Sparse phase retrieval of one-dimensional signals by Prony's method

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Abstract: In this paper, we show that sparse signals f representable as a linear combination of a finite number N of spikes at arbitrary real locations or as a finite linear combination of B-splines of order m with arbitrary real knots can be almost surely recovered from $\mathcal{O}(N^2)$ intensity measurements $|\mathcal{F}[f](\omega)|^2$ up to trivial ambiguities. The constructive proof consists of two steps, where in the first step the Prony method is applied to recover all parameters of the autocorrelation function and in the second step the parameters of f are derived. Moreover, we present an algorithm to evaluate f from its Fourier intensities and illustrate it at different numerical examples.

Key words: Sparse phase retrieval; sparse signals, non-uniform spline functions; finite support; Prony's method

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1. Introduction

Phase retrieval problems occur in many scientific fields, particularly in optics and communications. They have a long history with rich literature regarding uniqueness of solutions and existence of reliable algorithms for signal reconstruction, see e.g. [SEC⁺15] and references therein. Usually, the challenge in solving one-dimensional phase retrieval problems is to overcome the strong ambiguousness by determining appropriate further information on the solution signal. Previous literature on characterization of ambiguities of the phase retrieval problem with given Fourier intensities is often concerned with the discrete problem, where a signal $\mathbf{x}$ in $\mathbb{R}^N$ or $\mathbb{C}^N$ has to be recovered. For an overview on the complete characterization of nontrivial ambiguities is this discrete case as well as on appropriate additional signal information we refer to our survey [BP15a] and further recent results in [BP17, Bei17a, Bei17b].

Contribution of this paper. In this paper, we consider the continuous one-dimensional sparse phase retrieval problem to reconstruct a complex-valued signal from the modulus of its Fourier transform. Applications of this problem occur in electron microscopy, wave front sensing, laser optics [SST04, SSD⁺06] as well as in X-ray crystallography and speckle imaging [RCLV13]. For the posed problem, we will show that for sparse signals the given Fourier intensities are already sufficient for an almost sure unique recovery, and we will give a construction algorithm to recover f.

We assume that the sparse signal is either of the form

$$f(t) = \sum_{j=1}^{N} c_j^{(0)} \, \delta(t - T_j) \tag{1.1}$$

or, for $m > 0$,

$$f(t) = \sum_{j=1}^{N} c_j^{(m)} \, B_{j,m}(t) \tag{1.2}$$

with $c_j^{(m)} \in \mathbb{C}$, $T_j \in \mathbb{R}$ for $j = 1, \dots, N$, where δ denotes the Delta distribution, and $B_{j,m}$ is the B-spline of order m being determined by the (real) knots $T_j < T_{j+1} < \dots < T_{j+m}$. We want to recover these signals from the Fourier intensities $|\widehat{f}(\omega)|^2$ and will show that only $\mathcal{O}(N^2)$ samples are needed to recover f, i.e. all coefficients $c_j^{(m)}$, $j = 1, \dots, N$ and knots T_j, $j = 1, \dots, N + m$, almost surely up to trivial ambiguities. The proposed procedure is constructive and consists in two steps. In a first step, we employ Prony's method in oder to recover all parameters of the (squared) Fourier intensity function $|\mathcal{F}[f](\omega)|^2$. In a second step, we recover the parameters T_j and the complex coefficients c_j that determine the desired signal.

Related work on sparse phase retrieval. While the general phase retrieval problem has been extensively studied for a long tome, the special case of sparse phase retrieval grew to a strongly emerging field of research only recently, particularly often connected with ideas from compressed sensing. Most of the papers consider a discrete setting, where the N-dimensional real or complex k-sparse vector $\mathbf{x}$ has to be reconstructed from measurements of the more general form $|\langle \mathbf{a}_j, \mathbf{x} \rangle|^2$ with vectors $\mathbf{a}_j$ forming the rows of a measurement matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$. The needed number M of measurements depends on the sparsity k.

If $\mathbf{A}$ presents rows of a Fourier matrix, this setting is close to the sparse phase retrieval problem considered in optics, see e.g. [JOH13]. Here the problem is first rewritten as (non-convex) rank minimization problem, then a tight convex relaxation is applied and the optimization problem is solved by a re-weighted l_1-minimization method. The related approach in [ESM+15] employs the magnitudes of the short-time Fourier transform and applies the occurring redundancy for unique recovery of the desired signal. A corresponding reconstruction algorithm is then based on an adaptation of the GESPAR algorithm in [SBE14].

In [LV13], the measurement matrix $\mathbf{A}$ is taken with random rows and the PhaseLift approach [CSV13] leads to a convex optimization problem that recovers the sparse solution with high probability. Employing a thresholded gradient descent algorithm to a non-convex empirical risk minimization problem that is derived from the phase retrieval problem, Cai et al. [CLM16] have established the minimax optimal rates of convergence for noisy sparse phase retrieval under sub-exponential noise.

Other papers rely on the compressed sensing approach to construct special frame vectors $\mathbf{a}_j$ to ensure uniqueness of the phase retrieval problem with high probability, where the number of needed vectors is $\mathcal{O}(k)$, see e.g. [WX14, OE14, IVW17].

We would like to emphasize that all approaches employing general or random measurement matrices in phase retrieval are quite different in nature from our phase retrieval problem based on Fourier intensity measurements. In this paper, we want to stick on considering Fourier intensity measurements because of their particular relevance in practice.

Early attempts to exploit sparsity of a discrete signal for unique recovery using Fourier intensities go back to unpublished manuscripts by Yagle [Yaga, Yagb], where a variation of Prony's method is applied in a non-iterative algorithm to sparse signal and image reconstruction. Unfortunately, the algorithm proposed there not always determines the signal support correctly.

The continuous one-dimensional phase retrieval problem has been rarely discussed in the literature, see [Wal63, Hof64, RCLV13, Bei17b, BP15b]. In the preprint [RCLV13], the authors also considered the recovery of sparse continuous signals of the form (1.1). However, in that paper the sparse phase retrieval problem is in turn transferred into a turnpike problem that is computationally expensive to solve. Moreover there exist cases, where a unique solution cannot be found, see [Blo75]. Our method circumvents this problem by proposing an iterative procedure to fix the signal support (resp. the knots of the signal represented as a B-spline function) where the corresponding signal coefficients are evaluated simultaneously.

Organization of this paper. In Section 2, we shortly recall the mathematical formulation of the considered sparse phase retrieval problem and the notion of trivial ambiguities of the phase retrieval problem that always occur.

