Layer-adapted meshes vs. weak Dirichlet conditions in low-turbulent flow simulation

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1 Introduction

We apply a projection-based variational multiscale (VMS) method, originally published in [5], to the simulation of low-turbulent, wall-bounded, incompressible flow. Our approach relies on inf-sup stable finite element pairs for velocity/ pressure. The (semidiscrete) a-priori analysis of this method, given in [7], allows rather general nonlinear, piecewise constant coefficients of the subgrid models for the unresolved scales. The analysis takes advantage of divergence preserving interpolation which had been considered in [3] for the case of simplicial isotropic meshes. An extension to anisotropic quadrilateral meshes has been recently considered in [2].

Here we consider two approaches: (i) layer-adapted anisotropic quadrilateral grids (see, e.g., [4]), and (ii) weakly-enforced Dirichlet boundary conditions on isotropic meshes (see, e.g., [1]). For the coefficients of the projection-based VMS method we apply an approach motivated by arguments of the popular Smagorinsky model. Based on the results in [7, 2], we show the applicability of the recent a-priori analysis for VMS methods to both approaches. Numerical results for both variants are presented for the well-known benchmark of low-turbulent flow in a three-dimensional channel at $Re_\tau = 180$.

2 Variational multiscale model of Navier-Stokes problem

Let $\Omega \subset \mathbf{R}^3$ be a bounded polyhedral domain. The incompressible Navier-Stokes equations consists of finding velocity $\mathbf{u}$ and pressure p such that

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$$\begin{aligned} \partial_t \mathbf{u} - \nabla\cdot(2\nu\mathbf{D}\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } (0,T]\times\Omega \\ \nabla\cdot\mathbf{u} &= 0 \qquad \text{in } (0,T]\times\Omega \\ \mathbf{u}|_{t=0} &= \mathbf{u}^0 \qquad \text{in } \Omega \end{aligned}$$

with deformation tensor $\mathbf{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ and viscosity ν. For simplicity, we consider homogeneous Dirichlet boundary conditions and the solution spaces

$$V = [H_0^1(\Omega)]^3, \qquad Q = L_*^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}.$$

The variational formulation reads: find $(\mathbf{u}, p)\colon\ (0,T] \to V\times Q$ s.t. $\forall(\mathbf{v},q)\in V\times Q$

$$(\partial_t\mathbf{u},\mathbf{v}) + (2\nu\mathbf{D}\mathbf{u},\mathbf{D}\mathbf{v}) + b_S(\mathbf{u},\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) + (q,\nabla\cdot\mathbf{u}) = (\mathbf{f},\mathbf{v})$$

with

$$b_S(\mathbf{u},\mathbf{v},\mathbf{w}) := 1/2[((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})].$$

Let $\mathscr{T}_h$ be an admissible triangulation of Ω such that $\overline{\Omega} = \cup_{K\in\mathscr{T}_h}\overline{K}$. Here, we consider inf-sup stable velocity-pressure finite element (FE) spaces $V_h\times Q_h \subset V\times Q$ with the discrete inf-sup condition

$$\exists\beta \neq \beta(h) \ \ s.t. \ \ \inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in V_h}\frac{(q_h,\nabla\cdot\mathbf{v_h})}{\|q_h\|_0\|\nabla\mathbf{v}_h\|_0} \geq \beta > 0. \tag{1}$$

The Galerkin method reads: find $(\mathbf{u}_h, p_h)\colon\ (0,T]\to V_h\times Q_h$ s.t. $\forall(\mathbf{v}_h,q_h)\in V_h\times Q_h$

$$(\partial_t\mathbf{u}_h,\mathbf{v}_h) + (2\nu\mathbf{D}\mathbf{u}_h,\mathbf{D}\mathbf{v}_h) + b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) + (q_h,\nabla\cdot\mathbf{u}_h) = (\mathbf{f},\mathbf{v}_h).$$

