} = \|\mathbf{U}\|_{lps}\|\mathbf{E_h}\|_{lps}, \tag{26}$$

$$|III| = \mathcal{S}_{lps}\left(\varepsilon,\mathbf{E_h}\right) \leq \|\varepsilon\|_{lps}\|\mathbf{E_h}\|_{lps}, \tag{27}$$

$$-II \leq \|\varepsilon\|_G\|\mathbf{E_h}\|_G + IV \tag{28}$$

$$IV = \left(\mathbf{a}\cdot\nabla\varepsilon_{\mathbf{u}},\mathbf{e_u}\right) - \left(\left(\nabla\times\varepsilon_{\mathbf{b}}\right)\times\mathbf{d},\mathbf{e_u}\right) - \left(\nabla\times\left(\varepsilon_{\mathbf{u}}\times\mathbf{d}\right),\mathbf{e_b}\right) \tag{29}$$
$$- \left(\varepsilon_p,\nabla\cdot\mathbf{e_u}\right) + \left(\nabla\cdot\varepsilon_{\mathbf{u}},\mathbf{e}_p\right) + \left(\nabla\varepsilon_r,\mathbf{e_p}\right) - \left(\varepsilon_{\mathbf{b}},\nabla\mathbf{e}_r\right).$$

Then we can summarize estimates (25)-(29) as

$$\|\mathbf{E_h}\|_G^2 + \|\mathbf{E_h}\|_{lps}^2 \leq \left(\|\varepsilon\|_{lps} + \|\mathbf{U}\|_{lps}\right)\|\mathbf{E_h}\|_{lps} + \|\varepsilon\|_G\|\mathbf{E_h}\|_G + |IV|. \tag{30}$$

Integration by parts and Cauchy-Schwarz inequality give for the terms in IV:

$$\left(\mathbf{a}\cdot\nabla\varepsilon_{\mathbf{u}},\mathbf{e_u}\right) = -\Big(\mathbf{a}\cdot\nabla\mathbf{e_u},\varepsilon_{\mathbf{u}}\Big) \leq \Big(\sum_M \frac{\|\mathbf{a}\|_{\infty,M}^2}{\nu}\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\Big)^{\frac{1}{2}}\|\mathbf{E_h}\|_G,$$

$$-\Big(\varepsilon_p,\nabla\cdot\mathbf{e_u}\Big) \leq \Big(\sum_M \min\left(\frac{d}{\nu};\frac{1}{\tau_2}\right)\|\varepsilon_p\|_{0,M}^2\Big)^{\frac{1}{2}}\|\mathbf{E_h}\|_{lps},$$

$$-\left(\varepsilon_{\mathbf{b}},\nabla e_r\right) \leq \Big(\sum_M \frac{1}{\tau_5}\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}}\|\mathbf{E_h}\|_{lps}, \tag{31}$$

$$-\left(\nabla\times\left(\varepsilon_{\mathbf{u}}\times\mathbf{d}\right),\mathbf{e_b}\right) = \left(\varepsilon_{\mathbf{u}},\left(\nabla\times\mathbf{e_b}\right)\times\mathbf{d}\right) \leq \Big(\sum_M \frac{\|\mathbf{d}\|_{\infty,M}^2}{\lambda}\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\Big)^{\frac{1}{2}}\|\mathbf{E_h}\|_G.$$

The term $(\nabla\varepsilon_r,\mathbf{e_b})$ vanishes since $r = j^r r \equiv 0$. Moreover, term $-(e_p,\nabla\cdot\varepsilon_{\mathbf{u}})$ vanishes via $\nabla\cdot\mathbf{u} = 0$ and since $\mathbf{j^u u} \in V_h^{div}$. Let $\mathbf{d} \in \left[\mathrm{W}^{1,\infty}(\Omega)\right]^d$. By formula $\nabla\times(\mathbf{e}\times\mathbf{f}) = \mathbf{f}\cdot\nabla\mathbf{e} - \mathbf{f}\,(\nabla\cdot\mathbf{e}) - \mathbf{e}\cdot\nabla\mathbf{f} + \mathbf{e}\,(\nabla\cdot\mathbf{f})$, the inequalities of Cauchy, Schwarz and Poincare, it follows

$$-\left(\left(\nabla\times\varepsilon_{\mathbf{b}}\right)\times\mathbf{d},\mathbf{e_u}\right) = \sum_M \left(\varepsilon_{\mathbf{b}},\nabla\times\left(\mathbf{e_u}\times\mathbf{d}\right)\right)_M$$
$$\leq \Big(\sum_M \nu^{-1}\big(1+\sqrt{d}\big)^2\big(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}\big)^2\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}}\|\mathbf{E_h}\|_G. \tag{32}$$

