Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain

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Abstract

Considering the ambiguousness of the discrete-time phase retrieval problem to recover a signal from its Fourier intensities, one can ask the question: what additional information about the unknown signal do we need to select the correct solution within the large solution set? Based on a characterization of the occurring ambiguities, we investigate different a priori conditions in order to reduce the number of ambiguities or even to receive a unique solution. Particularly, if we have access to additional magnitudes of the unknown signal in the time domain, we can show that almost all signals with finite support can be uniquely recovered. Moreover, we prove that an analogous result can be obtained by exploiting additional phase information.

Key words: discrete one-dimensional phase retrieval for complex signals, compact support, additional magnitude and phase information in time domain

AMS Subject classifications: 42A05, 94A08, 94A12, 94A20

1. Introduction

The phase retrieval problem consists in recovering a complex-valued signal from the modulus of its Fourier transform. In other words, the phase of the signal in the frequency domain is lost. Recovery problems of this kind have many applications in physics and engineering as for example in crystallography [Mil90, Hau91, KH91], astronomy [DF87], and laser optics [SST04, SSD⁺06]. Finding an analytic or a numerical solution is generally challenging due to the well-known ambiguousness of the problem. In order to determine a meaningful solution, one hence requires further appropriate information about the unknown signal.

In this paper, we consider the phase retrieval problem in a discrete setting and restrict ourselves to the recovery of a complex-valued discrete-time signal $x := (x[n])_{n\in\mathbb{Z}}$ with finite support form its Fourier intensities $|\widehat{x}|$. All occurring ambiguities of this problem can be explicitly constructed via the zeros of the autocorrelation polynomial, see [BS79, OS89, BP15].

Depending on the application and the exact problem setting, different a priori conditions have been employed in the literature to extract special solutions of the phase retrieval problem. For real-valued signals, the interference with a known reference signal [KH90a, KH90b] can be exploited to reduce the complete solution set to at most two different signals. Further, the interference with an unknown reference signal has been considered in [KH93]. These approaches can also be used to recover complex-valued signals, see [RDN13, RSA+11, BP15]. Moreover, comparable results have been achieved by special reference signals that are strongly related to the unknown signal itself. For example, one can use a modulated version of the original signal, see [CESV13, ABFM14].

Other approaches are based on additional information in the frequency domain. In [HHLO83], beside the modulus $|\widehat{x}|$, the information whether the phase of Fourier transform $\widehat{x}(\omega)$ is contained in $[-\pi/2, \pi/2]$ or in $[-\pi, -\pi/2) \cup (\pi/2, \pi)$ has been studied. One may also replace the Fourier transform by the so-called short-time Fourier transform [NQL83a, NQL83b], where the original signal is overlapped with a small analysis window at different positions.

If the unknown discrete-time signal has a fixed support of the form $\{0, \dots, M-1\}$ and can thus be identified with M-dimensional vectors, then the Fourier intensities $|\widehat{x}(\omega_k)|$ at different points $\omega_k \in [-\pi, \pi)$ can be written as the intensity measurement $|\langle x, v_k \rangle|$ with $v_k := (e^{i\omega_k m})_{m=0}^{M-1}$. In a more general setting using a frame approach, the question arises how the vectors v_k have to be constructed, and how many vectors v_k are needed to ensure a unique recovery of x only from the intensities $|\langle x, v_k \rangle|$, see for instance [BCE06, BBCE09, ABFM14, BCM14, BH15] and references therein.

In some applications, like wave front sensing and laser optics [SST04], besides the Fourier intensity, the modulus of the unknown signal itself is known. For this specific one-dimensional phase retrieval problems, a multi-level Gauss-Newton method has been presented in [SSD+06, LT08, LT09] to determine a numerical solution. While this method worked well for the considered problems, its stability seems to depend on the given data sets [Bei13]. Further, for some rare cases, the algorithm converges to an approximate solution which is completely different from the original signal.

The occurring numerical problems in the above mentioned multi-level Gauss-Newton method have been the reason for our extensive studies of ambiguities of the complex discrete phase retrieval problem within this paper. Particularly, we will consider the question whether the additional knowledge of $|x| := (|x[n]|)_{n\in\mathbb{Z}}$ can indeed ensure a unique recovery of the unknown signal x. When we understand this uniqueness problem completely, then we can decide whether the behavior of the algorithm is due to ambiguities or just to ill-posedness of the problem. Besides modulus constraints in time domain, we will also consider the case that information about the phase $\arg x := (\arg x[n])_{n\in\mathbb{Z}}$ in time domain is available, and investigate how far this additional phase information can reduce the solution set.

The paper is organized as follows. In section 2, we briefly recall the characterization of the solution set in [BP15, Theorem 2.4], which is based on an appropriate factorization of the autocorrelation function. Distinguishing the trivial ambiguities, like rotations, shifts, and conjugations and reflections, from the non-trivial, we will observe that the discrete-time phase retrieval problem possesses only a finite set of relevant solutions.

In section 3, we investigate the recovery of a complex-valued signal x with fixed support $\{0, \ldots, N-1\}$ from its Fourier intensity $|\widehat{x}|$ where the modulus $|x[n]|$ of at least one signal component in time domain is also given. For that purpose, we generalize the findings in [XYC87, BP15] that almost each signal x is uniquely determined by its Fourier transform $|\widehat{x}|$ and its end point $x[N-1]$. We will show that already one given modulus $|x[n]|$ of the unknown signal can enforce uniqueness (up to trivial ambiguities) for almost every signal. Unfortunately, the uniqueness cannot be achieved for every signal even if we know the complete modulus $|x|$. We will construct examples where uniqueness is not obtained.

Finally, in section 4, we consider the phase retrieval problem with additional phase information $\arg x[n]$ at appropriate points in the time domain and prove that already two given phases can avoid the ambiguousness for almost all signals. However, similarly to the case of given moduli, even the complete phase information $\arg x := (\arg x[n])_{n\in\mathbb{Z}}$ may be not sufficient to ensure the uniqueness for special cases.

2. The phase retrieval problem

We consider the *one-dimensional discrete-time phase retrieval problem* where we wish to recover the complex-valued discrete-time signal $x := (x[n])_{n\in\mathbb{Z}}$ with finite support from its Fourier intensity

$$|\widehat{x}(\omega)| := |\mathcal{F}[x](\omega)| := \left|\sum_{n\in\mathbb{Z}} x[n]\, e^{-i\omega n}\right| \qquad (\omega \in \mathbb{R}).$$

Unfortunately, this problem is complicated because of the well-known ambiguousness. For example, we can easily construct further solutions by rotating, shifting, or reflecting and conjugating the original signal x, see [BP15].

Proposition 2.1. *Let x be a complex-valued signal with finite support. Then*

(i) *the rotated signal $(e^{i\alpha}\, x[n])_{n\in\mathbb{Z}}$ for $\alpha \in \mathbb{R}$*

(ii) *the time shifted signal $(x[n-n_0])_{n\in\mathbb{Z}}$ for $n_0 \in \mathbb{Z}$*

(iii) *the conjugated and reflected signal $(\overline{x[-n]})_{n\in\mathbb{Z}}$*

have the same Fourier intensity $|\widehat{x}|$.

Proof. This observation can be simply verified by determining the Fourier transforms of the considered signals. Obviously, we have

(i) $\mathcal{F}[(e^{i\alpha}\, x[n])_{n\in\mathbb{Z}}] = e^{i\alpha}\, \widehat{x}$;

(ii) $\mathcal{F}[(x[n-n_0])_{n\in\mathbb{Z}}] = \mathrm{e}^{-\mathrm{i}\omega n_0}\,\widehat{x}$;

(iii) $\mathcal{F}[(\overline{x[-n]})_{n\in\mathbb{Z}}] = \overline{\widehat{x}}$. □

Although the rotation, shift, and conjugation and reflection of a solution always results in a further solution, these signals are very closely related to the original signal. Therefore, we call these three kinds of ambiguities *trivial* as introduced in [Wan13]. However, besides the trivial ambiguities, the discrete-time phase retrieval problem usually possesses an extensive amount of further non-trivial solutions, see [BS79, BP15].