The idea of VMS was developed in 1995 by Hughes et al. Its application to scale separation in turbulence modeling started around 2000. The basic aim is to model the influence of smallest (unresolved) scales onto the small scales. Following an idea of Layton [5], we define a coarser FE space L_H for the deformation tensor where

$$\{0\} \subseteq L_H \subseteq L := \{\mathbf{L} = (l_{ij}) : l_{ij}\in L^2(\Omega)\forall i,j\in\{1,2,3\}\}.$$

Define the L^2-orthogonal projection operator $P_H : L \to L_H$ and the small scales via $\kappa(\mathbf{D}\mathbf{u}_h) := \mathbf{D}\mathbf{u}_h - P_H(\mathbf{D}\mathbf{u}_h)$ with fluctuation operator $\kappa := Id - P_H$. Then the VMS method reads: find $(\mathbf{u}_h, p_h)\colon\ (0,T]\to V_h\times Q_h$ s.t. $\forall(\mathbf{v}_h,q_h)\in V_h\times Q_h$:

$$\begin{aligned} &(\partial_t\mathbf{u}_h,\mathbf{v}_h) + 2\nu\,(\mathbf{D}\mathbf{u}_h,\mathbf{D}\mathbf{v}_h) + (\nu_T(\mathbf{u}_h)\kappa\mathbf{D}\mathbf{u}_h,\kappa\mathbf{D}\mathbf{v}_h) \\ &\qquad + b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) + (\nabla\cdot\mathbf{u}_h,q_h) = (\mathbf{f},\mathbf{v}_h) \end{aligned}$$

with cellwise constant $\nu_T^K(\mathbf{u}_h) := \nu_T(\mathbf{u}_h)|_K \geq 0$ for all $K\in\mathscr{T}_h$. The space

$$V_h^{\text{div}} := \{\mathbf{v}_h\in V_h : (\nabla\cdot\mathbf{v}_h,q_h) = 0\ \forall q_h\in Q_h\}$$

of discretely divergence free functions is not empty thanks to condition (1). Then the discrete problem reduces to: find $\mathbf{u}_h\colon\ (0,T]\to V_h^{\text{div}}$ s.t. $\forall\mathbf{v}_h\in V_h^{\text{div}}$:

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbf{D}\mathbf{u}_h, \mathbf{D}\mathbf{v}_h) + (\nu_T(\mathbf{u}_h)\kappa\mathbf{D}\mathbf{u}_h, \kappa\mathbf{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad (2)$$

with initial condition $\mathbf{u}_h(0) = I_h\mathbf{u}_0$ and an interpolation operator $I_h : V \to V_h^{\mathrm{div}}$. Let us start with the following stability result given in [7], Lemma 3.1.

Lemma 1. *Let* $\mathbf{f} \in [L^1(0,T;L^2(\Omega))]^3,\ \mathbf{u}_0 \in [L^2(\Omega)]^3$*; then we obtain for* $t \in (0,T]$ *control of kinetic energy and control of dissipation and subgrid terms, respectively:*

$$\|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))} \;\le\; K(\mathbf{f},\mathbf{u}_0) \;\equiv\; \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))}$$

$$\nu\|\mathbf{D}\mathbf{u}_h\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2}\int_0^t \sum_K \nu_T^K(\mathbf{u}_h)\|\kappa_u \mathbf{D}\mathbf{u}_h\|^2_{0,K}\,dt \le 3K^2(\mathbf{f},\mathbf{u}_0).$$

One technical trick in the error analysis is to rewrite the turbulence term as

$$\sum_K \nu_T^K(\mathbf{u}_h)\|\kappa_u \mathbf{D}\mathbf{v}_h\|^2_{0,K} = \sum_K \nu_T^K(\mathbf{u}_h)\Big(1 - \frac{\|P_H\mathbf{D}\mathbf{v}_h\|^2_{0,K}}{\|\mathbf{D}\mathbf{v}_h\|^2_{0,K}}\Big)\|\mathbf{D}\mathbf{v}_h\|^2_{0,K}$$

and to set

$$\nu_{\mathrm{mod}}(\mathbf{u}_h,\mathbf{v}_h) := 2\nu + \nu_T^K(\mathbf{u}_h)\Big(1 - \frac{\|P_H\mathbf{D}\mathbf{v}_h\|^2_{0,K}}{\|\mathbf{D}\mathbf{v}_h\|^2_{0,K}}\Big) \ge 2\nu.$$

