Computation of adaptive Fourier series by sparse approximation of exponential sums

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Problem statement

Let

$$f(t) = \frac{c_0(f)}{2} + \sum_{k=1}^{\infty} c_k(f)\mathrm{e}^{\mathrm{i}kt} + \overline{c_k(f)}\mathrm{e}^{-\mathrm{i}kt} \in L^2(0, 2\pi]).$$

Then $f(t) = 2\,\mathrm{Re}\, f^+(t)$ with

$$f^+(t) := \frac{c_0(f)}{2} + \sum_{k=1}^{\infty} c_k(f)\,\mathrm{e}^{ikt} = \sum_{k=0}^{\infty} c_k(f^+)\,\mathrm{e}^{ikt} \in H^2.$$

Decay of Fourier coefficients related to the smoothness of f.

Problem. Can we find an adaptive Fourier basis $\{B_k\}_{k=0}^{\infty}$ of H^2 such that

$$f^+(t) = \sum_{k=0}^{\infty} \langle f^+, B_k \rangle_{L^2} B_k,$$

where the decay of the adaptive Fourier coefficients is much faster?

Adaptive Fourier basis

Consider the Takenaka-Malmquist system

$$B_0(t) = \frac{\sqrt{1-|z_0|^2}}{1 - z_0\,\mathrm{e}^{it}}, \qquad B_\ell(t) = \frac{\sqrt{1-|z_\ell|^2}}{1 - z_\ell\,\mathrm{e}^{it}} \prod_{k=0}^{\ell-1} \frac{\mathrm{e}^{it} - \overline{z}_k}{1 - z_k\,\mathrm{e}^{it}}, \quad \ell = 1, 2, \ldots,$$

which is determined by the sequence of "zeros" $\{z_\ell\}_{\ell=0}^{\infty}$ with $|z_\ell| \le c < 1$.

Then

1. $\langle B_\ell, B_k \rangle_{L^2} = \delta_{\ell,k}$.
2. The system is complete in H^2 if and only if $\sum_{\ell \ge 0}(1 - |z_\ell|) = \infty$ (Akhiezer (1965)).
3. For $z_\ell = 0$, $\ell \ge 0$, we obtain the classical Fourier basis $B_\ell(t) = \mathrm{e}^{i\ell t}$.
4. General decay properties (Bultheel & Carrette (2003)).

Adaptive Fourier basis

For a given vector of pairwise different values $(z_0, z_1, \ldots, z_N)$ with $0 < |z_\ell| < 1$ let

$$P_\ell(t) := \frac{1}{(1 - z_\ell \mathrm{e}^{it})}, \quad \ell = 0, \ldots, N.$$

Theorem

The set of functions $\{B_\ell : \ell = 0, \ldots, N\}$ of the Takenaka-Malmquist system with the zeros $z_0, z_1, \ldots z_N$ in $\mathbb{D}_0$ spans the same subspace as $\{P_\ell : \ell = 0, \ldots, N\}$, respectively.

We will use this simpler basis $\{P_\ell\}_{\ell=0}^{\infty}$.

Adaptive Fourier sums and its convergence

Idea.
Approximate the Fourier coefficients $c_k(f^+)$ by exponential sums

$$\widetilde{c}_k^{(N)} = \sum_{\ell=0}^{N} a_\ell z_\ell^k, \quad k = 0, 1, \ldots$$

where we take $a_\ell \in \mathbb{C}$ and $z_\ell \in \mathbb{D}$ such that

$$\|\widetilde{\mathbf{c}}^{(N)} - \mathbf{c}(f^+)\|_{\ell^2}^2 := \sum_{k=0}^{\infty} |\tilde{c}_k^{(N)} - c_k(f^+)|^2 < \epsilon.$$

Then for $f_N^+(t) := \sum_{k=0}^{\infty} \widetilde{c}_k^{(N)} \, \mathrm{e}^{ikt}$ we obtain

$$\|f^+ - f_N^+\|_{L^2} = \|\mathbf{c}(f^+) - \widetilde{\mathbf{c}}^{(N)}\|_{\ell^2} < \epsilon.$$

Adaptive Fourier sums and its convergence

Taking the approximation $f_N^+(t) := \sum_{k=0}^{\infty} \tilde{c}_k^{(N)} \, \mathrm{e}^{ikt}$ with $\tilde{c}_k^{(N)} = \sum_{\ell=0}^{N} a_\ell z_\ell^k$ we have

$$\begin{aligned} f_N^+(t) &= \sum_{k=0}^{\infty} \tilde{c}_k^{(N)} \, \mathrm{e}^{ikt} = \sum_{k=0}^{\infty} \Big(\sum_{\ell=0}^{N} a_\ell z_\ell^k \Big) \, \mathrm{e}^{ikt} \\ &= \sum_{\ell=0}^{N} a_\ell \sum_{k=0}^{\infty} (z_\ell \mathrm{e}^{it})^k = \sum_{\ell=0}^{N} a_\ell \frac{1}{(1 - z_\ell \mathrm{e}^{it})} = \sum_{\ell=0}^{N} a_\ell \, P_\ell(t). \end{aligned}$$

Infinite Hankel matrices

We relate the problem $\min_{a_\ell, z_\ell} \left\| \mathbf{c}(f^+) - \left(\sum_{\ell=0}^{N} a_\ell z_\ell^k \right)_{k=0}^{\infty} \right\|_{\ell^2}$

to the problem of low-rank approximation of Hankel operators.

