# Reconstruction of Stationary and Non-stationary Signals by the Generalized Prony Method

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**Abstract.** We employ the generalized Prony method in [15] to derive new reconstruction schemes for a variety of sparse signal models using only a small number of signal measurements. By introducing generalized shift operators, we study the recovery of sparse trigonometric and hyperbolic functions as well as sparse expansions into Gaussians chirps and modulated Gaussian windows. Furthermore, we show how to reconstruct sparse polynomial expansions and sparse non-stationary signals with structured phase functions.

**Key words**: generalized Prony method, exponential sums, trigonometric sums, modulated Gaussian windows, Gaussian chirps, non-stationary signals, signal reconstruction, empirical mode decomposition.
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## 1 Introduction

The recovery of signals possessing a given structure is an important problem in various applications as e.g. wireless telecommunication [21], image super-resolution [23], nondestructive testing [5], and phase retrieval [3].

We assume that we can employ certain a priori knowledge about the underlying signal model, where a small number of parameters needs to be determined from given signal measurements in order to recover the structure of the signal. A prototype of such a signal is an exponential sum of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{x\alpha_j} \tag{1.1}$$

with unknown complex parameters $c_j$ and $\alpha_j$, $j = 1, \ldots, M$, which need to be recovered from measurement values of $f$. Here and in all other signal models we always assume that the parameters $\alpha_j$ are pairwise different and that all coefficients $c_j$ are

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nonzero. Otherwise the exponential sum can be suitably simplified to a sum with less terms.

It is well-known that this recovery problem can be solved by Prony's method using equidistant point evaluations $f(x_0 + kh)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ can be arbitrarily chosen, and $h \neq 0$ is a suitable step size. Indeed, the model (1.1) already covers many important applications. For example, taking $\alpha_j = -\mathrm{i}t_j$ with $t_j \in \mathbb{R}$, (1.1) is closely related to the model

$$g(t) = \sum_{j=1}^{M} c_j \, \delta(t - t_j), \tag{1.2}$$

a finite stream of Diracs. Since a shifted Dirac distribution $\delta(t - t_j)$ is a tempered distribution, its distributional Fourier transform is well-defined and equal to $\mathrm{e}^{-\mathrm{i}t_j x}$. The distributional Fourier transform of $g$ is here an extension of the classical Fourier transform $\widehat{g}(x) := \int_{-\infty}^{\infty} g(t)\, \mathrm{e}^{-\mathrm{i}tx} \mathrm{d}t$ for functions $g \in L_1(\mathbb{R})$ and equals to $f$ in (1.1) with $\alpha_j = -\mathrm{i}t_j$. Therefore, $g$ can be recovered from equidistant Fourier samples. More generally, considering a finite linear combination of arbitrary shifts of a given function $\phi$,

$$g(t) = \sum_{j=1}^{M} c_j \, \phi(t - t_j) \tag{1.3}$$

with $t_j \in \mathbb{R}$ leads by Fourier transform to the product $\left( \sum_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i}t_j x} \right) \widehat{\phi}(x)$ of an exponential sum of the form (1.1) with the Fourier transform $\widehat{\phi}(x)$, and can therefore also be recovered from suitable equidistant Fourier samples of $g$, see [9, 17, 19, 22]. Signals as given in (1.2) and (1.3) are said to have finite rate of innovation, [22]. Applying the Laplace transform to (1.1) with the restriction $\alpha_j \in \mathbb{R}$, we find

$$h(s) = \mathcal{L}f(s) = \sum_{j=1}^{M} \frac{c_j}{s - \alpha_j}.$$

Therefore, the rational function $h(s)$ can be reconstructed from equidistant samples of its inverse Laplace transform using Prony's method. Also, sparse expansions into shifted Lorentzian functions with the Fourier transform $\mathrm{e}^{-\mathrm{i}t_j x - \alpha_j |x|/2}$, where $t_j$ and $\alpha_j$ denote the shifts and the function width, can be recovered using the model (1.1), see [2].

However, many applications require more general signal models, for example

$$f(x) = \sum_{j=1}^{M} c_j(x) \, \mathrm{e}^{\mathrm{i}\phi_j(x)},$$

which we call *stationary* if and only if the *amplitudes* $c_j$ are constant and the *phase functions* $\phi_j$ are linear. Thus the exponential sum in (1.1) is stationary for $\alpha_j = \mathrm{i}t_j$ with $t_j \in \mathbb{R}$. In contrast, the signal is called *non-stationary* if these assumptions are no longer satisfied.