Section 3 is devoted to the special case of phase retrieval for signals of the form (1.1). Using Prony's method, we give a constructive proof for the unique recovery of the N-sparse signal f up to trivial ambiguities using $3/2\, N(N-1)+1$ Fourier intensity measurements. Here we have to assume that the knot differences $T_j - T_k$ are pairwise different.

In Section 4, the ansatz is generalized to the unique recovery of spline functions of the form (1.2) where we need to employ $3/2(N+m)(N+m-1)+1$ Fourier intensity measurements. In Section 5, we present an explicit algorithm for the considered sparse phase retrieval problem and illustrate it at different examples.

2. Trivial ambiguities of the phase retrieval problem

We wish to recover an unknown complex-valued signal $f\colon \mathbb{R} \to \mathbb{C}$ of the form (1.1) or (1.2) with compact support from its Fourier intensity $|\mathcal{F}[f]|$ given by

$$|\mathcal{F}[f](\omega)| := \left|\hat{f}(\omega)\right| := \left|\int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right| \qquad (\omega \in \mathbb{R}).$$

Unfortunately, the recovery of the signal f is complicated because of the well-known ambiguousness of the phase retrieval problem. Transferring [BP15a, Proposition 2.1] to our setting, we can recover f only up to the following ambiguities.

Proposition 2.1. *Let f be of a signal of the form* (1.1) *or a non-uniform spline function of the form* (1.2)*. Then*

(i) *the rotated signal* $\mathrm{e}^{\mathrm{i}\alpha} f$ *for* $\alpha \in \mathbb{R}$,

(ii) *the time shifted signal* $f(\cdot - t_0)$ *for* $t_0 \in \mathbb{R}$,

(iii) *and the conjugated and reflected signal* $\overline{f(-\cdot)}$

have the same Fourier intensity $|\mathcal{F}[f]|$.

Proof. Applying the properties of the Fourier transform, we have

(i) $\mathcal{F}[\mathrm{e}^{\mathrm{i}\alpha} f] = \mathrm{e}^{\mathrm{i}\alpha}\, \mathcal{F}[f]$;

(ii) $\mathcal{F}[f(\cdot - t_0)] = \mathrm{e}^{-\mathrm{i}\omega t_0}\, \mathcal{F}[f]$;

(iii) $\mathcal{F}[\overline{f[-\cdot]}] = \overline{\mathcal{F}[f]}$.

Considering the absolute value of each equation yields the assertion. □

Although the ambiguities in Proposition 2.1 always occur, they are of minor interest because of their close relation to the original signal. For this reason, we call ambiguities caused by rotation, time shift, conjugation and reflection, or by combinations of these *trivial*. In the following, we will show that for the considered sparse signals the remaining non-trivial ambiguities only occur in rare cases.

3. Phase retrieval for distributions with discrete support

Initially, we restrict ourselves to the recovery of signals f of the form (1.1) with complex-valued coefficients $c_j^{(0)}$ and spike locations $T_1 < \cdots < T_N$.

$$\widehat{f}(\omega) = \sum_{j=1}^{N} c_j^{(0)}\, \mathrm{e}^{-\mathrm{i}\omega T_j} \qquad (\omega \in \mathbb{R}),$$

and the known squared Fourier intensity $|\mathcal{F}[f]|^2$ can be represented by

$$\left|\widehat{f}(\omega)\right|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} c_j^{(0)} \overline{c}_k^{(0)}\, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)}. \tag{3.1}$$

Thus, in order to recover f being determined by the coefficients $c_j^{(0)} \in \mathbb{C}$ and the knots $T_j \in \mathbb{R}$, $j = 1, \ldots, N$, we will recover all parameters of the exponential sum in (3.1) in a first step and then derive the desired parameters of f in a second step.

3.1. First step: Parameter recovery by Prony's method

Assuming that the non-zero knot differences $T_j - T_k$ with $j \neq k$ are pairwise different, and denoting the distinct frequencies $T_j - T_k$ in increasing order by τ_ℓ with $\ell = -N(N-1)/2, \ldots, N(N-1)/2$, we can rewrite (3.1) as

$$P(\omega) := \left|\widehat{f}(\omega)\right|^2 = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell} = \gamma_0 + \sum_{\ell=1}^{N(N-1)/2} \left(\gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell} + \overline{\gamma}_\ell \, \mathrm{e}^{\mathrm{i}\omega\tau_\ell}\right) \tag{3.2}$$

with the related coefficients $\gamma_\ell := c_j^{(0)} \overline{c}_k^{(0)}$ for the non-zero frequencies $\tau_\ell = T_j - T_k$ and $\gamma_0 := \sum_{j=1}^N |c_j^{(0)}|^2$ for the zero frequency. Since $\tau_{-\ell} = -\tau_\ell$, the coefficients in (3.2) fulfill the conjugated symmetry $\gamma_{-\ell} = \overline{\gamma}_\ell$.

In order to recover the parameters τ_ℓ and the unknown coefficients γ_ℓ from the exponential sum (3.2) we employ Prony's method [Hil87, PT14]. Let $h > 0$ be chosen such that $h\tau_\ell < \pi$ for all $\ell = 1, \ldots, N(N-1)/2$.

Using the intensity values $P(hk) = \left|\widehat{f}(hk)\right|^2$, $k = 0, \ldots, 2N(N-1)+1$, the unknown parameters γ_ℓ and τ_ℓ in (3.2) can be determined by exploiting the algebraic Prony polynomial $\Lambda(z)$ defined by

$$\Lambda(z) := \prod_{\ell=-N(N-1)/2}^{N(N-1)/2} \left(z - \mathrm{e}^{-\mathrm{i}h\tau_\ell}\right) = \sum_{k=0}^{N(N-1)+1} \lambda_k \, z^k, \tag{3.3}$$

where λ_k denote the coefficients in the monomial representation of $\Lambda(z)$. Obviously, $\Lambda(z)$ is always a monic polynomial, which means that $\lambda_{N(N-1)+1} = 1$.