Approximate

$$\Gamma_{\mathbf{c}(f^+)} = \Gamma_{\mathbf{c}} := \begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ c_1 & c_2 & c_3 & \cdots \\ c_2 & c_3 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (c_{k+j})_{k,j=0}^{\infty}.$$

by an infinite Hankel matrix with rank $N+1$.

Theorem (Kronecker)

The infinite Hankel matrix $\Gamma_{\tilde{\mathbf{c}}} : \ell^2 \to \ell^2$ generated by $\tilde{\mathbf{c}} = (\tilde{c}_k)_{k=0}^{\infty} \in \ell^1$ of the form $\tilde{c}_k = \sum_{j=0}^{N} a_j z_j^k$, $k \in \mathbb{N}_0$, has finite rank $N+1$.

Theorem of Adamjan, Arov and Krein

Theorem (Adamjan, Arov and Krein (1971))

Let $\mathbf{c} \in \ell_1$ be a given sequence and let $\Gamma_{\mathbf{c}}$ be the corresponding infinite Hankel matrix on ℓ^2. Further, let $\sigma_0(\Gamma_{\mathbf{c}}) \geq \sigma_1(\Gamma_{\mathbf{c}}) \geq \sigma_2(\Gamma_{\mathbf{c}}) \geq \ldots$ denote the singular values of $\Gamma_{\mathbf{c}}$ in decreasing order. Then, for each $N \in \mathbb{N}_0$ there exists an infinite Hankel matrix $\Gamma_{\tilde{\mathbf{c}}}$ of rank $N+1$ such that

$$\|\Gamma_{\mathbf{c}} - \Gamma_{\tilde{\mathbf{c}}}\|_{\ell^2 \to \ell^2} = \sigma_{N+1}(\Gamma_{\mathbf{c}}).$$

In particular,

$$\|f^+ - f_N^+\|_{L^2} = \|\mathbf{c}(f^+) - \tilde{\mathbf{c}}\|_{\ell^2} \leq \|\Gamma_{\mathbf{c}(f^+)} - \Gamma_{\tilde{\mathbf{c}}}\|_{\ell^2 \to \ell^2} = \sigma_{N+1}(\Gamma_{\mathbf{c}(f^+)}).$$

Question: What do we know about the decay of the singular values $\sigma_N(\Gamma_{\mathbf{c}(f^+)})$ for $N \to \infty$?

Decay of singular values of Hankel operators

Theorem (Pushnitski & Yafaev (2015))

Let $\mathbf{c} = (c_k)_{k=0}^{\infty}$ *satisfy the decay conditions for* $k \to \infty$

$$c_k = \mathcal{O}(k^{-1}(\log k)^{-\alpha}), \quad \text{for } \alpha > 0, \tag{1}$$

$$\Delta^m c_k = \mathcal{O}(k^{-1-m}(\log k)^{-\alpha}), \ \text{for } \alpha > \frac{1}{2}, \ m = 0, \ldots, \lfloor \alpha \rfloor + 1. \tag{2}$$

Then

$$\sigma_N(\Gamma_{\mathbf{c}}) = \mathcal{O}(N^{-\alpha}), \quad N \to \infty.$$

Example: For $c_k = (1+k)^{-\gamma}$ with $\gamma > 1$ we even have exponential decay

$$\sigma_N(\Gamma_{\mathbf{c}}) = \mathrm{e}^{-\pi(\sqrt{2\gamma N} + o(\sqrt{N}))}$$

[Widom (1966)].

Decay in the adaptive Fourier basis

We arrive at

Theorem (Pl. & Pototskaia (2018))

Let $f^+(t) = \sum_{k=0}^{\infty} c_k(f^+)\, \mathrm{e}^{ikt}$ be a function in H^2. If $\mathbf{c} = (c_k(f^+))_{k=0}^{\infty}$ satisfies the decay conditions (1) and (2) then for each $N \in \mathbb{N}$, there exists a vector of zeros $z_0^{(N)}, \ldots, z_N^{(N)}$ determining a Takenaka-Malmquist system $\{B_k\}_{k=0}^{N}$ such that the N-th adaptive partial Fourier sum $f_N^+ = \sum_{k=0}^{N} \langle f, B_k \rangle_{L^2} B_k$ satisfies the asymptotic estimate

$$\|f^+ - f_N^+\|_{L^2} \leq CN^{-\alpha}$$

where C does not depend on N.