Non-stationary signals play an important role in many applications in signal processing, since natural signals usually change their behavior over time. Stationary signal analysis methods are usually not well suited for decomposing these signals, and some effort has been put into deriving new techniques for non-stationary signal decomposition. In [11] the empirical mode decomposition (EMD) has been proposed, a greedy non-parametric method to decompose non-stationary signals into so-called intrinsic mode functions. For applications and further investigation of EMD we refer to [12] and the references therein. Mathematically improved techniques include the synchrosqueezed wavelet transform [10] and the signal decomposition method using a signal separation operator [6]. Other recent attempts for signal reconstruction are based on hybrid methods that employ wavelet analysis and the finite rate of innovation approach, see [23].

Using more a priori information about the signal structure in a sparse model, we aim at a direct identification of the important signal components. In [4], the reconstruction of piecewise sinusoidal signals has been studied with finite rate of innovation methods based on the model (1.1), but this model is still restricted to linear phase functions. Recently, the generalized Prony method has been proposed in [15]. This approach allows the recovery of sparse sums of eigenfunctions of linear operators. It covers the reconstruction of exponential sums in (1.1) as a special case, where the exponentials are interpreted as eigenfunctions of the shift operator.

In this paper we want to exploit the generalized Prony method introduced in [15] and derive reconstruction procedures for different signal models that go essentially beyond the exponential sum in (1.1). Here, we particularly restrict ourselves to models that can be recovered just from direct measurement values, i.e., point evaluations of the signal. In Section 2.3 we will indicate how other sampling schemes can be also used instead.

Employing generalized shift operators we will present new recovery methods for a variety of signal models as e.g. linear combinations of Gaussian chirps, expansions into modulated Gaussian windows, and non-stationary trigonometric expansions. For each of these new models, we will present a reconstruction method and show which signal measurements are needed for the recovery.

The paper is organized as follows. In Section 2 we recall the Prony method and present its interpretation as a recovery method for sparse sums of eigenfunctions of the shift operator. We also describe some sampling schemes to reconstruct the exponential sum which we are allowed to use by exploiting the generalized Prony method. In Subsection 2.3, we introduce generalizations of the shift operator and show their basic properties.

Sections 3 − 6 are devoted to the investigation of various signal models that are based on the generalized shift operators. In Section 3 we consider sparse expansions into trigonometric and hyperbolic functions that can be reconstructed using the symmetric shift operator. In Section 4 we study the recovery of expansions into Gaussian chirps and modulated Gaussians.

In Section 5 we reconstruct sparse expansions into monomials and Gaussian chirps with different scaling. Moreover, sparse expansions into Chebyshev polynomials and linear combinations of non-stationary exponentials as e.g. $\mathrm{e}^{\alpha_j \cos x}$ can be recovered.

Section 6 is devoted to the recovery of further non-stationary signals whose components possess non-linear phase functions of the form $\alpha_j x^p + \beta_j$ with known real parameter $p > 0$ and unknown parameters $\alpha_j$, $\beta_j \in \mathbb{R}$ or with phase functions of the form $x^2 + \alpha_j x + \beta_j$.

Finally, we illustrate the signal models by some numerical examples in Section 7. Stability issues of the numerical methods will be more closely considered in a forthcoming paper.

# 2 Prony's Method for Exponential Sums Revisited

## 2.1 The Prony method

Let us first recall Prony's method to reconstruct $f(x)$ in (1.1) using equidistant samples of $f$. The function $f(x)$ in (1.1) can be interpreted as the solution of a linear difference equation with constant coefficients. This observation is the key to this recovering method. The main idea is to reconstruct the parameters $\mathrm{e}^{\alpha_j}$ in a first step, and the coefficients $c_j$ in a second step. Since we assumed that $\alpha_j \in \mathbb{C}$, we need to pay attention to the fact that the $\alpha_j$ may not be uniquely determined by $\mathrm{e}^{\alpha_j}$ due to the $2\pi$-periodicity of the complex exponential $\mathrm{e}^{\mathrm{i}x}$.