We briefly recall an explicit characterization of the complete solution set, which is based on the observations in [BP15]. Let the *autocorrelation signal* a of a signal x be defined by

$$a[n] := \sum_{k\in\mathbb{Z}} \overline{x[k]}\, x[k+n] \qquad (n \in \mathbb{Z}).$$

Further, let the *autocorrelation function* be the Fourier transform $\widehat{a}$ of the autocorrelation signal a. If N denotes the support length of the signal x, then the corresponding autocorrelation signal a possesses the support $\{-N+1,\dots,N-1\}$. Moreover, the autocorrelation signal a is always conjugate symmetric, i.e., $a[-n] = \overline{a[n]}$ for $n \in \mathbb{Z}$. Thus the autocorrelation function $\widehat{a}$ is a real-valued trigonometric polynomial of degree $N-1$ of the from

$$\widehat{a}(\omega) = \sum_{-N+1}^{N-1} a[n]\,\mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in \mathbb{R}).$$

Moreover, we have the relation

$$\widehat{a}(\omega) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[k+n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega n} = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega(n-k)} = |\widehat{x}(\omega)|^2.$$

Thus, knowing the Fourier intensity $|\widehat{x}|$ is equivalent to knowing the autocorrelation function. Further, the real-valued, non-negative trigonometric polynomial $\widehat{a}$ of degree $N-1$ is already determined by $2N-1$ samples from $[-\pi,\pi)$. Consequently, our initial phase retrieval problem is equivalent to the completely discrete formulation: Recover the complex-valued signal $x := (x[n])_{n\in\mathbb{Z}}$ with support of length N from the $2N-1$ values

$$\left|\widehat{x}\left(\tfrac{2\pi k}{N}\right)\right| \qquad (k = -N,\dots,N-1).$$

We summarize the considerations in [BP15] to characterize all solutions of $|\widehat{x}|^2 = \widehat{a}$ explicitly. Let $P(z)$, given by $P(\mathrm{e}^{-\mathrm{i}\omega}) = \mathrm{e}^{-\mathrm{i}\omega(N-1)}\,\widehat{a}(\omega)$, be the *associated polynomial* to $\widehat{a}$, i.e.

$$P(z) := \sum_{n=0}^{2N-2} a[n-N+1]\,z^n.$$

Here the conjugate symmetry $a[-n] = \overline{a[n]}$ of the coefficients implies that the zeros of the associated polynomial P appear in reflected pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ with respect to the unit circle, and that the zeros on the unit circle have even multiplicity. Since P is a polynomial of degree $2N-2$, there are precisely $N-1$ such zero pairs, yielding the factorization

$$P(z) = a[N-1] \prod_{j=1}^{N-1} \left(z - \gamma_j\right) \left(z - \overline{\gamma}_j^{-1}\right). \tag{1}$$

Now, we can employ (1) to factorize $\widehat{a}$,

$$\widehat{a}(\omega) = |\widehat{a}(\omega)| = \left|P\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| = |a[N-1]| \prod_{j=1}^{N-1} \left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right| \left|\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right|.$$

We rewrite the second linear factor by using the identity

$$\left|\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right| = |\overline{\gamma}_j|^{-1} \left|\overline{\gamma}_j - \mathrm{e}^{\mathrm{i}\omega}\right| = |\gamma_j|^{-1} \left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|$$

and have the representation

$$\widehat{a}(\omega) = |a[N-1]| \prod_{j=1}^{N-1} |\gamma_j|^{-1} \cdot \left| \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right) \right|^2. \tag{2}$$

Since one can similarly rewrite the first linear factor instead of the second, we obtain the following characterization of the solution set of the discrete-time phase retrieval problem, see [BP15, Theorem 2.4].

Theorem 2.2. *Let $\widehat{a}$ be a non-negative trigonometric polynomial of degree $N-1$. Then, each solution x of the discrete phase retrieval problem $|\widehat{x}|^2 = \widehat{a}$ with finite support has a Fourier representation of the form*

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha}\, \mathrm{e}^{-\mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right), \tag{3}$$

where α is a real number, n_0 is an integer, and where for each j the value β_j is chosen from the zero pair $(\gamma_j, \overline{\gamma}_j^{-1})$ of the associated polynomial to $\widehat{a}$.

Apart from the trivial rotation and shift ambiguity caused by the unimodular factor $\mathrm{e}^{\mathrm{i}\alpha}$ and the modulation $\mathrm{e}^{-\mathrm{i}\omega n_0}$ in Theorem 2.2, each solution y is characterized by the values $\beta_j \in (\gamma_j, \overline{\gamma}_j^{-1})$. Hence, we can uniquely identify each solution x up to rotations and shifts with the set $B := \{\beta_1, \ldots, \beta_{N-1}\}$, which we call the *corresponding zero set* of the solution x in the following. Obviously, we can construct up to 2^{N-1} different zero sets; the corresponding signals, however, do not have to be non-trivially different, see [BP15, Corollary 2.6].

Lemma 2.3. *Let x be a discrete-time signal of the form (3) with corresponding zero set $B := \{\beta_1, \dots, \beta_{N-1}\}$. Then, the conjugated and reflected signal $\overline{x[-\cdot]}$ corresponds to the zero set*

$$\left\{\overline{\beta}_1^{-1}, \dots, \overline{\beta}_{N-1}^{-1}\right\}.$$

Thus we can conclude that the phase retrieval problem to recover x can possesses up to 2^{N-2} non-trivially different solutions, [BP15, Corollary 2.6]. The number 2^{N-2} is here only an upper bound for the occurring non-trivial solutions since the actual number for a specific phase retrieval problem strongly depends on the zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ of the associated polynomial. For example, if all zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ lie on the unit circle, then every β_j in Theorem 2.2 is uniquely determined and the corresponding phase retrieval problem is uniquely solvable.

3. Using additional magnitudes of the unknown signal

To determine a unique solution within the solution set characterized by Theorem 2.2, we need additional information about the unknown signal. In [BP15], we have already shown that almost every signal x with support $\{0, \dots, N-1\}$ is uniquely determined by its Fourier intensity $|\widehat{x}|$ and the right end point $x[N-1]$. A similar observation has been done for real-valued signals by Xu et al. in [XYC87]. In this section, we will show that comparable results can be achieved by exploiting one absolute value $|x[n]|$ or even several absolute values $|x[n]|$ of the unknown signal x.

3.1. The modulus of an arbitrary signal value

We consider the phase retrieval problem to recover the signal x with support $\{0, \dots, N-1\}$ from its Fourier intensity $|\widehat{x}|$ and the modulus of one signal value $|x[N-1-\ell]|$ for some ℓ between 0 and $N-1$. We will show that this phase retrieval problem is almost surely uniquely solvable up to rotations.

Let $B := \{\beta_1, \dots, \beta_{N-1}\}$ be the corresponding zero set of a solution signal x as given in (3) and the subsequent comments. Further, let Λ be a subset of B. We introduce the *modified zero set*

$$B^{(\Lambda)} := \left\{\beta_1^{(\Lambda)}, \dots, \beta_{N-1}^{(\Lambda)}\right\}, \tag{4}$$

where the single elements are given by

$$\beta_j^{(\Lambda)} := \begin{cases} \overline{\beta}_j^{-1} & \beta_j \in \Lambda, \\ \beta_j & \text{else.} \end{cases}$$

Recall that by Theorem 2.2 each further non-trivial solution y of the discrete-time phase retrieval problem $|\widehat{x}|^2 = \widehat{a}$ corresponds to such a modified zero set.