The following semidiscrete a-priori estimate w.r.t. the mesh-dependent expression

$$|||\mathbf{v}_h|||_t^2 := \frac{1}{2}\,\|\mathbf{v}_h(t)\|_0^2 + \nu\,\|\mathbf{D}\mathbf{v_h}\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \sum_{K\in\mathscr{T}_h} \nu_T^K(\mathbf{u}_h)\,\|\kappa\mathbf{D}\mathbf{v}_h\|^2_{0,K}\,dt$$

is a variant of Theorem 3.5 in [7].

Theorem 1. *Under the assumptions of Lemma 1 and for a sufficiently smooth solution* $\mathbf{u}$ *of the Navier-Stokes model (see Ref. [7]) it holds for the solution* $\mathbf{u}_h$ *of (2)*

$$|||\mathbf{u}_h - I_h\mathbf{u}|||_t^2 \le Ce^{h(t)}\int_0^t g(s)\,ds, \qquad t\in(0,T) \quad (3)$$

with $h(t) := C\|\mathbf{D}\mathbf{u}(t)\|_0^4/[\min_K \nu^K_{mod}(\mathbf{u}_h(t),\mathbf{e}_h^u(t))]^3$ *and*

$$\begin{aligned} g(t) :=\; & \max_K \nu_T^K(\mathbf{u}_h(t))\Big(\|\kappa\mathbf{D}\mathbf{u}(t)\|_0^2 + \|\mathbf{D}(\mathbf{u} - I_h\mathbf{u}(t))\|_0^2\Big) \quad (4)\\ & + \frac{1}{\min_K \nu^K_{mod}(\mathbf{u}_h(t),\mathbf{e}_h^u(t))}\Big[\|\partial_t(\mathbf{u} - I_h\mathbf{u})(t)\|^2_{-1,\Omega} + \inf_{\tilde p_h\in Q_h}\|(p-\tilde p)(t)\|_0^2 \\ & + \Big(\|\mathbf{D}\mathbf{u}(t)\|_0^2 + \|\mathbf{u}_h(t)\|_0\|\mathbf{D}\mathbf{u}_h(t)\|_0\Big)\|\mathbf{D}(\mathbf{u} - I_h\mathbf{u})(t)\|_0^2\Big]. \end{aligned}$$

Let us consider Taylor-Hood elements with $V_h \times Q_h = [\mathbf{Q}_k]^3 \times \mathbf{Q}_{k-1}$ with $k \ge 2$ on isotropic meshes $\mathscr{T}_h$. Then we can apply the V_h^{div}-interpolation operator I_h of Girault/Scott [3] and the interpolation properties of the fluctuation operator κ. Under the requirement $\max_K \nu_T^K(\mathbf{u}_h(t)) \le Ch_K^2$, we obtain from (3)-(4) the estimate

$$|||\mathbf{u}_h - I_h\mathbf{u}_h|||_t^2 \leq C(\nu, \nu_T, T, \mathbf{u})h^{2k}, \ \ t \in (0,T]. \tag{5}$$

A possible choice is $\nu_T|_K = C_* \Delta^2 \|\kappa \mathbf{D}\mathbf{u}_h\|_{0,K} / \sqrt{vol(K)}$ for all $K \in \mathscr{T}_h$, user chosen constant C_*, and filter width $\Delta \sim h_K$, see Ref. [7].