Therefore, we assume an a priori known bound $|\mathrm{Im}\,\alpha_j| < T$ and choose a sampling with step size $h < \frac{\pi}{T}$ such that $|\alpha_j h| < \pi$. Then, $\alpha_j$ can be uniquely recovered from $\mathrm{e}^{\alpha_j h}$. Now, we define the characteristic polynomial (Prony polynomial)

$$P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{\alpha_j h}) = \prod_{j=1}^{M} (z - \lambda_j) \tag{2.1}$$

with $\lambda_j := \mathrm{e}^{\alpha_j h}$. Based on the monomial representation

$$P(z) = \sum_{k=0}^{M} p_k\, z^k = z^M + \sum_{k=0}^{M-1} p_k\, z^k,$$

we consider the homogeneous linear difference equation of order $M$ for $f$ in (1.1),

$$\begin{aligned} \sum_{k=0}^{M} p_k\, f(h(k+m)) &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j\, \lambda_j^{(k+m)} = \sum_{j=1}^{M} c_j \lambda_j^m \Big( \sum_{k=0}^{M} p_k\, \lambda_j^k \Big) \\ &= \sum_{j=1}^{M} c_j \lambda_j^m\, P(\lambda_j) = 0, \end{aligned}$$

which is satisfied for all $m \in \mathbb{Z}$. Exploiting that $p_M = 1$, we derive the linear system

$$\sum_{k=0}^{M-1} p_k\, f(h(k+m)) = -f(h(M+m)), \qquad m \in \mathbb{Z} \tag{2.2}$$

from the given function values $f(h\ell)$, $\ell = 0, \ldots, 2M-1$. The coefficient matrix $\mathbf{H} = (f(h(k+m)))_{k,m=0}^{M-1}$ in (2.2) has Hankel structure, and the linear system in (2.2) is uniquely solvable, provided that the values $\lambda_j = \mathrm{e}^{\alpha_j h}$ in (1.1) are pairwise different and $c_j \neq 0$ for $j = 1, \ldots, M$. This can be easily seen from the factorization

$$\mathbf{H} = \mathbf{V}\, \mathrm{diag}\,(c_1, \ldots, c_M)\, \mathbf{V}^T,$$

where $\mathbf{V}$ denotes the Vandermonde matrix $\mathbf{V} = (\lambda_j^k)_{k,j=0}^{M-1}$.

Having found the coefficients $p_k$ of the Prony polynomial $P(z)$ by solving (2.2), we can extract the zeros $\lambda_j = \mathrm{e}^{\alpha_j h}$, $j = 1, \ldots, M$ and finally determine the parameters $c_j$ in (1.1) by solving the (overdetermined) linear system

$$f(\ell) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\ell \alpha_j} = \sum_{j=1}^{M} c_j\, \lambda_j^\ell, \qquad \ell = 0, \ldots, 2M-1.$$

The described method has used the function values $f(h\ell)$, $\ell = 0, \ldots, 2M-1$. But a careful inspection of this approach shows that there is no need to start with $f(0)$. We can equivalently start with an arbitrary value $x_0 \in \mathbb{R}$ and apply the samples $f(x_0 + \ell h)$, $\ell = 0, \ldots, 2M-1$ to reconstruct (1.1). As before, we get

$$\begin{aligned} \sum_{k=0}^{M} p_k f(x_0 + h(k+m)) &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \mathrm{e}^{x_0 \alpha_j} \lambda_j^{(k+m)} = \sum_{j=1}^{M} c_j \mathrm{e}^{x_0 \alpha_j} \lambda_j^{m} \Big( \sum_{k=0}^{M} p_k \lambda_j^k \Big) \\ &= \sum_{j=1}^{M} c_j \, \mathrm{e}^{x_0 \alpha_j} \, \lambda_j^m \, P(\lambda_j) = 0, \end{aligned} \tag{2.3}$$

which leads to a similar Hankel system as in (2.2). For a recent survey on Prony's method and its applications we refer to [18].

## 2.2 Revisiting Prony's Method Using the Shift Operator

Following the ideas in [15], we now want to look at the exponential sum in (1.1) from a different point of view, i.e. as an expansion into eigenfunctions of a suitably chosen operator.