Let the $(N-1)$-variate *elementary symmetric polynomial* $S_n : \mathbb{C}^{N-1} \to \mathbb{C}$ of degree n in the variables $\beta_1, \ldots, \beta_{N-1}$ be given by

$$S_n(\beta_1, \ldots, \beta_{N-1}) := \sum_{1 \le k_1 < \cdots < k_n \le N-1} \beta_{k_1} \cdots \beta_{k_n} \tag{5}$$

for n from 1 to $N-1$. Since the polynomials S_n are independent from the order of the variables, we simply denote the elementary symmetric polynomial S_n of a corresponding zero set B by $S_n(B)$. Further, let $S_0 := 1$ and $S_n := 0$ for $n < 0$ and $n \ge N$. With the help of the modified zero set and the elementary symmetric polynomials, we can now give a criterion whether a discrete-time signal x can be uniquely recovered from its Fourier intensity $|\widehat{x}|$ and the modulus $|x[N-1-\ell]|$ or not.

Theorem 3.1. *Let the complex-valued signal x with support $\{0, \ldots, N-1\}$ be a solution of $|\widehat{x}|^2 = \widehat{a}$ as in (3) with corresponding zero set $B := \{\beta_1, \ldots, \beta_{N-1}\}$, and ℓ be in $\{0, \ldots, N-1\}$. Then the signal x can be uniquely recovered from $|\widehat{x}|$ and $|x[N-1-\ell]|$ up to rotations if and only if*

$$|S_\ell(B)| \ne \left(\prod_{\beta_j \in \Lambda} |\beta_j| \right) \cdot \left| S_\ell\left(B^{(\Lambda)}\right) \right| \tag{6}$$

holds for each non-empty subset $\Lambda \subset B$ where Λ does not contain reflected zero pairs of the form $(\beta_j, \overline{\beta}_j^{-1})$ or zeros on the unit circle.

Proof. By normalizing the support of the signal x to $\{0, \ldots, N-1\}$, we avoid the trivial shift ambiguity, and the modulation $\mathrm{e}^{-\mathrm{i}\omega n_0}$ in Theorem 2.2 vanishes. Therefore, the Fourier transform of the original signal x has the form

$$\widehat{x}(\omega) = \sum_{n=0}^{N-1} x[n]\,\mathrm{e}^{-\mathrm{i}\omega n} = \mathrm{e}^{\mathrm{i}\alpha} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right).$$

Since $\widehat{x}$ is an algebraic polynomial in $\mathrm{e}^{-\mathrm{i}\omega}$, Vieta's formulae imply

$$|x[N-1-\ell]| = \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot |S_\ell(B)|\,.$$

We suppose now that the signal x cannot be uniquely recovered up to rotations. Then there exists a second solution $\breve{x}$ that is not a rotation of the signal x. Since the zero sets of x and $\breve{x}$ cannot coincide, there exists a subset Λ which does not contain reflected zero pairs or zeros on the unit circle such that $\breve{x}$ corresponds to the modified zero set $B^{(\Lambda)}$. Now, a comparison of the moduli $|x[N-1-\ell]|$ and $|\breve{x}[N-1-\ell]|$ yields

$$\sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot |S_\ell(B)| = \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j^{(\Lambda)}\right|^{-1}} \cdot \left|S_\ell\left(B^{(\Lambda)}\right)\right|.$$

Simplifying this equation, we find

$$|S_\ell(B)| = \left(\prod_{\beta_j \in \Lambda} \left|\beta_j\right|\right) \cdot \left|S_\ell\left(B^{(\Lambda)}\right)\right|.$$

Hence, the unknown signal x can only be recovered uniquely up to rotations if and only if there exists no subset Λ fulfilling the above equation. □

In the special case when the solution signal x has support $\{0, \ldots, N-1\}$ of odd length N, and we have given the modulus of the centered value $|x[(N-1)/2]|$, then we need to pay special attention because $|x[(N-1)/2]|$ does not change under the reflection and conjugation of the complete signal. Since the reflected and conjugated signal corresponds to the reflection of the complete zero set, see Lemma 2.3, the uniqueness condition in Theorem 3.1 cannot be satisfied except if B is invariant under the reflection at the unit circle. To avoid this trivial ambiguity, we assume that the second solution $\breve{x}$ in the proof of Theorem 3.1 is not equal to the reflected and conjugated signal x. In other words, the subset Λ should not be extendable to the complete set B by adding reflected zero pairs or zeros on the unit circle. Adapting the proof of Theorem 3.1, we obtain the following slightly weaker statement.

Corollary 3.2. *Let the complex-valued signal x with support $\{0, \ldots, N-1\}$ of odd length N be a solution of $|\widehat{x}|^2 = \widehat{a}$ as in (3) with corresponding zero set $B := \{\beta_1, \ldots, \beta_{N-1}\}$. Then the signal x can be uniquely recovered from $|\widehat{x}|$ and $|x[(N-1)/2]|$ up to rotations and conjugate reflections if and only if*

$$\left|S_{\frac{N-1}{2}}(B)\right| \neq \left(\prod_{\beta_j \in \Lambda} \left|\beta_j\right|\right) \cdot \left|S_{\frac{N-1}{2}}\left(B^{(\Lambda)}\right)\right|$$

holds for each non-empty proper subset $\Lambda \subset B$ where Λ does not contain reflected zero pairs or zeros on the unit circle and cannot be extended to the complete set B by adding zeros of this kind.

We want to show that the conditions in Theorem 3.1 and Corollary 3.2 are almost always satisfied for given measurements $|\widehat{x}|$ and $|x[N-1-\ell]|$. For this purpose, we identify the elementary symmetric polynomial $S_\ell(B)$ with a real $(2N-2)$-variate polynomial in the variables

$$\boldsymbol{\beta} \coloneqq (\Re\beta_1, \Im\beta_1, \dots, \Re\beta_{N-1}, \Im\beta_{N-1})^{\mathrm{T}} \in \left(\mathbb{R}^2 \setminus \{0\}\right)^{N-1} \tag{7}$$

and show that for every non-empty $\Lambda \subset B$ and each $\ell \in \{0, \dots, N-1\}$ the exceptional zero sets are contained in the zero locus of a non-trivial $(2N-2)$-variate algebraic polynomial.

Lemma 3.3. *Let $B \coloneqq \{\beta_1, \dots, \beta_{N-1}\} \subset \mathbb{C}^{N-1}$, and let ℓ be in $\{0, \dots, N-1\}$. Then, for each non-empty subset $\Lambda \subset B$, the zero sets B satisfying*

$$|S_\ell(B)| = \Big(\prod_{\beta_j \in \Lambda} |\beta_j|\Big) \cdot \left|S_\ell\left(B^{(\Lambda)}\right)\right| \tag{8}$$

with $B^{(\Lambda)}$ in (4) can be identified with the zero locus of a non-trivial polynomial in $2N-2$ variables whenever $\ell \neq (N-1)/2$. For the case $\ell = (N-1)/2$, the assertion holds true if Λ is a proper subset of B.

Proof. Using the definition of the elementary symmetric function (5), the condition (8) implies

$$\Bigg|\sum_{1 \le k_1 < \dots < k_\ell \le N-1} \beta_{k_1} \cdots \beta_{k_\ell}\Bigg|^2 = \Bigg|\prod_{\beta_j \in \Lambda} \overline{\beta}_j\Bigg|^2 \cdot \Bigg|\sum_{1 \le k_1 < \dots < k_\ell \le N-1} \beta^{(\Lambda)}_{k_1} \cdots \beta^{(\Lambda)}_{k_\ell}\Bigg|^2, \tag{9}$$

where the empty sums for $\ell = 0$ have been set to one. With the substitution $\beta_j = \Re\beta_j + \mathrm{i}\,\Im\beta_j$, the left-hand side of (9) becomes a real algebraic polynomial $p_1(\boldsymbol{\beta})$. Since the reflection of a zero β_j at the unit circle is simply given by $\overline{\beta}_j^{-1}$, the reflected zeros in the modified zero set $B^{(\Lambda)}$ on the right-hand side completely cancel with the prefactor. Hence, the right-hand side of (9) is also an algebraic polynomial $p_2(\boldsymbol{\beta})$.