Using the definition of the Prony polynomial $\Lambda(z)$ in (3.3), we observe that

$$\begin{aligned} \sum_{k=0}^{N(N-1)+1} \lambda_k \, P(h(k+m)) &= \sum_{k=0}^{N(N-1)+1} \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \lambda_k \gamma_\ell \, \mathrm{e}^{-\mathrm{i}h(k+m)\tau_\ell} \\ &= \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \, \Lambda\!\left(\mathrm{e}^{-\mathrm{i}h\tau_\ell}\right) = 0 \end{aligned}$$

for $m = 0, \ldots, N(N-1)$. Consequently, the vector of remaining coefficients $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{N(N-1)})^{\mathrm{T}}$ of the Prony polynomial $\Lambda(z)$ can be determined by solving the linear equation system

$$\boldsymbol{H}\boldsymbol{\lambda} = -\boldsymbol{h} \tag{3.4}$$

with $\boldsymbol{H} := (P(h(k+m)))_{m,k=0}^{N(N-1)}$ and $\boldsymbol{h} := (P(h(N(N-1)+1+m)))_{m=0}^{N(N-1)}$. Since the Hankel matrix $\boldsymbol{H}$ can be written as

$$\boldsymbol{H} = \boldsymbol{V}^{\mathrm{T}} \operatorname{diag}(\gamma_{-N(N-1)/2}, \dots, \gamma_{N(N-1)/2})\, \boldsymbol{V}$$

with the Vandemonde matrix $\boldsymbol{V} := (\mathrm{e}^{-hk\tau_\ell})_{\ell=-N(N-1)/2,\,k=0}^{N(N-1)/2,\,N(N-1)+1}$, the linear equation system (3.4) possesses a unique solution if and only if the unimodular values $\mathrm{e}^{-\mathrm{i}h\tau_\ell}$ differ pairwise for $\ell = -N(N-1)/2, \dots, N(N-1)/2$. This assumption has been ensured by choosing an h such that $h\tau_\ell \in (-\pi, \pi)$, since the τ_ℓ had been supposed to be pairwise different.

Knowing the coefficients λ_k of $\Lambda(z)$, we can determine the unknown frequencies τ_ℓ by evaluating the roots of the Prony polynomial (3.3). The coefficients γ_ℓ can now be computed by solving the over-determined equation system

$$\sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell\, \mathrm{e}^{-\mathrm{i}hk\tau_\ell} = P(hk) \qquad (k = 0, \dots, 2N(N-1)+1) \tag{3.5}$$

with a Vandermonde-type system matrix.

The procedure summarized above is the usual Prony method, adapted to the non-negative exponential sum $P(\omega)$ in (3.2). In the numerical experiments in Section 5, we will apply the approximate Prony method (APM) in [PT10]. APM is based on the above considerations but it is numerically more stable and exploits the special properties $\gamma_{-\ell} = \overline{\gamma}_\ell$ and $\tau_{-\ell} = -\tau_\ell$ for $\ell = 0, \dots, N(N-1)/2$.

Let us now investigate the question, how many intensity values are at least necessary for the recovery of $P(\omega)$ in (3.2). Counting the number of unknowns of $P(\omega)$ in (3.2), we only need to recover the $3/2\, N(N-1)+1$ real values γ_0 and $\operatorname{Re}\gamma_\ell$, $\operatorname{Im}\gamma_\ell$, τ_ℓ, for $\ell = 1, \dots N(N-1)/2$. We will show now that using the special structure of the real polynomial $P(\omega)$ in (3.2) and of the Prony polynomial $\Lambda(z)$ in (3.3), we indeed need only $3/2\, N(N-1)+1$ exact equidistant real measurements $P(kh)$, $k = 0, \dots, 3/2\, N(N-1)$ to recover all parameters determining $P(\omega)$. This can be seen as follows.

Reconsidering $\Lambda(z)$ in (3.3) with $\tau_0 = 0$ and $\tau_\ell = -\tau_{-\ell}$, we obtain

$$\begin{aligned} \Lambda(z) &= (z-1) \prod_{\ell=1}^{N(N-1)/2} \left(z - \mathrm{e}^{\mathrm{i}h\tau_\ell}\right)\left(z - \mathrm{e}^{-\mathrm{i}h\tau_\ell}\right) \\ &= (z-1) \prod_{\ell=1}^{N(N-1)/2} \left(z^2 - 2z\cos(h\tau_\ell) + 1\right) = \sum_{k=0}^{N(N-1)+1} \lambda_k\, z^k, \end{aligned}$$

where all occurring coefficients λ_k are real. Moreover, since

$$z^{-(N(N-1)+1)/2}\Lambda(z) = (z^{1/2} - z^{-1/2}) \prod_{\ell=1}^{N(N-1)/2} \left(z - 2\cos(h\tau_\ell) + z^{-1}\right)$$

is antisymmetric, it follows that

$$\lambda_{N(N-1)+1-k} = -\lambda_k \qquad (k = 0, \dots, {}^{N(N-1)}/_{2}),$$

and particularly $\lambda_{N(N-1)+1} = -\lambda_0 = 1$. In order to determine the unknown coefficients λ_k, $k = 1, \dots, {}^{N(N-1)}/_{2}$ of

$$\Lambda(z) = \sum_{k=0}^{N(N-1)/2} \lambda_k \left(z^k - z^{N(N-1)+1-k}\right),$$

we employ (3.2) and observe that for $m = 0, \dots, {}^{N(N-1)}/_{2} - 1$,

$$\begin{aligned}
&\sum_{k=0}^{N(N-1)/2} \lambda_k \left[P(h(k+m)) - P(h(N(N-1)+1+m-k))\right] \\
&= \sum_{k=0}^{N(N-1)/2} \lambda_k \Bigg[\sum_{\ell=1}^{N(N-1)/2} \gamma_\ell \left(\mathrm{e}^{-\mathrm{i}h(k+m)\tau_\ell} - \mathrm{e}^{-\mathrm{i}h(N(N-1)+1+m-k)\tau_\ell}\right) \\
&\quad + \sum_{\ell=1}^{N(N-1)/2} \overline{\gamma}_\ell \left(\mathrm{e}^{\mathrm{i}h(k+m)\tau_\ell} - \mathrm{e}^{\mathrm{i}h(N(N-1)+1+m-k)\tau_\ell}\right)\Bigg] \\
&= \sum_{\ell=1}^{N(N-1)/2} \gamma_\ell\, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \sum_{k=0}^{N(N-1)/2} \lambda_k \left(\mathrm{e}^{-\mathrm{i}hk\tau_\ell} - \mathrm{e}^{-\mathrm{i}h(N(N-1)+1-k)\tau_\ell}\right) \\
&\quad + \sum_{\ell=1}^{N(N-1)/2} \overline{\gamma}_\ell\, \mathrm{e}^{\mathrm{i}hm\tau_\ell} \sum_{k=0}^{N(N-1)/2} \lambda_k \left(\mathrm{e}^{\mathrm{i}hk\tau_\ell} - \mathrm{e}^{\mathrm{i}h(N(N-1)+1-k)\tau_\ell}\right) \\
&= \sum_{\ell=1}^{N(N-1)/2} \gamma_\ell\, \mathrm{e}^{-\mathrm{i}hm\tau_\ell} \Lambda(\mathrm{e}^{-\mathrm{i}h\tau_\ell}) + \sum_{\ell=1}^{N(N-1)/2} \overline{\gamma}_\ell\, \mathrm{e}^{\mathrm{i}hm\tau_\ell} \Lambda(\mathrm{e}^{\mathrm{i}h\tau_\ell}) = 0.
\end{aligned}$$