We consider the usual shift operator $S_h \colon C(\mathbb{R}) \to C(\mathbb{R})$ acting on the vector space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$, given by

$$S_h \, f \coloneqq f(\cdot + h), \qquad h \in \mathbb{R} \setminus \{0\}. \tag{2.4}$$

Then, for each $\alpha \in \mathbb{C}$ we have

$$(S_h \mathrm{e}^{\alpha \cdot})(x) = \mathrm{e}^{\alpha(h+x)} = \mathrm{e}^{\alpha h} \, \mathrm{e}^{\alpha x},$$

i.e., $\mathrm{e}^{\alpha x}$ is an eigenfunction of $S_h$ with the eigenvalue $\mathrm{e}^{\alpha h}$. Therefore, we can interpret the signal $f(x)$ in (1.1) as a sparse linear combination of eigenfunctions of the shift operator $S_h$ and obtain

$$S_h f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j (x+h)} = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j h} \, \mathrm{e}^{\alpha_j x} = \sum_{j=1}^{M} c_j \, \lambda_j \, \mathrm{e}^{\alpha_j x}$$

with $\lambda_j \coloneqq \mathrm{e}^{\alpha_j h}$, and the eigenvalues $\lambda_j$ of the "active" eigenfunctions in the exponential sum are the zeros of the Prony polynomial $P(z)$ in (2.1).

As we have seen before, the eigenfunction $\mathrm{e}^{\alpha_j x}$ is only determined uniquely by its eigenvalue $\mathrm{e}^{\alpha_j h}$, if we have the a priori information that $\mathrm{Im}\, \alpha_j$ is contained in a fixed interval of length $\frac{2\pi}{h}$, since the eigenspace of $\mathrm{e}^{\alpha_j h}$ is very large. For $\alpha_j \in \mathbb{C}$ the relation

$$S_h \mathrm{e}^{\alpha_j x} = \mathrm{e}^{\alpha_j h} \, \mathrm{e}^{\alpha_j x}$$

also implies

$$S_h \mathrm{e}^{(\alpha_j + \frac{2\pi \mathrm{i} k}{h})x} = \mathrm{e}^{(x+h)(\alpha_j + \frac{2\pi \mathrm{i} k}{h})} = \mathrm{e}^{\alpha_j h} \, \mathrm{e}^{(\alpha_j + \frac{2\pi \mathrm{i} k}{h})x}.$$

Therefore, for each eigenvalue $\mathrm{e}^{\alpha_j h}$ we find the eigenspace spanned by the eigenfunctions $\{\mathrm{e}^{x(\alpha_j + \frac{2\pi \mathrm{i} k}{h})} : k \in \mathbb{Z}\}$.

We can now reinterpret Prony's method for the reconstruction of the exponential sum in (1.1) as follows. Assume that all wanted parameters $\alpha_j$ satisfy $|\mathrm{Im}\, \alpha_j| < T$ and $h < \frac{\pi}{T}$.

Then, a given finite linear combination of $M$ eigenfunctions of the shift operator $S_h$ with the corresponding $M$ pairwise different eigenvalues $\mathrm{e}^{\alpha_j h}$, $j = 1, \ldots, M$, and with nonzero complex coefficients $c_j$ can be completely recovered from the values $S_h^\ell f(x_0) = S_{h\ell} f(x_0)$, $\ell = 0, \ldots, 2M-1$, with $x_0 \in \mathbb{R}$. Using the shift operator $S_h$ in (2.4) the homogeneous difference equation in (2.3) can be simply rewritten as

$$\begin{aligned} \sum_{k=0}^{M} p_k (S_{h(k+m)} f)(x_0) &= \sum_{k=0}^{M} p_k (S_h^{k+m} f)(x_0) = \sum_{k=0}^{M} p_k S_h^{k+m} \Big( \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cdot} \Big)(x_0) \\ &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \, (S_h^{k+m} \mathrm{e}^{\alpha_j \cdot})(x_0) = \sum_{j=1}^{M} c_j \, \sum_{k=0}^{M} p_k \, \lambda_j^{m+k} \, \mathrm{e}^{\alpha_j \, x_0} \\ &= \sum_{j=1}^{M} c_j \lambda_j^m \, \Big( \sum_{k=0}^{M} p_k \, \lambda_j^k \Big) \mathrm{e}^{\alpha_j x_0} = 0. \end{aligned}$$

As before, the coefficients $p_k$ of the Prony polynomial can be computed by the Hankel system as in (2.2) with the coefficient matrix $\mathbf{H} = \Big( (S_{h(k+\ell)} f)(x_0) \Big)_{k,\ell=0}^{M-1}$, and the procedure to evaluate all parameters in (1.1) is the same as before.