Thus the vectors $\boldsymbol{\beta}$ satisfying (9) form the zero locus of a real $(2N-2)$-variate algebraic polynomial $p_1 - p_2$. We only have to show that the two polynomials p_1 and p_2 on both sides of (9) do not coincide. Without loss of generality we assume that Λ contains the first J zeros of B. Determining the real and imaginary parts of a summand

$$\beta_{k_1} \cdots \beta_{k_\ell} = (\Re\beta_{k_1} + \mathrm{i}\Im\beta_{k_1}) \cdots \left(\Re\beta_{k_\ell} + \mathrm{i}\Im\beta_{k_\ell}\right)$$

on the left-hand side of (9), we obtain a homogeneous polynomial of degree ℓ in the real variables $\Re\beta_{k_1}, \Im\beta_{k_1}, \dots, \Re\beta_{k_\ell}, \Im\beta_{k_\ell}$. Thus the polynomial p_1 is a $(2N-2)$-variate real homogeneous polynomial of degree 2ℓ. By contrast, we show that p_2 is not homogeneous of degree 2ℓ since it contains monomial terms of different degree. We distinguish the following cases.

(i) For numbers ℓ and J with $\ell + J \leq N - 1$, we always find increasing indicies $k_1 < \cdots < k_\ell$ such that $k_1 > J$. Then the product $\beta_{k_1}^{(\Lambda)} \cdots \beta_{k_\ell}^{(\Lambda)}$ in (9) simply becomes $\beta_{k_1} \cdots \beta_{k_\ell}$, and hence no zeros cancel with the prefactor. This implies that the corresponding monomials in $p_2(\boldsymbol{\beta})$ are exactly of degree $2(\ell + J) \neq 2\ell$.

(ii) If the numbers ℓ and J fulfil $\ell + J > N - 1$ and $J \leq \ell$, then we consider the summand with the indicies $k_1 = 1, \dots, k_\ell = \ell$. Now, the first J modified zeros $\beta_j^{(\Lambda)}$ cancel with the prefactor. Since the real and imaginary parts of the obtained summand $\beta_{J+1} \cdots \beta_\ell$ consist of monomials of degree $\ell - J$, we have at least one monomial of degree $2(\ell - J) \neq \ell$ in p_2.

(iii) Let finally $\ell + J > N - 1$ and $J > \ell$. We consider again the summands with indices $k_1 = 1, \dots, k_\ell = \ell$ on the right-hand side of (9). This yields the summand $\overline{\beta}_{\ell+1} \cdots \overline{\beta}_J$, which corresponds to monomials of degree $2(J - l)$ in $p_2(\boldsymbol{\beta})$, since all modified zeros cancel with the prefactor. Thus, for $J \neq 2\ell$, these terms have a degree different from 2ℓ.

For the special case $J = 2\ell$ we consider the indices $k_1 = N - \ell, \dots, k_\ell = N - 1$ that correspond to the summand

$$\overline{\beta}_1 \cdots \overline{\beta}_{N-\ell-1} \, \beta_{J+1} \cdots \beta_{N-1}$$

with $2N - 2 - \ell - J = 2N - 2 - 3\ell$ different complex variables. Thus, the corresponding terms of $p_2(\boldsymbol{\beta})$ has degree $4N - 4 - 2\ell - 2J = 4N - 4 - 6\ell$, being different from 2ℓ for $J \neq N - 1$. Hence, assuming that $J \neq N - 1$, i.e., Λ is a proper subset of B, there exists a monomial term of degree different from 2ℓ also in the special case $J = 2\ell$. □

Now we can conclude the following recovery result.

Theorem 3.4. *Let ℓ be an arbitrary integer between 0 and $N - 1$. Then almost every complex-valued signal x with support $\{0, \dots, N - 1\}$ can be uniquely recovered from $|\widehat{x}|$ and $|x[N - 1 - \ell]|$ up to rotations if $\ell \neq (N-1)/2$. In the case $\ell = (N-1)/2$, the reconstruction is almost surely unique up to rotations and conjugate reflections.*

Proof. From Lemma 3.3 we can conclude that for all possible choices of Λ and ℓ the exceptional zero sets which do not satisfy the uniqueness conditions in Theorem 3.1 and Corollary 3.2 are contained in the union of finitely many zero loci of $(2N - 2)$-variate real algebraic polynomials and thus form a set E of Lebesgue measure zero. It remains to show that this observation can be transferred to the components of the corresponding signals, which can be represented by

$$x[N-1-n] = (-1)^n \, C \, S_n(\beta_1, \dots, \beta_{N-1}), \quad , n = 0, \dots, N-1, \tag{10}$$

where C is an appropriate constant.

In order to prove this statement, we will apply the following variant of Sard's theorem [Sch69, Theorem 3.1.]. Let $F\colon D \to \mathbb{R}^n$ be a continuously differentiable mapping where D is an open set in $\mathbb{R}^n$. Then the image $F(E)$ of every measurable set $E \subset D$ is measurable, and the Lebesgue measure λ of the image $F(E)$ is bounded by

$$\lambda(F(E)) \le \int_E |\det \boldsymbol{J}_F(y)| \, \mathrm{d}y,$$

where $\boldsymbol{J}_F$ is the Jacobian of F.

We reconsider the equations in (10) as a mapping $F\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ with

$$\begin{aligned} &F(\Re\beta_1, \Im\beta_1, \dots, \Re\beta_{N-1}, \Im\beta_{N-1}, \Re C, \Im C) \\ &\quad = (\Re x[0], \Im x[0], \dots, \Re x[N-1], \Im x[N-1])^{\mathrm{T}}. \end{aligned}$$

Due to the fact that F is continuously differentiable. We now apply Sard's theorem and conclude that the set $F(E \times (\mathbb{R}^2 \setminus \{0\}))$ of all signals which cannot be recovered uniquely up to rotations also have Lebesgue measure zero. Since the remaining signals can be uniquely reconstructed up to rotations, the assertion follows. □

3.2. The moduli of the entire signal

Next, we will investigate the question whether every signal x can be uniquely recovered up to rotations if more then one modulus $|x[n]|$ or even all moduli $(|x[n]|)_{n\in\mathbb{Z}}$ are given. Phase retrieval problems of this kind, where the complete modulus of the signal is known, have been numerically studied in [GS72, SSD+06, LT08, LT09]. Based on our findings in the last subsection, we immediately obtain the following statement.

Corollary 3.5. *Almost every complex-valued signal x with support $\{0, \dots, N-1\}$ can be uniquely recovered from $|\widehat{x}|$ and $(|x[n]|)_{n=0}^{N-1}$ up to rotations.*

Unfortunately, the additional knowledge of more than one modulus of the signal in time domain does not ensure uniqueness of the solution of the corresponding phase retrieval problem in general, even if the complete modulus $|x| := (x[n])_{n\in\mathbb{Z}}$ of the signal x is given.

Theorem 3.6. *For every integer $N > 3$, there exists a signal x with support $\{0, \dots, N-1\}$ such that x cannot be uniquely recovered from $|\widehat{x}|$ and $|x|$ up to rotations.*

Proof. We consider the signal x with support $\{0, \dots, N-1\}$ whose corresponding zero set is given by $B := \{\eta_1, -\eta_1^{-1}, \mathrm{i}\eta_2, \dots, \mathrm{i}\eta_2\}$ with $\eta_1, \eta_2 \in \mathbb{R}$ and $\eta_1, \eta_2 > 1$. In other words, we choose $\beta_1 := \eta_1$, $\beta_2 := -\eta_1^{-1}$, and $\beta_3 := \cdots := \beta_{N-1} = \mathrm{i}\eta_2$. Using Theorem 2.2, we can write the Fourier transform of x as

$$\widehat{x}(\omega) = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{i}\omega n} = C\left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} + \eta_1^{-1}\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\eta_2\right)^{N-3}$$

where $C := \mathrm{e}^{\mathrm{i}\alpha}\sqrt{|a[n-1]|\,|\eta_2|^{N-3}}$. Further, we consider the signal y given in the frequency domain by

$$\widehat{y}(\omega) = \sum_{n=0}^{N-1} y[n]\mathrm{e}^{-\mathrm{i}\omega n} = C\left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_1^{-1}\right)\left(\mathrm{e}^{-\mathrm{i}\omega} + \eta_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\eta_2\right)^{N-3}$$

with support $\{0, \dots, N-1\}$ and corresponding zero set $\{\eta_1^{-1}, -\eta_1, \mathrm{i}\eta_2, \dots, \mathrm{i}\eta_2\}$. Obviously, we have $|\widehat{x}(\omega)| = |\widehat{y}(\omega)|$ since