3 Applications with layer-adapted meshes

Theorem 1 can be applied to layer-adapted meshes as well. Here we consider tensor-product meshes $\mathscr{T}_h$ in $d \in \{2,3\}$ dimensions where the transformation from reference cell $\hat{K} = (-1,1)^d$ to another cell $K \in \mathscr{T}_h$ can be described by the transformation $x = \text{diag}(h_{1,K}, \ldots, h_{d,K})\widehat{x} + a_K$ with the local mesh sizes $h_{i,K}$ into direction i and a shift $a_K \in \mathbf{R}^d$. Assume that the mesh size in direction d is locally the smallest one

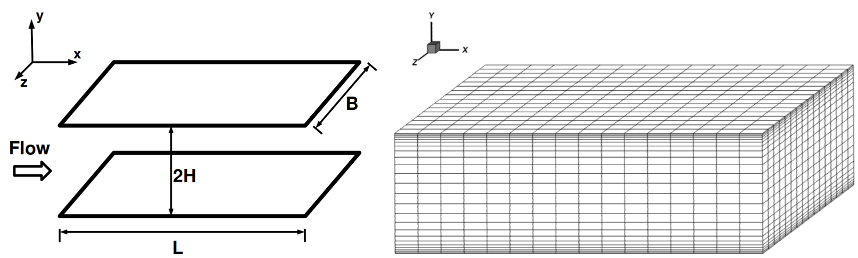

Fig. 1 Channel flow: domain (left) and mesh (right)

s.t. $0 < h_{d,K} \leq h_{i,K}\ \forall i \in \{1, \ldots, d-1\}, \forall K \in \mathscr{T}_h$ whereas the mesh sizes in the other directions are isotropic. Moreover, we suppose no abrupt change in the element sizes of neighboring cells, i.e. $h_{i,K'} \leq C h_{i,K} \leq C h_{i,K'} \forall K, K' \in \mathscr{T}_h,\ \overline{K}' \cap \overline{K} \neq \emptyset$. Such a construction is considered in Fig. 1 (right).

In [2] we obtained for a divergence-preserving interpolator $I_h : V \to V_h^{\text{div}}$ the following (presumably suboptimal) interpolation estimate:

$$\|\mathbf{v} - I_h\mathbf{v}\|_{H^m(K)}^2 \leq C\gamma_i^2 (h_{1,K}/h_{d,K})^2 \sum_{|\alpha|=l-m} h_K^{2\alpha} |D^\alpha \mathbf{v}|_{H^m(\omega(K))}^2 \ \ \forall \mathbf{v} \in H^l(\omega(K))^d$$

for $m \in \{0,1\}$. Here γ_i is the maximal aspect ratio of a patch $\omega(K)$ containing K, and $h_{1,K}/h_{d,K}$ is the local aspect ratio of K. The latter estimate can be applied to bound (3)-(4) on layer-adapted meshes.

We now consider a channel flow in $\Omega = (0,H) \times (0,L) \times (0,B)$ with $H = 1$, $L = 4\pi$, $B = \frac{4\pi}{3}$, see Fig. 1 (left). The viscosity $\nu = 1.5 \times 10^{-5}$ corresponds to $Re_\tau = 180$. The initial value for the velocity $\mathbf{u}$ is taken as the known mean profile with random noise. We apply a BDF(2)-scheme with time step $\partial t = 0.86$ and the Taylor-Hood element $\mathbf{Q}_2/\mathbf{Q}_1$ on an anisotropic Cartesian mesh with $16 \times 24 \times 16$ cells and $y(j) = H(\tanh(2(2j/N-1))/(\tanh(2)+1),\ j \in \{0, \ldots, N = 24\}$, see Fig. 1 (right).

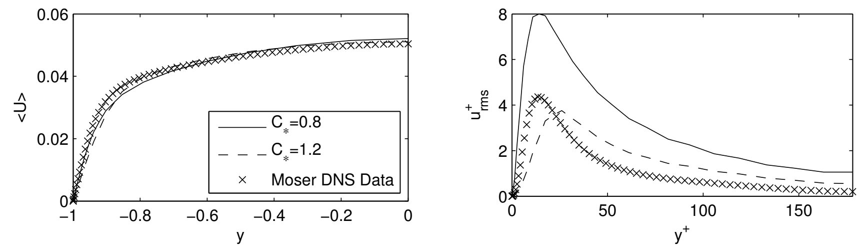

Fig. 2 Channel flow with VMS and $\mathbf{Q}_2/\mathbf{Q}_1$ vs $\mathbf{Q}_0^{disc}$