To make this procedure work, we have essentially used two properties of the shift operator, namely:

- $S_h$ is a linear operator,
- $\mathrm{e}^{\alpha x}$ is the unique eigenfunction of $S_h$ to the eigenvalue $\mathrm{e}^{\alpha h}$ for each $\alpha \in \mathbb{C}$ with $\mathrm{Im}\, \alpha \in (-\pi/h, \pi/h)$.

## 2.3 Sampling Schemes for Recovering Exponential Sums

As we have seen, the exponential sum in (1.1) can be completely reconstructed using the equidistant function values $f(x_0 + h\ell)$, $\ell = 0, \ldots, 2M-1$. These values can be understood as the application of a point evaluation functional $F_{x_0}$ with

$$F_{x_0}(S_h^\ell f) = F_{x_0}(S_{h\ell} f) := f(h\ell + x_0).$$

In this paper, we are especially interested in deriving signal models that can be completely recovered using only signal values. However, in this section we want to emphasize that the generalized Prony method in [15] allows a higher flexibility of sampling schemes. In [15], it has been shown that instead of using a point evaluation functional we can also employ another linear functional $F$ to $S_{h\ell} f$ satisfying the assumption that all eigenfunctions of the shift operator (which may play an active role in the exponential sum to be recovered) do not vanish under the action of $F$.

Using this generalized approach, we replace our measurements $(S_h^\ell f)(x_0) = f(x_0 + h\ell)$ by $F(S_h^\ell f)(x)$. Then, the recovery of the parameters $\alpha_j$ can be still achieved by evaluating the coefficients of the Prony polynomial $P(z)$ in the first step,

$$\begin{aligned} \sum_{k=0}^{M} p_k F(S_h^{k+m} f) &= \sum_{k=0}^{M} p_k F\Big( S_h^{k+m} \Big( \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cdot} \Big) \Big) = \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \, F(S_h^{k+m} \mathrm{e}^{\alpha_j \cdot}) \\ &= \sum_{j=1}^{M} c_j \, \sum_{k=0}^{M} p_k \, \lambda_j^{m+k} \, F(\mathrm{e}^{\alpha_j \cdot}) = \sum_{j=1}^{M} c_j \lambda_j^m \, \Big( \sum_{k=0}^{M} p_k \, \lambda_j^k \Big) F(\mathrm{e}^{\alpha_j \cdot}) = 0. \end{aligned}$$

Therefore, we obtain the Hankel system

$$\sum_{k=0}^{M-1} p_k\, F(S_h^{k+m} f) = \sum_{k=0}^{M-1} p_k\, F(f(\cdot + h(k+m))) = -F(f(\cdot + h(M+m))) \quad (2.5)$$

for $m \in \mathbb{Z}$, where the Hankel matrix $\mathbf{H} = \left(F(S_h^{k+m} f)\right)_{k,m=0}^{M-1}$ is invertible, since we have

$$\mathbf{H} = \mathbf{V}\, \mathrm{diag}\,(c_1, \ldots, c_M)\, \mathrm{diag}\,(F(\mathrm{e}^{\alpha_1 \cdot}), \ldots, F(\mathrm{e}^{\alpha_M \cdot}))\, \mathbf{V}^T$$

with the Vandermonde matrix $\mathbf{V} = (\lambda_j^k)_{k,j=0}^{M-1} = (\mathrm{e}^{\alpha_j h k})_{k,j=0}^{M-1}$.

To illustrate the variety of possible sampling schemes, we give two examples.

1. Assume that we know a priori that the parameters in (1.1) satisfy $\mathrm{Im}\, \alpha_j \in (0,T)$ and choose $0 < h < \frac{2\pi}{T}$. Then we can consider

$$Ff := \int_{x_0}^{x_0+h} f(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, \chi_{[0,h]}(x - x_0)\, \mathrm{d}x = \langle f, \chi_{[0,h]}(\cdot - x_0)\rangle,$$

where $\chi_{[0,h]}$ denotes the characteristic function on $[0,h]$, and the condition

$$F\mathrm{e}^{\alpha \cdot} = \int_{x_0}^{x_0+h} \mathrm{e}^{\alpha x} \mathrm{d}x \neq 0$$

is obviously satisfied for all $\alpha \in (0,T)$ since $\mathrm{e}^{\alpha h} \neq 1$. With this sampling functional it is sufficient to take the values

$$F(S_h^{\ell} f) = \int_{x_0}^{x_0+h} f(x + h\ell)\, \mathrm{d}x = \int_{x_0+h\ell}^{x_0+h(\ell+1)} f(x)\, \mathrm{d}x, \qquad \ell = 0, \ldots, 2M-1,$$

to reconstruct $f$.