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} + \eta_1^{-1}\right)\right| = \left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_1^{-1}\right)\left(\mathrm{e}^{-\mathrm{i}\omega} + \eta_1\right)\right|.$$

Moreover, we can show that $|x[n]| = |y[n]|$ for all $n \in \{0, \dots, N-1\}$. Expanding the factorization of $\widehat{x}(\omega)$ and $\widehat{y}(\omega)$, we obtain

$$\begin{aligned}|x[N-1-\ell]| &= |C|\left|\binom{N-3}{\ell-2}(-1)(-\mathrm{i}\eta_2)^{\ell-2} + \binom{N-3}{\ell-1}(-\eta_1 + \eta_1^{-1})(-\mathrm{i}\eta_2)^{\ell-1} + \binom{N-3}{\ell}(-\mathrm{i}\eta_2)^{\ell}\right| \\ &= |C|\,|\eta_2|^{\ell-2}\left|\binom{N-3}{\ell-2}(-1) + \binom{N-3}{\ell-1}(-\eta_1 + \eta_1^{-1})(-\mathrm{i}\eta_2) + \binom{N-3}{\ell}(-\eta_2^2)\right|\end{aligned}$$

for $\ell = 2, \dots, N-3$, while

$$|y[N-1-\ell]| = |C|\,|\eta_2|^{\ell-2}\left|\binom{N-3}{\ell-2}(-1) + \binom{N-3}{\ell-1}(\eta_1 - \eta_1^{-1})(-\mathrm{i}\eta_2) + \binom{N-3}{\ell}(-\eta_2^2)\right|.$$

Thus, we indeed have $|x[N-1-\ell]| = |y[N-1-\ell]|$. The assertion follows analogously for the remaining indices $\ell = 0, 1, N-2, N-1$. □

4. Using additional phase information

Now, we study the question, whether a priori phase information about the unknown signal x can also enforce uniqueness of the solution of the discrete phase retrieval problem $|\widehat{x}|^2 = \widehat{a}$, where $\widehat{a}$ is the given non-negative trigonometric polynomial of degree $N-1$. Obviously, knowing the phase of only one component $x[n]$ of the signal is not sufficient because of the trivial rotation ambiguity. Thus, we need at least the phase of two components.

Firstly, we consider the right and left end point of a signal x given by

$$x[N-1] = \mathrm{e}^{\mathrm{i}\alpha} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}}$$

and

$$x[0] = (-1)^{N-1}\, \mathrm{e}^{\mathrm{i}\alpha} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot \prod_{j=1}^{N-1} \beta_j$$

as characterized in Theorem 2.2. The end points of all further solutions are obtained by changing the zero set $B := \{\beta_1, \dots, \beta_{N-1}\}$, where the corresponding zeros of a subset $\Lambda \subset B$ are reflected at the unit circle. Since the additional rotation by α can be individually chosen for each ambiguity, we can assume without loss of generality that the phase of the right end point coincides for all non-trivial solutions. Observing that the phase of a complex number is invariant under reflection at the unit circle, we can conclude that the phases of the left end point of all solutions are also equal. Therefore, knowing the phases of the two end points does not reduce the set of non-trivial ambiguities. Nonetheless, we will show that additional phase information for two components of the unknown signal which are not the two end points can really enforce almost surely uniqueness of solutions the phase retrieval problem.

4.1. Phase of an arbitrary point and the end point

First, we consider the special case, where we have a priori information about $\arg x[N-1]$ and a further value $\arg x[N-1-\ell]$ for one $\ell \in \{1, \dots, N-2\}$. We proceed similarly as in section 3. First, we characterize the signals that cannot be uniquely reconstructed. Then we show that the exceptional zero sets corresponding to solution ambiguities are contained in an appropriate algebraic variety. Finally, we conclude that ambiguities can only arise in rare special cases. Under the assumption that the unknown signal x possesses the support $\{0, \dots, N-1\}$ and can thus be written in the form (3) with $n_0 = 0$, we obtain the following uniqueness condition, where S_ℓ again denotes the elementary symmetric polynomial in (5).

Theorem 4.1. *Let x be a complex-valued signal with support $\{0, \dots, N-1\}$ and corresponding zero set $B := \{\beta_1, \dots, \beta_{N-1}\}$, and let ℓ be an integer between 1 and $N-2$. Then the signal x cannot be uniquely recovered from $|\hat{x}|$, $\arg x[N-1]$, and $\arg x[N-1-\ell]$ if and only if there exists a non-empty subset $\Lambda \subset B$, where Λ does not contain reflected zero pairs or zeros on the unit circle, such that B and $B^{(\Lambda)}$ satisfy*

$$\Re S_\ell(B)\, \Im S_\ell(B^{(\Lambda)}) - \Im S_\ell(B)\, \Re S_\ell(B^{(\Lambda)}) = 0 \tag{11}$$

and

$$\Re S_\ell(B)\,\Re S_\ell(B^{(\Lambda)}) + \Im S_\ell(B)\,\Im S_\ell(B^{(\Lambda)}) \geq 0. \tag{12}$$

Proof. We assume that the phase retrieval problem to recover x from $|\hat{x}|$, $\arg x[N-1-\ell]$, and $\arg x[N-1]$ possesses at least one further solution y. By Theorem 2.2, there is a subset $\Lambda \subset B$ such that the second solution y corresponds to the modified zero set $B^{(\Lambda)}$. Since x and y are different, we can assume that Λ is non-empty and does not contain reflected zero pairs or zeros on the unit circle.

We recall that the components of a signal with support $\{0, \dots, N-1\}$ are given by

$$x[N-1-\ell] = (-1)^\ell \, e^{i\alpha} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot S_\ell(B) \tag{13}$$

for $\ell \in \{0, \dots, N-1\}$ due to Vieta's formulae. For the components $y[N-1-\ell]$ of the ambiguity, we have an analogous representation where the corresponding zero set B is replaced by $B^{(\Lambda)}$. Since $S_0(B) = S_0(B^{(\Lambda)}) = 1$, the phases of the end points $x[N-1]$ and $y[N-1]$ can only coincide if x and y have the same rotation factor $e^{i\alpha}$.

The second phase condition $\arg x[N-1-\ell] = \arg y[N-1-\ell]$ is now equivalent to

$$\arg S_\ell(B) = \arg S_\ell(B^{(\Lambda)}).$$

Thus, the value $S_\ell(B^{(\Lambda)})$ has to lie on the real ray from the origin through $S_\ell(B)$ in the complex plane, i.e.,

$$\Re S_\ell(B)\,\Im S_\ell(B^{(\Lambda)}) - \Im S_\ell(B)\,\Re S_\ell(B^{(\Lambda)}) = 0.$$

and

$$\Re S_\ell(B)\,\Re S_\ell(B^{(\Lambda)}) + \Im S_\ell(B)\,\Im S_\ell(B^{(\Lambda)}) \geq 0.$$

This assertion also holds when one or both signal values $x[N-1-\ell]$ or $y[N-1-\ell]$ are zero, and the corresponding phases are not uniquely defined. □

We show now that the non-uniqueness condition is only rarely satisfied.

Lemma 4.2. *Let $B = \{\beta_1, \dots, \beta_{N-1}\} \in \mathbb{C}^{N-1}$, and let ℓ be in $\{1, \dots, N-2\}$. Then for each non-empty subset $\Lambda \subset B$, the zero sets B satisfying*

$$\Re S_\ell(B)\,\Im S_\ell(B^{(\Lambda)}) - \Im S_\ell(B)\,\Re S_\ell(B^{(\Lambda)}) = 0 \tag{14}$$

with $B^{(\Lambda)}$ in (4) can be identified with the zero locus of a non-trivial polynomial in $2N-2$ variables.