As reference data from direct numerical simulation we refer to [6]. We are mainly interested in reference values of means, the main channel profile $\langle U \rangle = \lim_{\delta\to\infty} \frac{1}{\delta}\int_{t_0}^{t_0+\delta} U\, dt$ with $U(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{x})\cdot\mathbf{e}_1$ and the scaled Reynolds stress $u' = U - \langle U \rangle$, $y^+ = yu_\tau/\nu$, see Fig. 2. The numerical results are reasonable on the given coarse mesh $\mathscr{T}_h$. We did not observe a significant influence of the maximal aspect ratio which here is $\frac{1}{16}L/y(1) \approx 55$.

4 Applications with weak Dirichlet boundary conditions

An anisotropic mesh refinement becomes more and more expensive with increasing Reynolds number Re_τ. An alternative is to consider weakly enforced Dirichlet conditions on isotropic meshes. Here we follow the framework of Bazilevs et al. [1]. Fig. 3 shows the difference of strongly and weakly enforced Dirichlet data.

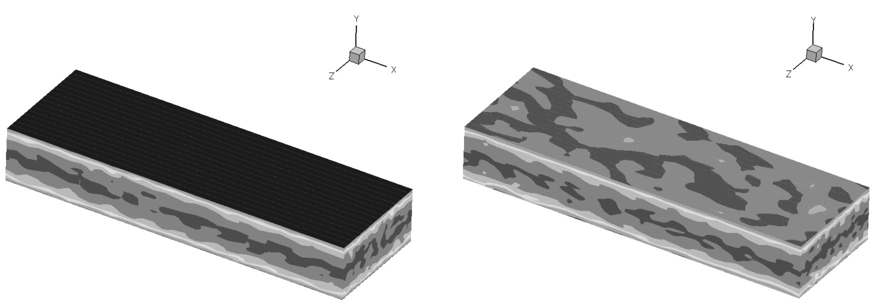

Fig. 3 Strongly (left) and weakly (right) imposed Dirichlet boundary conditions

We assume for simplicity the case of homogeneous Dirichlet data $\mathbf{u} = \mathbf{0}$. Moreover, we divide the boundary $\partial\Omega = \Gamma_{\text{in}} \dot\cup \Gamma_{\text{out}} \dot\cup \Gamma_0$ with $\Gamma_{\text{in}} : \mathbf{u}_h\cdot\mathbf{n} < 0$, $\Gamma_{\text{out}} : \mathbf{u}_h\cdot\mathbf{n} > 0$, and $\Gamma_0 : \mathbf{u}_h\cdot\mathbf{n} = 0$. Following Ref. [1] we consider the modified problem: find $\mathbf{u}_h \colon (0,T] \to V_h \subset [H^1(\Omega)]^d$, $p_h \colon (0,T] \to Q_h$ s.t. $\forall(\mathbf{v}_h, q_h) \in V_h \times Q_h$:

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbf{D}\mathbf{u}_h, \mathbf{D}\mathbf{v}_h) + (\nu_T\,\kappa\mathbf{D}\mathbf{u}_h, \kappa\mathbf{D}\mathbf{v}_h) + b_S\,(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \\ + B_{\text{wall}}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) - (p_h, \nabla\cdot\mathbf{v}_h) + (\nabla\cdot\mathbf{u}_h, q_h) = (\mathbf{f}, \mathbf{v}_h)$$

with

$$B_{\text{wall}}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = -\,(2\nu\mathbf{D}\mathbf{u}_h\cdot\mathbf{n}, \mathbf{v}_h)_{\partial\Omega} + (p_h, \mathbf{v}_h\cdot\mathbf{n})_{\partial\Omega} - (q_h, \mathbf{u}_h\cdot\mathbf{n})_{\partial\Omega} \\ -\frac{1}{2}((\mathbf{u}_h\cdot\mathbf{n})\mathbf{u}_h, \mathbf{v}_h)_{\partial\Omega} - (\mathbf{u}_h, 2\nu\mathbf{D}\mathbf{v}_h\cdot\mathbf{n})_{\partial\Omega} + \frac{1}{2}(\mathbf{u}_h, (\mathbf{u}_h\cdot\mathbf{n})\mathbf{v}_h)_{\Gamma_{\text{out}}} \\ + (\mathbf{u}_h, \tau_B\mathbf{v}_h)_{\partial\Omega} + (\mathbf{u}_h\cdot\mathbf{n}, (C_B\nu/h - \tau_B)\,\mathbf{v}_h\cdot\mathbf{n})_{\partial\Omega}\,.$$