2. With the assumption $\alpha_j = -\mathrm{i}t_j$ and $t_j \in (-1,1)$ we consider the functional

$$Ff := \int_{-\infty}^{\infty} f(x)\, \overline{\Phi(x)}\, \mathrm{d}x = \langle f,\, \Phi\rangle$$

with $\Phi(x) = \frac{1}{\pi}\mathrm{sinc}\,(x) = \frac{1}{\pi}\frac{\sin(x)}{x}$ for $x \in \mathbb{R} \setminus \{0\}$ and $\Phi(0) = \frac{1}{\pi}$. The Fourier transform of the sinc kernel is the box function $\widehat{\Phi}(t) = \chi_{[-1,1]}(t)$ and the Parseval identity implies for $g$ in (1.2) with $\widehat{g}(x) = f(x)$ that

$$Ff(\cdot + h\ell) = \langle f(\cdot + h\ell),\, \Phi\rangle = \langle \widehat{g}(\cdot + h\ell),\, \Phi\rangle = \langle g,\, \widehat{\Phi}\, \mathrm{e}^{\mathrm{i}\ell h \cdot}\rangle.$$

In particular, since $g$ is a stream of diracs, it follows that

$$Ff(\cdot + h\ell) \;=\; \sum_{j=1}^{M} c_j\, \langle \delta(\cdot - t_j), \widehat{\Phi}\, \mathrm{e}^{\mathrm{i}\ell h \cdot}\rangle = \sum_{j=1}^{M} c_j\, \chi_{[-1,1]}(t_j) \mathrm{e}^{\mathrm{i}\ell h t_j} = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\mathrm{i}\ell h t_j}.$$

Therefore, also the samples $\langle f(\cdot + h\ell), \Phi\rangle$, $\ell = 0, \ldots, 2M-1$, are sufficient to recover $f$ in (1.1).

## 2.4 Generalized Shift Operators

We want to generalize the shift operator in order to be able to recover many more signal models beyond exponential sums. In particular, we consider the following linear operators.

**A)** Let $S_{h,-h}\colon C(\mathbb{R}) \to C(\mathbb{R})$ denote the *symmetric shift operator* for given $h \leq 0$ by

$$S_{h,-h}f(x) := \frac{1}{2}\Big(f(x-h) + f(x+h)\Big) = \frac{1}{2}(S_{-h} + S_h)f(x). \tag{2.6}$$

**B)** Let $H\colon \mathbb{R} \to \mathbb{C}$ be a given continuous function that does not vanish in the sampling domain $[a,b] \subseteq \mathbb{R}$. Further, let the function $G\colon [a,b] \to \mathbb{R}$ be continuous and strictly monotone in the sampling domain $[a,b] \subseteq \mathbb{R}$ and let $G^{-1}$ denote its inverse function. We introduce the shift operator $S_{H,G,h}\colon C([a,b]) \to C(\mathbb{R})$ for $h \in \mathbb{R}$ by

$$S_{H,G,h}f(x) := \frac{H(x)}{H(G^{-1}(G(x)+h))}\, f(G^{-1}(G(x)+h)). \tag{2.7}$$

In the special case $G(x) = x$ this shift operator simplifies to $S_{H,h}$ with

$$S_{H,h}f(x) := H(x) S_h \left(\frac{f}{H}\right)(x) = \frac{H(x)}{H(x+h)}\, f(x+h). \tag{2.8}$$

If $H(x)$ is a nonvanishing constant function, the shift operator $S_{H,G,h}$ simplifies to

$$S_{G,h}f(x) := f(G^{-1}(G(x)+h)). \tag{2.9}$$

These operators have the following properties.