We then summarize equations (30)-(32). Using Young's inequality, we obtain

$$\|\mathbf{E_h}\|_G^2 + \|\mathbf{E_h}\|_{lps}^2 \leq S_1^2 + S_2^2,$$

with

$$S_1 := \|\varepsilon\|_G + \Big(\sum_M \frac{1}{\nu}\|\mathbf{a}\|_{\infty,M}^2\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\Big)^{\frac{1}{2}} + \Big(\sum_M \frac{1}{\lambda}\|\mathbf{d}\|_{\infty,M}^2\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\Big)^{\frac{1}{2}}$$
$$+\Big(\sum_M \nu^{-1}\big(1+\sqrt{d}\big)^2\big(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}\big)^2\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}},$$

$$S_2 := \|\varepsilon\|_{lps} + \|\mathbf{U}\|_{lps} + \Big(\sum_M \min\Big(\frac{d}{\nu};\frac{1}{\tau_2}\Big)\|\varepsilon_{\mathbf{p}}\|_{0,M}^2\Big)^{\frac{1}{2}} + \Big(\sum_M \frac{1}{\tau_5}\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}}.$$

The approximation properties of the FE spaces, see [5], and the local L^2-projector yield for $\mathbf{U} \in [H^{k+1}(\Omega)]^d \times [H^{k+1}(\Omega)]^d \times H^k(\Omega) \times H^k(\Omega)$ that

$$\begin{aligned} S_1^2 &\le C\sum_M h_M^{2k}\Big[\Big(\nu\big(1+\frac{\|\mathbf{a}\|_{\infty,M}^2 h_M^2}{\nu^2}\big) + \lambda\frac{\|\mathbf{d}\|_{\infty,M}^2 h_M^2}{\lambda^2}\big)\Big)|\mathbf{u}|_{k+1,\omega_M}^2 \\ &\quad + \Big(\lambda + \frac{h_M^2}{\nu}\big(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}\big)^2\Big)|\mathbf{b}|_{k+1,\omega_M}^2\Big], \qquad (33)\\ S_2^2 &\le C\sum_M h_M^{2s}\Big[\big(\tau_2 d^2 + \tau_1|\mathbf{a_M}|^2 + \tau_4|\mathbf{d_M}|^2\big)|\mathbf{u}|_{s+1,\omega_M}^2 \\ &\quad + \min\Big(\frac{d}{\nu};\frac{1}{\tau_2}\Big)|p|_{s,\omega_M}^2 + \big(\tau_3|\mathbf{d_M}|^2 + \tau_6 d^2 + \frac{h_M^2}{\tau_5}\big)|\mathbf{b}|_{s+1,\omega_M}^2\Big] \qquad (34)\end{aligned}$$

where ω_M denotes an appropriate patch around cell M.

Denote the local fluid and magnetic Reynolds numbers by

$$Re_{f,M} := \|\mathbf{a}\|_{\infty,M} h_M/\nu, \qquad Re_{m,M} := \|\mathbf{d}\|_{\infty,M} h_M/\lambda.$$

respectively. We will call an error estimate to be of order k if the coefficients multiplying corresponding Sobolev norms of the solutions are of order h^k uniformly w.r.t. the problem data. In this case, sufficient conditions can be found by the following (mild) restrictions on the local mesh width h_M

$$\sqrt{\nu}Re_{f,M} \le C, \quad \sqrt{\lambda}Re_{m,M} \le C, \quad h_M(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}) \le C\sqrt{\nu} \qquad (35)$$

and on the stabilization parameters (by using (7))

$$0 \le \tau_1 \le Ch_M^{2(k-s)}/|\mathbf{a_M}|^2, \quad 0 \le \tau_3, \tau_4 \le Ch_M^{2(k-s)}/|\mathbf{d_M}|^2, \quad Ch_M^2 \le \tau_5. \qquad (36)$$