Here, C_B, τ_B are user-chosen constants and we consider a choice at the end of this section. Let us modify the error analysis from Section 2, starting with a stability estimate. Please note that we obtain additional control of certain boundary terms.

Lemma 2. *Assume* $\mathbf{f}\in L^1(0,T;L^2(\Omega))$, $\mathbf{u}_0\in[L^2(\Omega)]^d$, $C_B\geq\tau_B h/\nu>0$ *and* $0<\mu\leq\tau_B-4C_I^2\nu h^{-1}$. *Then we obtain*

$$\|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))} \leq K(\mathbf{f},\mathbf{u}_0) := \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))}$$
$$|||\mathbf{u}_h|||_t^2 + \int_0^t \mu\,\|\mathbf{u}_h\|_{0,\partial\Omega}^2 + \frac{1}{2}\left\|\sqrt{|\mathbf{u}_h\cdot\mathbf{n}|}\mathbf{u}_h\right\|_{0,\Gamma_{in}}^2 dt \leq K^2(\mathbf{f},\mathbf{u}_0).$$

Proof. The new part in the proof is the treatment of the boundary terms. Therefore, we plug in $\mathbf{v}_h=\mathbf{u}_h$ and $q_h=p_h$ and obtain

$$B_{\text{wall}}(\mathbf{u}_h, p_h; \mathbf{u}_h, p_h) = -\,(2\nu\mathbf{D}\mathbf{u}_h\cdot\mathbf{n}, \mathbf{u}_h)_{\partial\Omega} + (p_h, \mathbf{u}_h\cdot\mathbf{n})_{\partial\Omega} \\ -\frac{1}{2}((\mathbf{u}_h\cdot\mathbf{n})\mathbf{u}_h, \mathbf{u}_h)_{\partial\Omega} - (\mathbf{u}_h, 2\nu\mathbf{D}\mathbf{u}_h\cdot\mathbf{n})_{\partial\Omega} - (p_h, \mathbf{u}_h\cdot\mathbf{n})_{\partial\Omega} \\ + \frac{1}{2}(\mathbf{u}_h, (\mathbf{u}_h\cdot\mathbf{n})\mathbf{u}_h)_{\Gamma_{\text{out}}} + (\mathbf{u}_h, \tau_B\mathbf{u}_h)_{\partial\Omega} + (\mathbf{u}_h\cdot\mathbf{n}, (C_B\nu/h-\tau_B)\,\mathbf{u}_h\cdot\mathbf{n})_{\partial\Omega} \\ \geq C_B\nu/h\,\|\mathbf{u}_h\cdot\mathbf{n}\|_{0,\partial\Omega}^2 + \tau_B\left(\|\mathbf{u}_h\cdot\tau_1\|_{0,\partial\Omega}^2 + \|\mathbf{u}_h\cdot\tau_2\|_{0,\partial\Omega}^2\right) \\ -\nu\,\|\mathbf{D}\mathbf{u}_h\|_0^2 - 4C_I^2\nu/h\,\|\mathbf{u}_h\|_{0,\partial\Omega}^2 + \frac{1}{2}\left\|\sqrt{|\mathbf{u}_h\cdot\mathbf{n}|}\mathbf{u}_h\right\|_{0,\Gamma_{\text{in}}}^2 \\ \geq \left(\tau_B - 4\nu C_I^2/h\right)\|\mathbf{u}_h\|_{0,\partial\Omega}^2 + \frac{1}{2}\left\|\sqrt{|\mathbf{u}_h\cdot\mathbf{n}|}\mathbf{u}_h\right\|_{0,\Gamma_{\text{in}}}^2 - \nu\,\|\mathbf{D}\mathbf{u}_h\|_0^2,$$

where τ_1, τ_2 are the directions orthogonal to $\mathbf{n}$ and C_I is from a trace theorem. With the usual way to handle the other terms, e.g. in Ref. [7], the claim is proven.