Therefore, the vector of unknown coefficients $\boldsymbol{\lambda} \coloneqq (\lambda_1, \dots, \lambda_{N(N-1)/2})^{\mathrm{T}}$ can be already evaluated from the system

$$\begin{aligned}
&\sum_{k=1}^{N(N-1)/2} \lambda_k \left[P(h(k+m)) - P(h(N(N-1)+1+m-k))\right] \\
&\qquad = \left[P(hm) - P(h(N(N-1)+1+m))\right] \qquad\qquad (m = 0, \dots, {}^{N(N-1)}/_{2} - 1).
\end{aligned}$$

The parameters τ_ℓ are then extracted from the zeros of $\Lambda(z)$, and the coefficients γ_ℓ, $\ell = 0, \dots, {}^{N(N-1)}/_{2}$, are computed as in (3.5) but with $k = 0, \dots, {}^{3}/_{2}\, N(N-1)$.

3.2. Second step: Unique signal recovery

Having determined the parameters τ_ℓ as well as the corresponding coefficients γ_ℓ of (3.2), we want to reconstruct the parameters T_j and $c_j^{(0)}$, $j = 1, \ldots, N$, of f in (1.1) in a second step.

Theorem 3.1. *Let f be a signal of the form* (1.1)*, whose knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \ldots, N\}$ with $j \neq k$, and whose coefficients satisfy $|c_1^{(0)}| \neq |c_N^{(0)}|$. Further, let h be a step size such that $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k. Then f can be uniquely recovered from its Fourier intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, {}^3\!/\!_2\, N(N-1)$ up to trivial ambiguities.*

Proof. Applying Prony's method to the given data $|\mathcal{F}[f](h\ell)|$, we can compute the frequencies τ_ℓ and the related coefficients γ_ℓ of the squared Fourier intensity (3.2). Again, we assume that the frequencies τ_ℓ occur in increasing order and, further, denote the list of positive frequencies by $\mathcal{T} := \{\tau_\ell\}_{\ell=1}^{N(N-1)/2}$.

Obviously, the maximal distance $\tau_{N(N-1)/2}$ now corresponds to the length $T_N - T_1$ of the unknown f in (1.1). Due to the trivial shift ambiguity, we can assume without loss of generality that $T_1 = 0$ and $T_N = \tau_{N(N-1)/2}$. Further, the second largest distance $\tau_{(N(N-1)/2)-1}$ corresponds either to $T_{N-1} - T_1$ or to $T_N - T_2$. Due to the trivial reflection and conjugation ambiguity, we can assume that $T_{N-1} = \tau_{(N(N-1)/2)-1}$. By definition, there exists a $\tau_{\ell^*} > 0$ in our sequence of parameters $\mathcal{T}$ such that $\tau_{\ell^*} + \tau_{(N(N-1)/2)-1} = \tau_{N(N-1)/2}$, and τ_{ℓ^*} hence corresponds to the knot difference $T_N - T_{N-1}$. Thus, we obtain

$$c_N^{(0)} \overline{c}_1^{(0)} = \gamma_{N(N-1)/2}, \qquad c_{N-1}^{(0)} \overline{c}_1^{(0)} = \gamma_{(N(N-1)/2)-1}, \quad \text{and} \quad c_N^{(0)} \overline{c}_{N-1}^{(0)} = \gamma_{\ell^*}.$$

These equations lead us to

$$c_N^{(0)} = \frac{\gamma_{N(N-1)/2}}{\overline{c}_1^{(0)}}, \qquad c_{N-1}^{(0)} = \frac{\gamma_{(N(N-1)/2)-1}}{\overline{c}_1^{(0)}},$$

and thus to

$$\left| c_1^{(0)} \right|^2 = \frac{\gamma_{N(N-1)/2} \overline{\gamma_{(N(N-1)/2)-1}}}{\gamma_{\ell^*}}.$$

Since f can only be recovered up to a global rotation, we can assume that $c_1^{(0)}$ is real and non-negative, which allows us to determine the coefficients $c_1^{(0)}$, $c_N^{(0)}$, and $c_{N-1}^{(0)}$ in a unique way.

To determine the remaining coefficients and support knots of f, we notice that the third largest distance $\tau_{(N(N-1)/2)-2}$ corresponds either to $T_N - T_2$ or to $T_{N-2} - T_1$. As before, we always find a frequency τ_{ℓ^*} such that $\tau_{(N(N-1)/2)-2} + \tau_{\ell^*} = \tau_{N(N-1)/2}$.
Case 1: If $\tau_{(N(N-1)/2)-2} = T_N - T_2$, then we have

$$\tau_{\ell^*} = \tau_{N(N-1)/2} - \tau_{(N(N-1)/2)-2} = (T_N - T_1) - (T_N - T_2) = T_2 - T_1$$

with the related coefficient $\gamma_{\ell^*} = c_2^{(0)} \overline{c}_1^{(0)}$. Moreover, we have $\gamma_{(N(N-1)/2)-2} = c_N^{(0)} \overline{c}_2^{(0)}$ such that

$$c_2^{(0)} = \frac{\gamma_{\ell^*}}{\overline{c}_1^{(0)}} = \frac{\overline{\gamma}_{(N(N-1)/2)-2}}{\overline{c}_N^{(0)}}. \tag{3.6}$$