Condition (23) implies the latter condition on τ_5. Moreover, (34) suggests the balance $\tau_5\tau_6 \sim h_M^2$, thus (see also [1],[2])

$$\tau_5 \sim L_0^2/\lambda, \qquad \tau_6 \sim h_M^2\lambda/L_0^2. \qquad (37)$$

A balance of the terms with the div-div parameter τ_2 leads to the practically unfeasible formula $\tau_2 \sim \max\big(0; |p|_{k,M}/|\mathbf{u}|_{k+1,M} - \nu\big)$. A reasonable compromise is to set

$$\tau_2 \sim 1. \qquad (38)$$

Theorem 3. *Assume that the solution $(\mathbf{u}, \mathbf{b}, p)$ of (3) belongs to $[H^{k+1}(\Omega)]^d \times [H^{k+1}(\Omega)]^d \times H^k(\Omega)$ and that $\mathbf{j^u u} \in V_h^{div}$. Further, let the LPS parameters be chosen according to condition (36)-(37) and that the local mesh width h_M is chosen such that (35) is valid. Then we obtain (using $r \equiv 0$)*

$$\|\mathbf{U_h} - \mathbf{JU}\|_{\mathrm{G}}^2 + \|\mathbf{U_h} - \mathbf{JU}\|_{\mathrm{lps}}^2 \le C\sum_M h_M^{2k}\Big(|\mathbf{u}|_{k+1,\omega_M}^2 + |\mathbf{b}|_{k+1,\omega_M}^2 + |p|_{k,\omega_M}^2\Big).$$

Numerical results for the magnetic part, i.e. $\mathbf{u} \equiv 0, p \equiv 0$, show the relevance of the parameter design (37) for Taylor-Hood type pairs $C_h \times S_h$. In particular, this is valid if the magnetic field $\mathbf{b}$ does not belong to $[H^1(\Omega)]^d$. Such singular solutions can be well approximated on meshes with suitable macro-element structure, like cross-box elements, see [1]. Our results confirm this for Taylor-Hood type pairs $C_h \times S_h$ as well.

Numerical experiments for the fluid part, i.e. $\mathbf{b} = \mathbf{0}, r = 0$, see [4], show: The mesh conditions (35) are much less restrictive than the typical ones on the local Peclet number $Pe_M := h_M \|\mathbf{a}\|_{\infty,M}/\nu \le 1$ in the Galerkin method for advection-diffusion problems. The div-div stabilization term is very important for robust estimates in case of Taylor-Hood elements. Compared to the Galerkin method, much better local mass conservation clearly improves the H^1- and L^2-error rates for velocity $\mathbf{u}_h$. Increasing values of $Re_f := \|\mathbf{a}\|_\infty C_P/\nu$ can lead to order reduction. Nevertheless, the choice of the div-div parameters τ_2 is still a question of ongoing discussion. It turns out that the SUPG-stabilization is much less important than div-div stabilization, thus showing the surprising robustness of the Galerkin-FEM with div-div stabilization in case of inf-sup stable pairs $V_h \times Q_h$.

4 Improved error estimates

The restrictions (35) on the mesh width are not convincing. Let us assume the following orthogonality conditions

$$(\mathbf{v} - \mathbf{j}^{\mathbf{u}}\mathbf{v}, \zeta_{\mathbf{h}}) = 0 \quad \forall \mathbf{v} \in V \quad \text{and} \quad \forall \zeta_{\mathbf{h}} \in [D_h^{\mathbf{u}}(M)], \tag{39}$$

$$\left(\mathbf{c} - \mathbf{j}^{\mathbf{b}}\mathbf{c}, \eta_{\mathbf{h}}\right) = 0 \quad \forall \mathbf{c} \in C \quad \text{and} \quad \forall \eta_{\mathbf{h}} \in \left[D_h^{\mathbf{b}}(M)\right]. \tag{40}$$

Sufficient conditions on $\mathcal{T}_h, \mathcal{M}_h$, the FE and projection spaces for (39)-(40) can be found in [6] or [4]. In particular, for the one-level approach with $\mathcal{T}_h = \mathcal{M}_h$, one has to enrich the velocity space by local bubble functions [6]. Another implication is that $\mathbf{j}^{\mathbf{u}}\mathbf{u} \notin V_h^{div}$, hence the mixed term $(e_p, \nabla \cdot \varepsilon_{\mathbf{u}})$ has to be considered. Moreover, a careful selection of the pressure spaces Q_h is required. The critical mixed term vanishes for continuous pressure space $Q_h = \mathbb{P}_{k-1}$. In case of discontinuous space $Q_h = \mathbb{P}_{-(k-1)}$, one can introduce additional pressure jump terms across interior edges to handle it, see [4].