Now we are in the position to state the following semidiscrete a-priori error estimate.

Theorem 2. *Assume* $V_h=[\mathbf{Q}_k]^3$, $Q_h=\mathbf{Q}_{k-1}$, $k\geq 2$ *and* $\nu_T^K(\mathbf{u}_h)\in\mathscr{O}(h^2)$. *Under the assumptions of Lemma 2 and sufficient smoothness assumptions on the data, see Ref. [7], it holds*

$$|||\mathbf{u}_h - I_h\mathbf{u}|||_t^2 + \int_0^t \mu\,\|\mathbf{u}_h - I_h\mathbf{u}\|_{0,\partial\Omega}^2 + \frac{1}{2}\left\|\sqrt{|\mathbf{u}_h\cdot\mathbf{n}|}\,(\mathbf{u}_h - I_h\mathbf{u})\right\|_{0,\Gamma_{in}}^2 dt \leq Ch^{2k}$$

for every $t \in (0,T]$ *with* $C = C(\nu, \nu_T, T, \mathbf{u})$.

Proof. Let us show the key steps. Please note that $B_{\text{wall}}(\mathbf{u}, p; \mathbf{v}_h, q_h) = 0$. The boundary terms can be treated like linear terms since $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Consider $\mathbf{e}_u := \mathbf{u}_h - I_h\mathbf{u}$ and $\varepsilon_u := \mathbf{u} - I_h\mathbf{u}$ with an interpolation operator $I_h : H^1(\Omega) \to \widetilde{V}_h^{\text{div}}$ with

$$\widetilde{V}_h^{\text{div}} := \{\mathbf{v}_h \in V_h : (\nabla \cdot \mathbf{v}_h, q_h) = 0,\ (\mathbf{v}_h \cdot \mathbf{n}, q_h)_{\partial\Omega} = 0\ \forall q_h \in Q_h\}.$$

Such an operator can be constructed by minor modifications of the operator from Ref. [3]. With the exception of the boundary terms, everything can be estimated exactly as in the case of strong boundary conditions, see Ref. [7] formula (15). Hence, we focus on the boundary terms

$$B_{\text{wall}}(\mathbf{u}_h - \mathbf{u}, p_h - p; \mathbf{e}_u, e_p) = B_{\text{wall}}(\mathbf{e}_u, e_p; \mathbf{e}_u, e_p) - B_{\text{wall}}(\varepsilon_u, \varepsilon_p; \mathbf{e}_u, e_p),$$

where the first term on the right hand side can be treated like in Lemma 2 to get

$$\begin{aligned} B_{\text{wall}}(\mathbf{e}_u, e_p; \mathbf{e}_u, e_p) \geq{}& \left(\tau_B - 4\nu C_I^2/h\right) \|\mathbf{e}_u\|_{0,\partial\Omega}^2 + (C_B\nu/h - \tau_B)\|\mathbf{e}_u \cdot \mathbf{n}\|_{0,\partial\Omega}^2 \\ &+ \frac{1}{2}\left\|\sqrt{|\mathbf{u}_h \cdot \mathbf{n}|}\mathbf{e}_u\right\|_{0,\Gamma_{\text{in}}}^2 - \nu\|\mathbf{D}\mathbf{e}_u\|_0^2. \end{aligned}$$