Case 2: If $\tau_{(N(N-1)/2)-2} = T_{N-2} - T_1$, then we have

$$\tau_{\ell^*} = \tau_{N(N-1)/2} - \tau_{(N(N-1)/2)-2} = (T_N - T_1) - (T_{N-2} - T_1) = T_N - T_{N-2}$$

with the related coefficient $\gamma_{\ell^*} = c_N^{(0)} \overline{c}_{N-2}^{(0)}$ and $\gamma_{(N(N-1)/2)-2} = c_{N-2}^{(0)} \overline{c}_1^{(0)}$. Thus,

$$c_{N-2}^{(0)} = \frac{\overline{\gamma}_{\ell^*}}{\overline{c}_N^{(0)}} = \frac{\gamma_{(N(N-1)/2)-2}}{\overline{c}_1^{(0)}}. \tag{3.7}$$

However, only one of the two equalities in (3.6) and (3.7) can be true, since if both were true then $\gamma_{\ell^*} \overline{c}_N^{(0)} = \overline{c}_1^{(0)} \overline{\gamma}_{(N(N-1)/2)-2}$ and $\overline{c}_1^{(0)} \overline{\gamma}_{\ell^*} = \overline{c}_N^{(0)} \gamma_{(N(N-1)/2)-2}$ lead to

$$\left| \frac{c_N^{(0)}}{c_1^{(0)}} \right| = \left| \frac{\gamma_{(N(N-1)/2)-2}}{\gamma_{\ell^*}} \right| = \left| \frac{c_1^{(0)}}{c_N^{(0)}} \right|$$

contradicting the assumption that $|c_N^0| \neq |c_1^0|$. Consequently, either the equation in (3.6) or the equation in (3.7) holds true and we can either determine T_2 with $c_2^{(0)}$ or T_{N-2} with $c_{N-2}^{(0)}$. Removing all parameters τ_ℓ from the sequence of distances $\mathcal{T}$ that correspond to the differences $T_j - T_k$ of the recovered knots, we can repeat this approach to find the remaining coefficients and knots of f inductively. □

If we identify the space of complex-valued signals of the form (1.1) with the real space $\mathbb{R}^{3N}$, the condition that two knot differences $T_{j_1} - T_{k_1}$ and $T_{j_2} - T_{k_2}$ are equal for fixed indices j_1, j_2, k_1, and k_2 defines a hyper plane with Lebesgue measure zero. An analogous observation follows for the condition $|c_1^{(0)}| = |c_N^{(0)}|$. The signals excluded in Theorem 3.1 hence form a negligible null set.

Corollary 3.2. *Almost all signals f in* (1.1) *can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.*

Remark 3.3. 1. Since the proof of Theorem 3.1 is constructive, it can be used to recover an unknown signal (1.1) analytically and numerically. If the number N of spikes is known beforehand then the assumption of Theorem 3.1 can be simply checked during the computation. If the assumption regarding pairwise different distances $T_j - T_k$ is not satisfied, then the application of Prony's method in the first step yields less than

$N(N-1)+1$ pairwise distinct parameters τ_ℓ. The second assumption $|c_N^0| \neq |c_1^0|$ can be verified in the second step, where $c_1^{(0)}$, $c_{N-1}^{(0)}$, and $c_N^{(0)}$ are evaluated.

2. A similar phase retrieval problem had been transferred to a turnpike problem in [RCLV13]. The turnpike problem deals with the recovery of the knots T_j from an unlabeled set of distances. Although this problem is solvable under certain conditions, a backtracing algorithm can have exponential complexity in the worst case, see [LSS03].

4. Retrieval of spline functions with arbitrary knots

In this section, we generalize our findings to spline functions of order $m \geq 1$. Let us recall that the B-splines $B_{j,m}$ in (1.2) being generated by the knot sequence $T_1 < \cdots < T_{N+m}$ are recursively defined by

$$B_{j,m}(t) := \tfrac{t-T_j}{T_{j+m-1}-T_j}\, B_{j,m-1}(t) + \tfrac{T_{j+m}-t}{T_{j+m}-T_{j+1}}\, B_{j+1,m-1}(t)$$

with

$$B_{j,1}(t) := \mathbb{1}_{[T_j,T_{j+1})}(t) := \begin{cases} 1 & t \in [T_j, T_{j+1}), \\ 0 & \text{else}, \end{cases}$$

see for instance [Boo78, p. 131]. Further, we notice that for $0 \leq k \leq m-2$ the kth derivative of the spline f in (1.2) is given by

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) = \sum_{j=1}^{N+k} c_j^{(m-k)}\, B_{j,m-k}(t), \tag{4.1}$$

where the coefficients $c_j^{(m-k)}$ are recursively defined by

$$c_j^{(m-k)} := (m-k)\, \frac{c_j^{(m-k+1)} - c_{j-1}^{(m-k+1)}}{T_{j+m-k} - T_j} \qquad (j = 1, \dots, N+k),$$

with the convention that $c_0^{(m-k+1)} = c_{N+k}^{(m-k+1)} = 0$, see [Boo78, p. 139]. For $k = m-1$, equation (4.1) coincides with a step function, i.e., with the right derivative of the linear spline $f^{(m-2)}$. Further, in a distributional manner, the mth derivative of f is given by

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) = \sum_{j=1}^{N+m} c_j^{(0)}\, \delta(t - T_j) \tag{4.2}$$

with the coefficients

$$c_1^{(0)} := c_1^{(1)}, \quad c_{N+m}^{(0)} := -c_{N+m-1}^{(1)}, \quad c_j^{(0)} := c_j^{(1)} - c_{j-1}^{(1)} \qquad (j = 2, \dots, N+m-1),$$

and the Dirac delta distribution δ.

Applying the Fourier transform to (4.2), we now obtain

$$\widehat{f^{(m)}}(\omega) = (\mathrm{i}\omega)^m \,\widehat{f}(\omega) = \sum_{j=1}^{N+m} c_j^{(0)} \mathrm{e}^{-\mathrm{i}\omega T_j}. \tag{4.3}$$

and thus

$$\omega^{2m} \left|\widehat{f}(\omega)\right|^2 = \sum_{j=1}^{N+m} \sum_{k=1}^{N+m} c_j^{(0)} \overline{c}_k^{(0)} \, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)}. \tag{4.4}$$

Since the exponential sum on the right-hand side of (4.4) has exactly the same structure as the exponential sum in (3.2), we can immediately generalize Theorem 3.1 by considering

$$P(\omega) := \omega^{2m} \left|\widehat{f}(\omega)\right|^2 = \sum_{\ell=-(N+m)(N+m-1)/2}^{(N+m)(N+m-1)/2} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell}. \tag{4.5}$$