(39)-(40) allow modified estimates of the skew-symmetric terms

$$(\mathbf{a} \cdot \nabla \varepsilon_{\mathbf{u}}, \mathbf{e}_{\mathbf{u}}) = -\Big(\kappa_h^u (\mathbf{a} \cdot \nabla \mathbf{e}_{\mathbf{u}}), \varepsilon_{\mathbf{u}}\Big) \le \Big(\sum_M \frac{1}{\tau_1} \|\varepsilon_{\mathbf{u}}\|_{0,M}^2\Big)^{\frac{1}{2}} \|\mathbf{E}_{\mathbf{h}}\|_{lps},$$

$$-\left((\nabla \times \varepsilon_{\mathbf{b}}) \times \mathbf{d}, \mathbf{e}_{\mathbf{u}}\right) = \Big(\varepsilon_{\mathbf{b}}, \kappa_h^b \big(\nabla \times (\mathbf{e}_{\mathbf{u}} \times \mathbf{d})\big)\Big) \le \Big(\sum_M \frac{1}{\tau_4} \|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}} \|\mathbf{E}_{\mathbf{h}}\|_{lps},$$

$$-\left(\nabla \times (\varepsilon_{\mathbf{u}} \times \mathbf{d}), \mathbf{e}_{\mathbf{b}}\right) = \left(\varepsilon_{\mathbf{u}}, \kappa_h^b \left((\nabla \times \mathbf{e}_{\mathbf{b}}) \times \mathbf{d}\right)\right) \le \Big(\sum_M \frac{1}{\tau_3} \|\varepsilon_{\mathbf{u}}\|_{0,M}^2\Big)^{\frac{1}{2}} \|\mathbf{E}_{\mathbf{h}}\|_{lps}.$$

Then, a modification of (33) leads to

$$S_1^2 \leq C \sum_M h_M^{2k} \Big[\Big(\nu + \frac{h_M^2}{\tau_1} + \frac{h_M^2}{\tau_3} \Big) |\mathbf{u}|^2_{k+1,M} + \Big(\lambda + \frac{h_M^2}{\tau_4} \Big) |\mathbf{b}|^2_{k+1,M} \Big]. \qquad (41)$$

Preserving the choice of div-div parameters according to (38) and of (37), a calibration of the parameters in (41) and (34) gives

$$Ch_M^2 \leq \tau_1 \leq C/|\mathbf{a}_M|^2, \qquad Ch_M^2 \leq \tau_3, \tau_4 \leq C/|\mathbf{d}_M|^2, \qquad (42)$$

and allows to omit the restrictions (35). A careful estimation has to consider the approximation of $\mathbf{a_M} \sim \mathbf{a}$ and $\mathbf{d_M} \sim \mathbf{d}$. For simplicity, we assume here elementwise constant fields $\mathbf{a}|_M = \mathbf{a_M}$ and $\mathbf{d}|_M = \mathbf{b_M}$.

Theorem 4. *Let the orthogonality conditions (39)-(40) be valid. Assume that the solution* $(\mathbf{u}, \mathbf{b}, p)$ *of (3) belongs to* $[H^{k+1}(\Omega)]^d \times [H^{k+1}(\Omega)]^d \times H^k(\Omega)$*. Further, let the LPS parameters be chosen according to conditions (38), (37) and (42). Then we obtain the quasi-optimal error estimate in Theorem 3 without the mesh-width restrictions (35).*

Numerical experiments for the fluid part, i.e. $\mathbf{b} = \mathbf{0}, r = 0$, show: One can omit restriction (35) if conditions (39)-(40) are valid, see [4]. The experiments indicate that optimal error estimates for the H^1- and L^2-error rates for the velocity $\mathbf{u}_h$ are obtained which are robust with respect to Re_f.

Corresponding numerical experiments for the magnetic part and the full MHD problem are in preparation and will be reported elsewhere.

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