Computation of adaptive Fourier sums

Theorem (AAK (1971))

Let $(\sigma_{N+1}, \mathbf{u}^{(N+1)})$ *be a fixed singular pair of* $\Gamma_{\mathbf{c}}$ *with* $\sigma_{N+1} \notin \{\sigma_k\}_{k \neq N+1}$ *and* $\sigma_{N+1} \neq 0$.

- *Then*

$$P_{\mathbf{u}^{(N+1)}}(x) := \sum_{k=0}^{\infty} u_k^{(N+1)} x^k$$

has exactly $N+1$ *zeros* $z_0^{(N)}, \ldots, z_N^{(N)}$ *in* $\mathbb{D}$.

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has exactly $N+1$ *zeros* $z_0^{(N)}, \ldots, z_N^{(N)}$ *in* $\mathbb{D}$.

- *There are coefficients* $a_0^{(N)}, \ldots, a_N^{(N)} \in \mathbb{C}$ *such that for*
$$\tilde{\mathbf{c}} = (\tilde{c}_j)_{j=0}^{\infty} = \left(\sum_{k=0}^{N} a_k^{(N)} (z_k^{(N)})^j \right)_{j=0}^{\infty}$$
we have
$$\|\Gamma_{\mathbf{c}} - \Gamma_{\tilde{\mathbf{c}}}\|_{\ell^2 \to \ell^2} = \sigma_{N+1}.$$

Computation of adaptive Fourier sums

Input: samples $c_k = c_k(f^+)$, $k = 0, \ldots, L$, for a sufficiently large L.

1. Use a Prony-like method to find a very good approximation of c_k, $k = 0, \ldots, L$, in the form

$$\breve{c}_k = \sum_{j=0}^{M} \breve{a}_j \, \breve{z}_j^k, \qquad k \in \mathbb{N}, \; M > N.$$

2. Compute the $M+1$ nonzero singular values $\sigma_0 > \ldots > \sigma_M$ of $\Gamma_{\breve{\mathbf{c}}}$ and the singular vector $\mathbf{u}^{(N+1)}$ to σ_{N+1}.
3. Compute the zeros $\tilde{z}_0^{(N)}, \ldots, \tilde{z}_N^{(N+1)}$ of $P_{\mathbf{u}^{(N+1)}}(x) := \sum_{k=0}^{\infty} u_k^{(N+1)} x^k$ in $\mathbb{D}$.

Output: $\tilde{z}_0^{(N)}, \ldots, \tilde{z}_N^{(N)}$ determining the adaptive basis.

Computation of adaptive Fourier sums

Step 2 can be reduced to solve a con-eigenvalue problem of size $M + 1$, see
Beylkin & Monzon (2005)
Pl. & Pototskaia (2016)

Step 3 can be solved using the rational representation of

$$P_{\mathbf{u}^{(N+1)}}(x) := \sum_{k=0}^{\infty} u_k^{(N+1)} x^k,$$

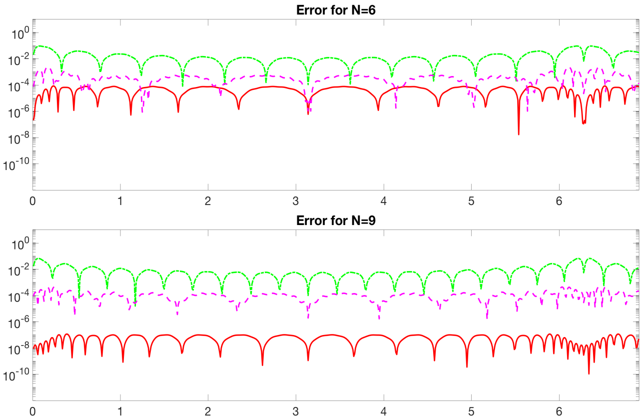
since $\mathbf{u}^{(N+1)} = (u_k^{(N+1)})_{k=0}^{\infty}$ can be exactly represented by an exponential sum of length $M + 1$.

Numerical example

Consider $f(t) = \sum_{k=1}^{\infty} c_k(f) \sin(kt)$ with $c_k(f) = \frac{1}{k^2}$.
Use 101 classical Fourier coefficients.

N	σ_N	$\|f - f_N\|_{L_2}$	$\|f - f_N^{(G)}\|_{L_2}$
1	1.1078e-01	1.1057e-01	8.2932e-02
2	2.2710e-02	2.2693e-02	6.0175e-02
3	6.0753e-03	6.0538e-03	1.1511e-02
4	1.6706e-03	1.6635e-03	6.8640e-03
5	3.9237e-04	3.8919e-04	1.5419e-03
6	7.4187e-05	7.3599e-05	1.0114e-03
7	1.1081e-05	1.0949e-05	6.1344e-04
8	1.2559e-06	1.2382e-06	4.5046e-04
9	9.8620e-08	9.6716e-08	2.9683e-04
10	4.2914e-09	9.6368e-09	2.8196e-04

Numerical example



Error $|f(t) - S_N(f)(t)|$ (green), $|f(t) - f_N^{(G)}(t)|$ (magenta) and $|f(t) - f_N(t)|$ (red) for $t \in [-\pi, \pi]$.

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\thankyou