**Theorem 2.1.** *Let the operators $S_{h,-h}$ and $S_{H,G,h}$ be defined as above. Then we have*

$$\begin{aligned} S_{h_2,-h_2}(S_{h_1,-h_1}f) &= S_{h_1,-h_1}(S_{h_2,-h_2}f) \\ &= \tfrac{1}{2}\left(S_{h_1+h_2,-(h_1+h_2)}f + S_{h_1-h_2,-(h_1-h_2)}f\right), \qquad (2.10) \\ S_{H,G,h_1}(S_{H,G,h_2}f) &= S_{H,G,h_2}(S_{H,G,h_1}f) = S_{H,G,h_1+h_2}f, \qquad (2.11) \end{aligned}$$

*In particular,*

$$\begin{aligned} S^k_{h,-h}f &= \frac{1}{2^{k-1}} \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{\ell} (S_{(k-2\ell)h,-(k-2\ell)h}f + \delta_{k/2,\lfloor k/2\rfloor} \frac{1}{2^k}\binom{k}{k/2} f, \\ S^k_{H,G,h}f &= S_{H,G,kh}f, \qquad S^k_{G,h}f = S_{G,kh}f, \qquad S^k_{H,h}f = S_{H,kh}f, \qquad (2.12) \end{aligned}$$

*where $\delta_{k/2,\lfloor k/2\rfloor}$ is the Kronecker symbol with $\delta_{k/2,\lfloor k/2\rfloor} = 1$ for even $k$ and $\delta_{k/2,\lfloor k/2\rfloor} = 0$ for odd $k$.*

**Proof.** 1. We find for the symmetric shift

$$\begin{aligned} & S_{h_2,-h_2}(S_{h_1,-h_1}f)(x) = S_{h_2,-h_2}\left(\frac{1}{2}(f(x+h_1) + f(x-h_1))\right) \\ = \; & \frac{1}{4}\left(f(x+h_1+h_2) + f(x-h_1+h_2) + f(x+h_1-h_2) + f(x-h_1-h_2)\right) \\ = \; & \frac{1}{2}\left((S_{h_1+h_2,-(h_1+h_2)}f)(x) + (S_{h_1-h_2,-(h_1-h_2)}f)(x)\right). \end{aligned}$$

Repeated application of the operator $S_{h,-h}$ yields

$$\begin{aligned} S^k_{h,-h} f &= \frac{1}{2^k}(S_{-h}+S_h)^k(f) = \frac{1}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell} S^{\ell}_{-h} S_h^{k-\ell} f \\ &= \frac{1}{2^k}\sum_{\ell=0}^{\lfloor (k-1)/2\rfloor}\binom{k}{\ell}(S_{(k-2\ell)h}+S_{-(k-2\ell)h})f + \delta_{k/2,\lfloor k/2\rfloor}\frac{1}{2^k}\binom{k}{k/2} f \\ &= \frac{1}{2^{k-1}}\sum_{\ell=0}^{\lfloor (k-1)/2\rfloor}\binom{k}{\ell} S_{(k-2\ell)h,-(k-2\ell)h} f + \delta_{k/2,\lfloor k/2\rfloor}\frac{1}{2^k}\binom{k}{k/2} f. \end{aligned}$$

2. Assume that $H$ is a nonvanishing continuous function in $[a,b]\subset\mathbb{R}$, $G$ is a continuous, strictly monotonous function in $[a,b]$, and $x, x+h_1, x+h_2, x+h_1+h_2 \in [a,b]$. Then,

$$\begin{aligned} & S_{H,G,h_2}(S_{H,G,h_1} f)(x) \\ = \;& S_{H,G,h_2}\left(\frac{H(\cdot)}{H(G^{-1}(G(\cdot)+h_1))} f(G^{-1}(G(\cdot)+h_1))\right)(x) \\ = \;& \frac{H(x)}{H(G^{-1}(G(x)+h_2))}\,\frac{H(G^{-1}(G(x)+h_2))}{H(G^{-1}(G(x)+h_2+h_1))} f(G^{-1}(G(x)+h_1+h_2)) \\ = \;& \frac{H(x)}{H(G^{-1}(G(x)+h_2+h_1))} f(G^{-1}(G(x)+h_1+h_2)) \\ = \;& (S_{H,G,h_1+h_2} f)(x) = S_{H,G,h_1}(S_{H,G,h_2} f)(x). \end{aligned}$$

In particular, it follows that

$$\begin{aligned} S_{H,h_2}(S_{H,h_1} f)(x) &= S_{H,h_1}(S_{H,h_2} f)(x) = S_{H,h_1+h_2} f(x), \\ S_{G,h_2}(S_{G,h_1} f)(x) &= S_{G,h_1}(S_{G,h_2} f)(x) = S_{G,h_1+h_2} f(x). \end{aligned}$$

□

As we have seen in Section 2.2, the Prony method can be interpreted as a reconstruction method for finite expansions into the eigenfunctions $\mathrm{e}^{\alpha x}$ of the shift operator $S_h$. Now we want to consider the eigenfunctions of $S_{h,-h}$ and of $S_{H,G,h}$.