Proof. Again, we identify the corresponding zero set B of the signal x with the real $(2N-2)$-dimensional vector $\boldsymbol{\beta}$ as in (7). With the substitution $\beta_j = \Re\beta_j + \mathrm{i}\,\Im\beta_j$, the reflected zeros at the unit circle are now given by

$$\Re\overline{\beta}_j^{-1} = \frac{\Re\beta_j}{[\Re\beta_j]^2+[\Im\beta_j]^2} \quad \text{and} \quad \Im\overline{\beta}_j^{-1} = \frac{\Im\beta_j}{[\Re\beta_j]^2+[\Im\beta_j]^2}.$$

Thus, the elementary symmetric polynomials

$$S_\ell(B^{(\Lambda)}) = \sum_{1\le k_1<\cdots<k_n\le N-1} \left(\Re\beta_{k_1}^{(\Lambda)} + \mathrm{i}\Im\beta_{k_1}^{(\Lambda)}\right)\cdots\left(\Re\beta_{k_\ell}^{(\Lambda)} + \mathrm{i}\Im\beta_{k_\ell}^{(\Lambda)}\right)$$

corresponding to the modified zero sets are rational $2(N-2)$-variate functions in the real variables $\Re\beta_j$ and $\Im\beta_j$ ($j = 1,\dots,N-1$), where the denominator of the individual summands contains the moduli of the reflected zeros. Multiplying (14) with

$$\Pi_\Lambda \coloneqq \prod_{\beta_j\in\Lambda}\left(\left[\Re\beta_j\right]^2 + \left[\Im\beta_j\right]^2\right),$$

we thus obtain the equivalent condition

$$\Pi_\Lambda\left(\Re S_\ell(B)\,\Im S_\ell(B^{(\Lambda)}) - \Im S_\ell(B)\,\Re S_\ell(B^{(\Lambda)})\right) = 0 \tag{15}$$

with an algebraic polynomial in the variables $\Re\beta_j$ and $\Im\beta_j$ on the left-hand side.

In order to show that the polynomial (15) cannot vanish everywhere, we use the following idea: we choose a specific monomial in the left summand

$$\Pi_\Lambda\,\Re S_\ell(B)\,\Im S_\ell(B^{(\Lambda)}), \tag{16}$$

and show that this monomial does not occur in the right summand

$$\Pi_\Lambda\,\Im S_\ell(B)\,\Re S_\ell(B^{(\Lambda)}). \tag{17}$$

Assuming that we reflect the first J zeros of B, i.e., $\Lambda \coloneqq \{\beta_1,\dots,\beta_J\}$, we distinguish the following two major cases.

(i) Firstly, we assume that $N-1 > \ell \ge J \ge 1$ and consider the specific monomial

$$p_1(\boldsymbol{\beta}) \coloneqq \Im\beta_1\left(\prod_{k=2}^{\ell}[\Re\beta_k]^2\right)\Re\beta_{\ell+1}.$$

Here p_1 uniquely arises in (16) from the factor $\Re\beta_2\cdots\Re\beta_{\ell+1}$ in $\Re S_\ell(B)$ and the factor $\Im\beta_1\,\Re\beta_2\cdots\Re\beta_\ell/\Pi_\Lambda$ in $\Im S_\ell(B^{(\Lambda)})$. However, p_1 is not contained in (17) since otherwise $\Re S_\ell(B^{(\Lambda)})$ has to contain the factor $\Im\beta_1\,\Re\beta_2\cdots\Re\beta_\ell/\Pi_\Lambda$ such that Π_Λ cancels out. However, this is impossible because the nominator of the real part

$\Re S_\ell(B)$ consists only of monomials with an even number of 'imaginary variables' $\Im \beta_j$.

(ii) Let now $1 \le \ell \le J \le N-1$. First we assume that $2\ell \ge N-1$ and investigate the monomial

$$p_2(\boldsymbol{\beta}) := \Im\beta_1 \Bigg(\prod_{k=2}^{N-1-\ell} \Re\beta_k\Bigg) \Bigg(\prod_{k=N-\ell}^{\ell} [\Re\beta_k]^2\Bigg) \Bigg(\prod_{k=\ell+1}^{J} [\Re\beta_k]^3\Bigg) \Bigg(\prod_{k=J+1}^{N-1} \Re\beta_{J+1}\Bigg),$$

where the three last products can be empty. Observe that p_2 possesses the degree $2J$. Taking the definition of Π_Λ into account, we can conclude that the monomial p_2 uniquely arises from $\Re\beta_{N-\ell}\cdots\Re\beta_{N-1}$ in $\Re S_\ell(B)$, and

$$\frac{\Im\beta_1 \Re\beta_2 \cdots \Re\beta_\ell}{\prod_{j=1}^{\ell}\Big(\big[\Re\beta_j\big]^2 + \big[\Im\beta_j\big]^2\Big)}$$

in $\Im S_\ell(B^{(\Lambda)})$. Using an analogous argumentation as before, the monomial p_2 cannot be a term in (17) since then the nominator of $\Re S_\ell(B^{(\Lambda)})$ has to contain a monomial with only one 'imaginary variable' $\Im\beta_j$.

It remains to show that the polynomial is also non-trivial for $2\ell < N-1$. Here we examine the monomial

$$\Im\beta_1 \Bigg(\prod_{k=2}^{\ell} \Re\beta_k\Bigg) \Bigg(\prod_{k=\ell+1}^{N-\ell-1} [\Re\beta_{\ell+1}]^2\Bigg) \Bigg(\prod_{k=N-\ell}^{J} [\Re\beta_k]^3\Bigg) \Bigg(\prod_{k=J+1}^{N-1} \Re\beta_{J+1}\Bigg)$$

in the case $N-\ell \le J$ and otherwise the monomial

$$\Im\beta_1 \Bigg(\prod_{k=2}^{\ell} \Re\beta_k\Bigg) \Bigg(\prod_{k=\ell+1}^{J} [\Re\beta_{\ell+1}]^2\Bigg) \Bigg(\prod_{k=N-\ell}^{N-1} \Re\beta_{N-\ell}\Bigg).$$

Again these monomials occur in (16) but not in (17). □

Considering the union of the constructed zero loci in Lemma 4.2 for all possible subsets Λ, we can conclude that the additional phase information can indeed enforce uniqueness of the reconstruction for almost every signal.

Theorem 4.3. *Let ℓ be an arbitrary integer between 1 and $N-2$. Then almost every complex-valued signal x with support $\{0,\ldots,N-1\}$ can be uniquely recovered from $|\widehat{x}|$, $\arg x[N-1]$, and $\arg x[N-1-\ell]$.*

Proof. In Lemma 4.2, we have observed that the zero sets satisfying the condition (11) in Theorem 4.1 for a specific subset Λ lie in the zero locus of an algebraic polynomial. Since this polynomial is non-trivial, these zero sets form a set with zero Lebesgue measure. Using Vieta's formulare and Sard's theorem, we can deduce the assertion similarly as in the proof of Theorem 3.4. □

4.2. Phase of two arbitrary points

Let us now consider the phase retrieval problem where we have a priori information about the phase at two inner signal points. More precisely, we consider the recovery of an unknown signal x from its Fourier intensity $|\widehat{x}|$ and the phases $\arg x[N-1-\ell_1]$ and $\arg x[N-1-\ell_2]$ with $\ell_1, \ell_2 \in \{1, \dots, N-2\}$ and $\ell_1 \neq \ell_2$. Similarly as in Theorem 4.1, we can characterize the corresponding zero sets of all signals that cannot be uniquely reconstructed, where we again apply the notations of section 2.