Theorem 4.1. *Let f be a spline function of the form* (1.2) *of order m, whose knot distances $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N+m\}$ with $j \neq k$, and whose coefficients satisfy $|c_1^{(0)}| \neq |c_{N+m}^{(0)}|$. Further, let h be a step size such that $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k. Then f can be uniquely recovered from its Fourier intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2(N+m)(N+m-1)$ up to trivial ambiguities.*

Proof. The statement can be established by proceeding in the same manner as in Section 3. First we apply Prony's method to the given samples $(h\ell)^{2m}|\mathcal{F}[f](h\ell)|^2$ with $\ell = 0, \dots, 3/2(N+m)(N+m-1)$ in order to determine the coefficients and frequencies of $P(\omega)$ in (4.5). In a second step, the values $c_j^{(0)}$ and T_j in (4.3) can be determined analytically as discussed in Theorem 3.1. Reversing the definition of $c_j^{(m-k)}$, we can finally compute the unknown coefficients $c_j^{(m)}$ by

$$c_j^{(1)} = c_j^{(0)} + c_{j-1}^{(1)} \qquad (j = 1, \dots, N+m-1)$$

and

$$c_j^{(m-k+1)} = \frac{T_{j+m-k} - T_j}{m-k} \, c_j^{(m-k)} + c_{j-1}^{(m-k+1)} \qquad (j = 1, \dots, N+k-1)$$

with $c_0^{(1)} := 0$ and $c_0^{(m-k+1)} := 0$, which finishes the proof. □

Corollary 4.2. *Almost all spline functions f of order m in* (1.2) *can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.*

5. Numerical experiments

Since the proofs of Theorem 3.1 and Theorem 4.1 are constructive, they can be straightforwardly transferred to numerical algorithms to recover a spline function from its Fourier intensity. However, the classical Prony method introduced in subsection 3.1 is numerically unstable with respect to inexact measurements and to frequencies lying close together. For this reason, there are numerous approaches to improve the classical method. In order to verify Theorem 3.1 and Theorem 4.1 numerically, we apply the so-called approximate Prony method (APM) proposed by Potts and Tasche in [PT10, Algorithm 4.7] for recovery of parameters of an exponential sum of the form

$$P(\omega) = \sum_{\ell=-M}^{M} \gamma_\ell \, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell} \tag{5.1}$$

with $\tau_\ell = -\tau_{-\ell}$ and $\gamma_\ell = \overline{\gamma}_{-\ell}$. The algorithm can be summarized as follows, where the exact number $2M+1$ of the occurring frequencies in (5.1) needs not be known beforehand.

Algorithm 5.1 (Approximate Prony method [PT10]).
Input: upper bound $L \in \mathbb{N}$ of the number $2M+1$ of exponentials; measurements $P(hk)$ with $k = 0, \dots, 2\check{M}$ and $\check{M} \geq L$; accuracies ε_1, ε_2, and ε_3.

1. Compute a right singular vector $\boldsymbol{\lambda}^{(1)} := (\lambda_k^{(1)})_{k=0}^{L}$ corresponding to the smallest singular value of the rectangular Hankel matrix $\boldsymbol{H} := (P(h(k+m)))_{k,m=0}^{2N-L,L}$.
2. Evaluate the roots $z_j^{(1)} = r_j^{(1)} \, \mathrm{e}^{\mathrm{i}\omega_j^{(1)}}$ of the polynomial $\Lambda^{(1)}(z) := \sum_{k=0}^{L} \lambda_k^{(1)} \, z^k$ with $\omega_j^{(1)} \in [0, \pi)$ and $|r_j^{(1)} - 1| \leq \varepsilon_1$.
3. Compute a right singular vector $\boldsymbol{\lambda}^{(2)} := (\lambda_k^{(2)})_{k=0}^{L}$ corresponding to the second smallest singular value of the rectangular Hankel matrix $\boldsymbol{H} := (P(h(k+m)))_{k,m=0}^{2N-L,L}$.
4. Evaluate the roots $z_j^{(2)} = r_j^{(2)} \, \mathrm{e}^{\mathrm{i}\omega_j^{(2)}}$ of the polynomial $\Lambda^{(2)}(z) := \sum_{k=0}^{L} \lambda_k^{(2)} \, z^k$ with $\omega_j^{(2)} \in [0, \pi)$ and $|r_j^{(2)} - 1| \leq \varepsilon_1$.
5. Determine all frequencies of the form $\omega_\ell := 1/2 \, (\omega_j^{(1)} + \omega_k^{(2)})$ if there exist indices j and k with $|\omega_j^{(1)} - \omega_k^{(2)}| \leq \varepsilon_2$, and denote the number of found frequencies by $\widetilde{M}$.
6. Compute the coefficients γ_ℓ as least squares solution of the over-determined linear system
$$\sum_{\ell=-\widetilde{M}}^{\widetilde{M}} \gamma_\ell \, \mathrm{e}^{\mathrm{i}hk\tau_\ell} = P(hk) \qquad (k = 0, \dots, 2\check{M})$$

with $\tau_\ell = -\tau_{-\ell} = \omega_\ell/h$ by using the diagonal preconditioner

$$\boldsymbol{D} := \operatorname{diag}\left(\tfrac{1-|k|}{\widetilde{M}+1}\right)_{k=-\widetilde{M}}^{\widetilde{M}}.$$

7. Delete all pairs (τ_ℓ, γ_ℓ) with $|\gamma_\ell| \le \varepsilon_3$.
8. Repeat step 6 with respect to the remaining frequencies τ_ℓ.

Output: coefficients γ_ℓ and frequencies τ_ℓ.