The structure of $B_{\text{wall}}(\varepsilon_u, \varepsilon_p; \mathbf{e}_u, e_p)$ is a combination of interpolation errors and terms for the left hand side. We estimate

$$\begin{aligned} B_{\text{wall}}(\varepsilon_u, \varepsilon_p; \mathbf{e}_u, e_p) \leq{}& \frac{\nu}{16}\|\mathbf{D}\mathbf{e}_u\|_0^2 + \frac{\mu}{2}\|\mathbf{e}_u\|_{0,\partial\Omega}^2 + (C_B\nu/h - \tau_B)\|\mathbf{e}_u \cdot \mathbf{n}\|_{0,\partial\Omega}^2 \\ &+ \frac{1}{4}\left\|\sqrt{|\mathbf{u}_h \cdot \mathbf{n}|}\mathbf{e}_u\right\|_{0,\Gamma_{\text{in}}}^2 + \frac{4\nu^2 C_I^2}{\mu h}\|\mathbf{D}\varepsilon_u\|_0^2 \\ &+ \left(16\nu C_I^2/h + \tau_B^2/\mu\right)\|\varepsilon_u\|_{0,\partial\Omega}^2 + \frac{1}{2}(C_B\nu/h - \tau_B)\|\varepsilon_u \cdot \mathbf{n}\|_{0,\partial\Omega}^2 \\ &+ \frac{1}{4}\left\|\sqrt{|\mathbf{u}_h \cdot \mathbf{n}|}\varepsilon_u\right\|_{0,\Gamma_{\text{in}}}^2 + \frac{1}{2}(C_B\nu/h - \tau_B)^{-1}\left\|\varepsilon_p\right\|_{0,\partial\Omega}^2. \end{aligned}$$

To obtain this result we use $(e_p, (\mathbf{u} - I_h\mathbf{u}) \cdot \mathbf{n})_{\partial\Omega} = 0$, since we chose $I_h\mathbf{u} \in \widetilde{V}_h^{\text{div}}$. Together with the techniques in Ref. [7] and the stability result in Lemma 2, the claim is proven.

Let us discuss a choice of the parameters C_B and τ_B. The variant of weakly imposed boundary conditions from Ref. [1] is based on a wall function formulation, where $\tau_B = u_\tau^2/\left\|\mathbf{u}_{h,\tan}\right\|$ is determined to fulfill Spalding's law of the wall for a turbulent boundary layer with the wall-friction velocity $u_\tau^2 = \nu \frac{\partial <u_1>}{\partial y}|_{y=0}$ and the velocity vector $\mathbf{u}_{h,\tan}$ tangential to the wall. For the channel flow at $Re_\tau = 180$, one obtains $u_\tau \approx 0.0028$ and $\left\|\mathbf{u}_{h,\tan}\right\| \approx 0.043$, see Ref. [6]. For a better understanding of this parameter we refer to the theory on boundary layers. In the viscous sublayer it holds $1 = y^+/u^+ = y\tau_B/\nu$, where the '+' stands for the wall coordinates and y is the coordinate normal to the wall. This means that $\tau_B \sim \nu/h$ in the viscous sublayer and $\tau_B \geq C\nu/h$ away from layer, since $u^+ < y^+$ in these regions. From the analysis above we need the existence of $0 < \mu \leq \tau_B - 4C_I^2 \nu/h$. A possible choice of τ_B is

$$\tau_B = \max\left(u_\tau^2 / \left\|\mathbf{u}_{h,\tan}\right\|, 8C_I^2 \nu/h\right).$$

For the remaining parameter, the analysis leads to $C_B > \tau_B h/\nu$.

5 Summary and Outlook

We applied a projection-based variational multiscale method to the numerical simulation of wall-bounded flows at moderate Reynolds numbers. A semidiscrete a-priori error estimate from [7] is extended to layer-adapted meshes of tensor-product type and to weak Dirichlet boundary conditions on isotropic meshes. Based on the error analysis, model parameters for the weak boundary treatment are derived.

For the channel flow at $Re_\tau = 180$ with layer-adapted meshes of tensor-product type no instability was obtained for meshes with moderately high aspect ratio. The numerical results of [1] for a channel flow on isotropic meshes validate the application of a weak treatment of Dirichlet boundary condition at even higher Re_τ.

For higher Reynolds numbers, a weak treatment of Dirichlet boundary conditions on isotropic meshes seems to be more advantageous as it mimics the wall of the law. Despite the potential influence of large aspect ratios, layer-adapted meshes will be more and more expensive with increasing Reynolds numbers.

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