Theorem 4.4. *Let x be a complex-valued signal with support $\{0, \dots, N-1\}$ and corresponding zero set $B := \{\beta_1, \dots, \beta_{N-1}\}$, and let ℓ_1 and ℓ_2 be different integers between 1 and $N-2$. Then the signal x cannot be uniquely recovered from $|\widehat{x}|$, $\arg x[N-1-\ell_1]$, and $\arg x[N-1-\ell_2]$ if and only if there exists a non-empty subset $\Lambda \subset B$, where Λ does not contain reflected zero pairs or zeros on the unit circle, such that B and $B^{(\Lambda)}$ satisfy*

$$\begin{aligned}\Re\Big[S_{\ell_1}(B)\Big]\,\Im\Big[\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big]&\\ -\Im\Big[S_{\ell_1}(B)\Big]\,\Re\Big[\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big]&=0\end{aligned}$$

and

$$\begin{aligned}\Re\Big[S_{\ell_1}(B)\Big]\,\Re\Big[\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big]&\\ +\Im\Big[S_{\ell_1}(B)\Big]\,\Im\Big[\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big]&\geq 0.\end{aligned}$$

Proof. We assume that the phase retrieval problem to recover x from its Fourier intensity and the phases $\arg x[N-1-\ell_1]$ and $\arg x[N-1-\ell_2]$ has a further solution y. By Theorem 2.2, we find a subset $\Lambda \subset B$ so that y corresponds to the modified zero set $B^{(\Lambda)}$, where Λ does not contain reflected zero pairs or zeros on the unit circle.

Recall that the components of x and y can be written in the form (13), where for y we replace B by $B^{(\Lambda)}$ and the rotation factor $\mathrm{e}^{\mathrm{i}\alpha}$ by $\mathrm{e}^{\mathrm{i}\alpha_1}$. Due to the trivial rotation ambiguity, we can always rotate the second signal y such that the phases $\arg x[N-1-\ell_2] = \arg\big((-1)^{\ell_2}\,\mathrm{e}^{\mathrm{i}\alpha}\,S_{\ell_2}(B)\big)$ and $\arg y[N-1-\ell_2] = \arg\big((-1)^{\ell_2}\,\mathrm{e}^{\mathrm{i}\alpha_1}\,S_{\ell_2}(B^{(\Lambda)})\big)$ coincide. In other words, we choose

$$\alpha_1 = \arg\Big(\mathrm{e}^{\mathrm{i}\alpha_1}\,\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\Big).$$

Using the representation (13) for of $x[N-1-\ell_1]$ and $y[N-1-\ell_1]$, we can simplify the condition for the second given phase to

$$\arg\Big((-1)^{\ell_1}\,\mathrm{e}^{\mathrm{i}\alpha}\,S_{\ell_1}(B)\Big) = \arg\Big((-1)^{\ell_1}\,\mathrm{e}^{\mathrm{i}\alpha}\,\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big)$$

and thus to

$$\arg(S_{\ell_1}(B)) = \arg\Big(\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big).$$

Following the procedure in the proof of Theorem 4.1, the complex numbers $S_{\ell_1}(B)$ and $\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})$ have to lie on the same ray starting from the origin in the complex plane, which results in the linear equation and the inequality condition of the assertion. □

Remark 4.5. In the symmetric case, where $\arg x[\ell]$ and $\arg x[N-1-\ell]$ are given for an $\ell \in \{1,\dots,N-2\}$, the phase retrieval problem always yields a second solution y of the form

$$y := \mathrm{e}^{\mathrm{i}(\arg x[\ell]+\arg x[N-1-\ell])}\,\overline{x[N-1-\cdot]}$$

caused by rotation, shift, and conjugation and reflection of x. In particular, we obtain

$$\arg y[\ell] = \arg x[\ell] + \arg x[N-1-\ell] - \arg x[N-1-\ell]$$

and

$$\arg y[N-1-\ell] = \arg x[\ell] + \arg x[N-1-\ell] - \arg x[\ell],$$

which implies that y really is an ambiguity. Thus, Theorem 4.4 holds true in this case with $\Lambda = B$. In order to eliminate this special case, we will assume that Λ is a proper subset of B whenever $\ell_2 = N-1-\ell_1$. ○

Lemma 4.6. *Let $B = \{\beta_1,\dots,\beta_{N-1}\} \subset \mathbb{C}^{N-1}$, and let ℓ_1 and ℓ_2 be different integers in $\{1,\dots,N-2\}$. Then for each non-empty subset $\Lambda \subset B$, the zero sets B satisfying*

$$\begin{aligned}\Re\Big[S_{\ell_1}(B)\Big]\,\Im\Big[\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big]&\\ -\,\Im\Big[S_{\ell_1}(B)\Big]\,\Re\Big[\overline{S_{\ell_2}(B^{(\Lambda)})}\,S_{\ell_2}(B)\,S_{\ell_1}(B^{(\Lambda)})\Big] &= 0\end{aligned} \tag{18}$$

with $B^{(\Lambda)}$ in (4) can be identified with the zero locus of a non-trivial polynomial in $2N-2$ variables whenever $\ell_1+\ell_2 \neq N-1$. In the case $\ell_1+\ell_2 = N-1$, the statement holds for every proper subset $\Lambda \subset B$.

Proof. Similarly as in the proof of Lemma 4.6, we substitute $\beta_j = \Re\beta_j + \mathrm{i}\,\Im\beta_j$ and multiply the equation (18) with

$$\Pi_\Lambda := \prod_{\beta_j \in \Lambda} \left(\left[\Re\beta_j\right]^2 + \left[\Im\beta_j\right]^2\right)^2$$

to obtain the equivalent condition

$$\begin{aligned} \Pi_\Lambda \Big(\Re\left[S_{\ell_1}(B)\right] \Im\left[\overline{S_{\ell_2}(B^{(\Lambda)})}\, S_{\ell_2}(B)\, S_{\ell_1}(B^{(\Lambda)})\right] \\ - \Im\left[S_{\ell_1}(B)\right] \Re\left[\overline{S_{\ell_2}(B^{(\Lambda)})}\, S_{\ell_2}(B)\, S_{\ell_1}(B^{(\Lambda)})\right]\Big) = 0, \end{aligned} \tag{19}$$

whose left-hand side can be understood as an $2(N-1)$-variate algebraic polynomial in the real variables $\Re\beta_j$ and $\Im\beta_j$. We show that this polynomial is non-trivial by finding in any case a monomial that is contained in

$$\Pi_\Lambda\, \Im\left[S_{\ell_1}(B)\right] \Re\left[\overline{S_{\ell_2}(B^{(\Lambda)})}\, S_{\ell_2}(B)\, S_{\ell_1}(B^{(\Lambda)})\right] \tag{20}$$

but not in

$$\Pi_\Lambda\, \Re\left[S_{\ell_1}(B)\right] \Im\left[\overline{S_{\ell_2}(B^{(\Lambda)})}\, S_{\ell_2}(B)\, S_{\ell_1}(B^{(\Lambda)})\right]. \tag{21}$$

Since the polynomial in (19) depends on ℓ_1, ℓ_2, and J, this leads to a cumbersome case study. We discuss only one case in detail and state for all other cases the monomials that can be shown to arise in (20) but not in (21).

(i) Let $\ell_1 > \ell_2 \geq J$ and consider the monomial

$$[\Re\beta_1]^2 \left(\prod_{k=2}^{\ell_2} [\Re\beta_k]^4\right) [\Re\beta_{\ell_2+1}]^3 \left(\prod_{k=\ell_2+2}^{\ell_1} [\Re\beta_k]^2\right) \Im\beta_{\ell_1+1}, \tag{22}$$

where the product over the squared variables can be empty. The degree of this monomial is $2(\ell_1 + \ell_2)$. Consequently, the monomial in (22) is obtained from the term $\Re\beta_2 \cdots \Re\beta_{\ell_1} \Im\beta_{\ell_1+1}$ in $\Im[S_{\ell_1}(B)]$ and

$$\Pi_\Lambda^{-1}\, [\Re\beta_1]^2\, [\Re\beta_2]^3 \cdots [\Re\beta_k]^3\, [\Re\beta_{k+1}]^2\, \Re\beta_{k+2} \cdots \Re\beta_\ell$$

in $\Re[\overline{S_{\ell_2}(B^{(\Lambda)})}\, S_{\ell_2}(B)\, S_{\ell_1}(B^{(\Lambda)})]$. Since this factorization is unique, the considered monomial (22) cannot vanish within (20). Recalling that monomials in the real part of $S_{\ell_1}(B)$ contain an even number of 'imaginary variables' $\Im\beta_j$, we can simply conclude, that the monomial (22) does not occur in (21).