A second adaption of the proof of Theorem 4.1 concerns the reconstruction of the coefficients $c_j^{(m)}$ from the recovered coefficients $c_j^{(0)}$. In order to describe the relation between the coefficients as linear equation system, we define the rectangular matrices $\boldsymbol{C}^{(m-k)} \in \mathbb{R}^{(N+k-1)\times(N+k)}$ for $k = 0, \dots, m-1$ elementwise by

$$C_{j\ell}^{(m-k)} := \begin{cases} \frac{m-k}{T_{j+m-k}-T_j} & \ell = j, \\ \frac{k-m}{T_{j+m-k}-T_j} & \ell = j-1, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \qquad C_{j\ell}^{(0)} := \begin{cases} 1 & \ell = j, \\ -1 & \ell = j-1, \\ 0 & \text{else.} \end{cases}$$

Then, the recursion between the coefficients $c_j^{(m-k+1)}$ and $c_j^{(m-k)}$ can be stated as

$$\boldsymbol{C}^{(m-k)}\, \boldsymbol{c}^{(m-k+1)} = \boldsymbol{c}^{(m-k)},$$

where we use the coefficient vectors $\boldsymbol{c}^{(m-k)} := (c_j^{(m-k)})_{j=1}^{N+k}$. Instead of computing the coefficients stepwise from left to right, we can determine the coefficients $c_j^{(m)}$ by solving the over-determined linear equation system

$$\boldsymbol{C}^{(0)} \cdots \boldsymbol{C}^{(m-1)}\, \boldsymbol{c}^{(m)} = \boldsymbol{c}^{(0)}. \tag{5.2}$$

With these modifications, we recover a spline function of order m from its Fourier intensity by the following algorithm.

Algorithm 5.2 (Phase retrieval).
Input: Fourier intensities $|\mathcal{F}[f](hk)|$ with $k = 0, \dots, 2\breve{M}$, step size $h > 0$, order $m \ge 0$ of the spline function, upper bound L of the number $N + m$ of knots with $L(L-1) < \breve{M}$, accuracy ε.

1. Compute the squared Fourier intensity of the mth derivative of the spline at the given points by

$$|\mathcal{F}[f^{(m)}](hk)|^2 = (hk)^{2m} |\mathcal{F}[f](hk)|^2 \qquad (k = 0, \dots, 2\breve{M}).$$

2. Apply the approximate Prony method (Algorithm 5.1) to determine the knot distances τ_ℓ with $\ell = -(N+m)(N+m-1)/2, \dots, (N+m)(N+m-1)/2$ in increasing order and the corresponding coefficients γ_ℓ.
3. Update the reconstructed distances and coefficients by
$$\tau_\ell := \frac{\tau_\ell - \tau_{-\ell}}{2} \qquad \text{and} \qquad \gamma_\ell := \frac{\gamma_\ell + \overline{\gamma}_{-\ell}}{2}$$
for $\ell = 0, \dots, (N+m)(N+m-1)/2$.
4. Set $T_1 := 0$, $T_{N+m} := \tau_{(N+m)(N+m-1)/2}$, and $T_{N+m-1} := \tau_{((N+m)(N+m-1)/2)-1}$; find the index ℓ^* with $|\tau_{\ell^*} - T_{N+m} + T_{M+m-1}| \le \varepsilon$; and compute the corresponding coefficients by
$$c_1^{(0)} := \left| \frac{\gamma_{(N+m)(N+m-1)/2}\, \overline{\gamma}_{((N+m)(N+m-1)/2)-1)}}{\gamma_{\ell^*}} \right|^{\frac{1}{2}}$$
as well as
$$c_{N+m}^{(0)} := \frac{\gamma_{(N+m)(N+m-1)/2}}{\overline{c}_1^{(0)}} \qquad \text{and} \qquad c_N^{(0)} := \frac{\gamma_{((N+m)(N+m-1)/2)-1}}{\overline{c}_1^{(0)}}.$$
Initialize the lists of recovered knots and coefficients by
$$T := [T_1, T_{N+m}, T_{N+m-1}] \qquad \text{and} \qquad C^{(0)} := [c_1^{(0)}, c_{N+m}^{(0)}, c_{N+m-1}^{(0)}],$$
and remove the used knot distances from the set $\mathcal{T} := \{\tau_\ell\}_{\ell=0}^{(N+m)(N+m-1)/2}$.
5. For the maximal remaining distance τ_{k^*} in $\mathcal{T}$, determine the index ℓ^* with $|\tau_{k^*} + \tau_{\ell^*} - T_{M+n}| \le \varepsilon$.
   a) If $|\tau_{k^*} - \tau_{\ell^*}| \le \varepsilon$, the knot distance corresponds to the centre of the interval $[T_1, T_{M+n}]$. Thus append T by $T_{N+m}/2$ and $C^{(0)}$ by $\gamma_{k^*}/\overline{c}_1^{(0)}$.
   b) Otherwise, compute the values $d^{(\mathrm{r})} := \gamma_{k^*}/\overline{c}_1^{(0)}$ and $d^{(\mathrm{l})} := \gamma_{\ell^*}/\overline{c}_1^{(0)}$. If
   $$\left| c_{N+m}^{(0)}\, \overline{d}^{(\mathrm{r})} - \gamma_{\ell^*} \right| < \left| c_{N+m}^{(0)}\, \overline{d}^{(\mathrm{l})} - \gamma_{k^*} \right|,$$
   then assume that (3.7) with $d^{(\mathrm{r})}$, γ_{k^*}, $c_{N+m}^{(0)}$ instead of $c_{N-2}^{(0)}$, $\gamma_{(N(N-1)/2)-2}$, $c_N^{(0)}$ holds true and append T by $1/2\,(\tau_{k^*} + T_{N+m} - \tau_{\ell^*})$ and $C^{(0)}$ by $d^{(\mathrm{r})}$, else assume that (3.6) with $d^{(\mathrm{l})}$, γ_{k^*}, $c_{N+m}^{(0)}$ instead of $c_2^{(0)}$, $\gamma_{(N(N-1)/2)-2}$, $c_N^{(0)}$ holds true and append T by $1/2\,(\tau_{\ell^*} + T_{N+m} - \tau_{k^*})$ and $C^{(0)}$ by $d^{(\mathrm{l})}$.

   Remove all distances between the new knot and the already recovered knots from $\mathcal{T}$ and repeat step 5 until the set $\mathcal{T}$ is empty.
6. Determine the coefficients $c_j^{(m)}$ by solving the over-determined equation system (5.2).

Output: knots T_j and coefficients $c_j^{(m)}$ of the signal (1.1) ($m = 0$) or the spline function in (1.2) ($m > 0$).

Example 5.3. In the first numerical example, we consider a spike function as in (1.1) with 15 spikes. More precisely, the locations T_j and the coefficients $c_j^{(0)}$ of the true spike function f are given in Table 1. In order to recover f from the Fourier intensity measurements $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 1000$ and with $h \approx 3.655\,073 \cdot 10^{-2}$, we apply Algorithm 5.2 with the accuracies $\varepsilon := 10^{-3}$, $\varepsilon_1 := 10^{-5}$, $\varepsilon_2 := 10^{-7}$, and $\varepsilon_3 := 10^{-10}$. The results of the phase retrieval algorithm and the absolute errors of the knots and coefficients of the recovered spike function are shown in Figure 1. Although the approximate Prony method has to recover 211 knot differences, the knots and coefficients of f are reconstructed very accurately. ○

Example 5.4. In the second example, we apply Algorithm 5.2 to recover the piecewise quadratic spline function ($m = 3$) in (1.2) with the knots and coefficients in Table 2 from the Fourier intensity measurements $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 400$ and with $h \approx 3.088\,663 \cdot 10^{-2}$. As accuracies for the phase retrieval algorithm and the approximate Prony method, we choose $\varepsilon := 10^{-3}$, $\varepsilon_1 := 10^{-5}$, $\varepsilon_2 := 10^{-10}$, and $\varepsilon_3 := 10^{-10}$. In Figure 2, the recovered function is compared with the true signal. Again, the reconstructed knots and coefficients have only very small absolute errors. ○

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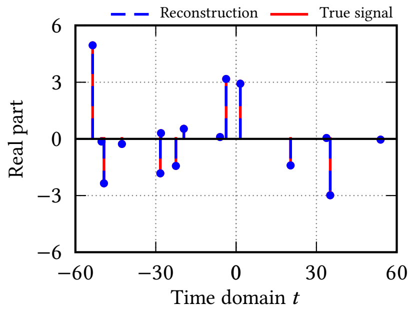
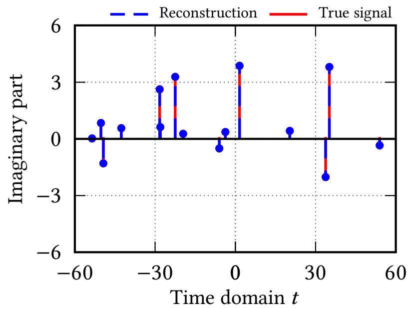
j	T_j	$c_j^{(0)}$	j	T_j	$c_j^{(0)}$	j	T_j	$c_j^{(0)}$
1	-53.5895	4.910 + 0.000i	6	-28.1475	0.278 + 0.598i	11	1.3755	2.887 + 3.828i
2	-50.2765	-0.165 + 0.814i	7	-22.6005	-1.450 + 3.246i	12	20.0945	-1.423 + 0.397i
3	-49.3765	-2.368 − 1.314i	8	-19.6495	0.508 + 0.243i	13	33.4525	0.023 − 2.039i
4	-42.6915	-0.293 + 0.541i	9	-6.1705	0.073 − 0.528i	14	34.8415	-2.997 + 3.767i
5	-28.3915	-1.841 + 2.589i	10	-3.8985	3.135 + 0.339i	15	53.5895	-0.064 − 0.368i

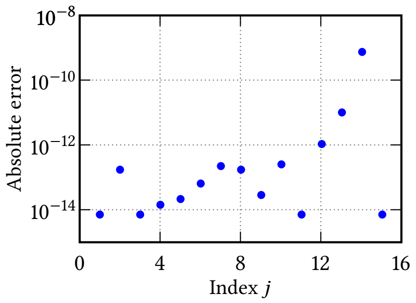
Table 1: Knots T_j and coefficients $c_j^{(0)}$ of the spike function in Example 5.3



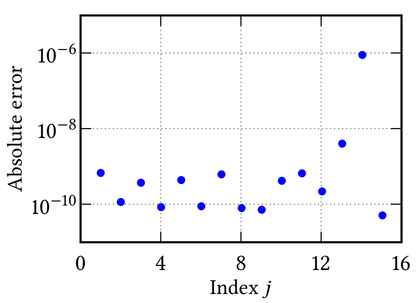
(a) Real part of the recovered and true spike function



(b) Imaginary part of the recovered and true spike function



(c) Absolute error of the recovered knots

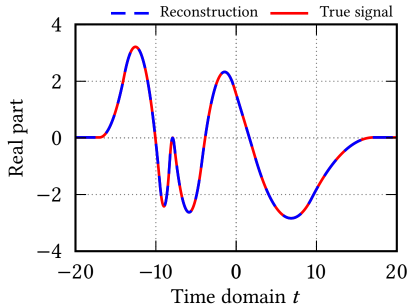


(d) Absolute error of the recovered coefficients

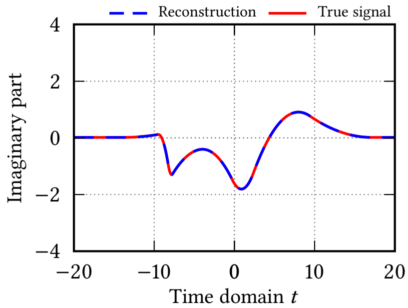
Figure 1: Results of Algorithm 5.2 for the spike function in Example 5.3

j	T_j	$c_j^{(3)}$	j	T_j	$c_j^{(3)}$	j	T_j	$c_j^{(3)}$
1	-17.022	5.342 + 0.000i	5	-7.745	3.597 − 0.334i	8	2.309	—
2	-13.921	-3.569 + 0.132i	6	-4.313	0.554 − 2.251i	9	9.318	—
3	-9.536	0.440 − 1.413i	7	-0.336	-4.072 + 1.433i	10	17.022	—
4	-8.301	-4.685 − 0.499i						

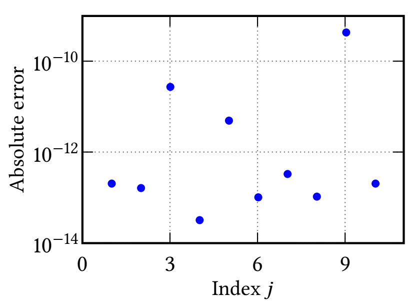
Table 2: Knots T_j and coefficients $c_j^{(3)}$ of the spline function in Example 5.4



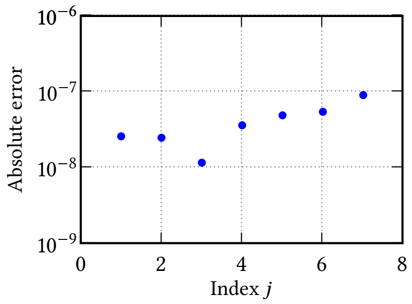
(a) Real part of the recovered and true spline function



(b) Imaginary part of the recovered and true spline function



(c) Absolute error of the recovered knots



(d) Absolute error of the recovered coefficients

Figure 2: Results of Algorithm 5.2 for the spline function in Example 5.4

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