(ii) $\ell_1 \geq J > \ell_2$ with $2\ell_2 \leq J$:

$$[\Re\beta_1]^2 \left(\prod_{k=2}^{\ell_2} [\Re\beta_k]^3\right) \left(\prod_{k=\ell_2+1}^{J-\ell_2} [\Re\beta_k]^4\right) \left(\prod_{k=J-\ell_2+1}^{J} [\Re\beta_k]^5\right) \left(\prod_{k=J+1}^{\ell_1} [\Re\beta_k]^2\right) \Im\beta_{\ell_1+1}$$

(iii) $\ell_1 \geq J > \ell_2$ with $2\ell_2 > J$:

$$[\Re\beta_1]^2 \left(\prod_{k=2}^{J-\ell_2} [\Re\beta_k]^3\right) \left(\prod_{k=J-\ell_2+1}^{\ell_2} [\Re\beta_k]^4\right) \left(\prod_{k=\ell_2+1}^{J} [\Re\beta_k]^5\right) \left(\prod_{k=J+1}^{\ell_1} [\Re\beta_k]^2\right) \Im\beta_{\ell_1+1}$$

(iv) $\ell_2 < \ell_1 < J$ with $2\ell_2 < 2\ell_1 \leq J$:

$$\left(\prod_{k=1}^{\ell_2} [\Re\beta_k]^2\right) \left(\prod_{k=\ell_2+1}^{\ell_1} [\Re\beta_k]^3\right) \left(\prod_{k=\ell_1+1}^{J-\ell_1} [\Re\beta_k]^4\right) [\Im\beta_{J-\ell_1+1}]^5 \cdot \left(\prod_{k=J-\ell_1+2}^{J-\ell_2} [\Re\beta_k]^5\right) \left(\prod_{k=J-\ell_2+1}^{J} [\Re\beta_k]^6\right)$$

(v) $\ell_2 < \ell_1 < J$ with $2\ell_2 \leq J < 2\ell_1$ and $\ell_1 + \ell_2 < J$:

$$\left(\prod_{k=1}^{\ell_2} [\Re\beta_k]^2\right) \left(\prod_{k=\ell_2+1}^{J-\ell_1} [\Re\beta_k]^3\right) \left(\prod_{k=J-\ell_1+1}^{\ell_1} [\Re\beta_k]^4\right) \cdot [\Im\beta_{\ell_1+1}]^5 \left(\prod_{k=\ell_1+2}^{J-\ell_2} [\Re\beta_k]^5\right) \left(\prod_{k=J-\ell_2+1}^{J} [\Re\beta_k]^6\right)$$

(vi) $\ell_2 < \ell_1 < J$ with $2\ell_2 \leq J < 2\ell_1$ and $\ell_1 + \ell_2 = J < N - 1$:

$$\left(\prod_{k=1}^{\ell_2} [\Re\beta_k]^2\right) [\Re\beta_{\ell_2+1}]^3 \left(\prod_{k=\ell_2+2}^{\ell_1} [\Re\beta_k]^4\right) \left(\prod_{k=\ell_1+1}^{J} [\Re\beta_k]^6\right) \Im\beta_{J+1}$$

(vii) $\ell_2 < \ell_1 < J$ with $2\ell_2 \leq J < 2\ell_1$ and $\ell_1 + \ell_2 > J$:

$$\left(\prod_{k=1}^{J-\ell_1} [\Re\beta_k]^2\right) [\Im\beta_{J-\ell_1+1}]^3 \left(\prod_{k=J-\ell_1+2}^{\ell_2} [\Re\beta_k]^3\right) \left(\prod_{k=\ell_2+1}^{J-\ell_2} [\Re\beta_k]^4\right) \cdot \left(\prod_{k=J-\ell_2+1}^{\ell_1} [\Re\beta_k]^5\right) \left(\prod_{k=\ell_1+1}^{J} [\Re\beta_k]^6\right)$$

(viii) $\ell_2 < \ell_1 < J$ with $2\ell_1 > 2\ell_2 > J$:

$$\left(\prod_{k=1}^{J-\ell_1} [\Re\beta_k]^2\right) [\Im\beta_{J-\ell_1+1}]^3 \left(\prod_{k=J-\ell_1+2}^{J-\ell_2} [\Re\beta_k]^3\right) \left(\prod_{k=J-\ell_2+1}^{\ell_2} [\Re\beta_k]^4\right) \cdot \left(\prod_{k=\ell_2+1}^{\ell_1} [\Re\beta_k]^5\right) \left(\prod_{k=\ell_1+1}^{J} [\Re\beta_k]^6\right)$$

Summarizing, one can show that the polynomial on the left-hand side of (19) is non-trivial for all possible combinations of ℓ_1, ℓ_2, and J. □

As before, Lemma 4.6 implies now the almost sure uniqueness of the solutions of the discrete phase retrieval problem.

Theorem 4.7. *Let ℓ_1 and ℓ_2 be different integers between 1 and $N-2$. Then almost every signal complex-valued x with support $\{0,\dots,N-1\}$ can be uniquely recovered from $|\widehat{x}|$, $\arg x[N-1-\ell_1]$, and $\arg x[N-1-\ell_2]$ whenever $\ell_1+\ell_2 \neq N-1$. In the case $\ell_1+\ell_2 = N-1$, the recovery is only unique up to conjugate reflections.*

Proof. The assertion follows in an analogous way as Theorem 4.3. Again, the corresponding zero sets satisfying the non-uniqueness conditions in Theorem 4.4 for a specific ℓ_1, ℓ_2, and Λ lie in the zero locus of an algebraic polynomial by Lemma 4.2. Due to the fact that there exist only finitely many different subsets Λ, the exceptional zero sets of signals without a unique reconstruction form a set with zero Lebesgue measure. With Vieta's formulae and Sard's theorem, the assertion follows. □

4.3. The phase of the entire signal

Finally, we will consider the question whether every signal x can be uniquely recovered from $|\widehat{x}|$ and the complete phase information $\arg x := (\arg x[n])_{n=0}^{N-1}$ in the time domain. Based on Theorem 4.3 and Theorem 4.7 we obviously have the following statement.

Corollary 4.8. *Almost every signal complex-valued x with support $\{0,\dots,N-1\}$ can be uniquely recovered from $|\widehat{x}|$ and $\arg x$.*

Unfortunately, the complete phase information of a signal fails to enforce the uniqueness of the phase retrieval problem for every signal.

Theorem 4.9. *For every integer $N > 2$, there exists a signal x with support $\{0,\dots,N-1\}$ such that x cannot be uniquely recovered from $|\widehat{x}|$ and $\arg x$.*

Proof. Let x be a signal with support $\{0,\dots,N-1\}$ of the form

$$\widehat{x}(\omega) = \sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\mathrm{i}\omega n} = \sqrt{|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right).$$

where all zeros β_j are real and negative. Since the linear factors $(e^{-i\omega} - \beta_j)$ only have positive coefficients, the components $x[n]$ of the constructed signal x are real and non-negative, i.e. $\arg x[n] = 0$ for all n. This observation remains valid for all arising ambiguities in Theorem 2.2. Hence, if the corresponding zero set contains at least two zeros unequal to -1, we find a further ambiguity with phase zero in the time domain. □

Acknowledgements

We grateful acknowledge the funding of this work by the DFG in the frame work of the SFB 755 'Nanoscale photonic imaging' and of the GRK 2088 'Discovering structure in complex data: Statistics meets Optimization and Inverse Problems.'

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