**Theorem 2.2.** *1. The symmetric shift operator $S_{h,-h}$ in (2.6) possesses the eigenfunctions $\cos(\alpha x)$, $\sin(\alpha x)$, $\cosh(\alpha x)$ and $\sinh(\alpha x)$ for $\alpha\in\mathbb{R}$.*

*2. Let $S_{H,G,h}$ be the generalized shift operator as given in (2.7) with given functions $H, G$ satisfying the properties in the definition. Then, $S_{H,G,h}$ possesses eigenfunctions of the form*

$$H(x)\,\mathrm{e}^{\alpha G(x)}, \qquad \alpha\in\mathbb{C}.$$

**Proof.** 1. We observe that for $h \ge 0$, we have

$$S_{h,-h}\cos(\alpha x) = \frac{1}{2}\left[\cos(\alpha(x+h)) + \cos(\alpha(x-h))\right] = \cos(\alpha h)\,\cos(\alpha x),$$

and

$$S_{h,-h}\sin(\alpha x) = \frac{1}{2}\left[\sin(\alpha(x+h)) + \sin(\alpha(x-h))\right] = \cos(\alpha h)\,\sin(\alpha x),$$

i.e., the symmetric shift operator $S_{h,-h}$ possesses the eigenfunctions $\cos(\alpha x)$ and $\sin(\alpha x)$ for all $\alpha\in\mathbb{R}$. Similar identities are well-known for $\cosh x$ and $\sinh x$.

2. Employing the definition of $S_{H,G,h}$ we obtain

$$\begin{aligned} S_{H,G,h}(H(\cdot)\,\mathrm{e}^{\alpha G(\cdot)})(x) &= \frac{H(x)}{H(G^{-1}(G(x)+h))}\, H(G^{-1}(G(x)+h))\,\mathrm{e}^{\alpha G(G^{-1}(G(x)+h))} \\ &= H(x)\,\mathrm{e}^{\alpha(G(x)+h)} = \mathrm{e}^{\alpha h}\, H(x)\,\mathrm{e}^{\alpha G(x)}, \end{aligned}$$

i.e., $H(x)\,\mathrm{e}^{\alpha G(x)}$ is an eigenfunction of $S_{H,G,h}$ to the eigenvalue $\mathrm{e}^{\alpha h}$. □

**Remark 2.3.** 1. The generalized shift operator $S_{H,G,h}$ generates a group structure with the property (2.11) and with the identity operator obtained for $h = 0$.

2. Property (2.11) plays an important role for the considered generalization of Prony's method since it simplifies the action of the $\ell$-th power of the generalized shift operator simply to a generalized shift $\ell h$. Therefore, we can employ $S_{H,G,h}$ similarly as in Section 2.2 and need only function values for the recovery of finite eigenfunction expansions. Indeed, application of powers of $S_{H,G,h}$ to a finite expansion $f(x) = \sum\limits_{j=1}^{M} c_j\, H(x)\,\mathrm{e}^{\alpha_j G(x)}$ at $x = x_0$ gives

$$\begin{aligned} (S^{\ell}_{H,G,h}\, f)(x_0) &= S^{\ell}_{H,G,h}\Big(\sum_{j=0}^{M} c_j\, H(\cdot)\,\mathrm{e}^{\alpha_j G(\cdot)}\Big)(x_0) \\ &= \sum_{j=0}^{M} c_j\, H(x_0)\,\mathrm{e}^{\alpha_j h\ell}\,\mathrm{e}^{\alpha_j G(x_0)} = \sum_{j=0}^{M} \tilde{c}_j\,\mathrm{e}^{\alpha_j h\ell} \end{aligned}$$

with $\tilde{c}_j = c_j\, H(x_0)\mathrm{e}^{\alpha_j G(x_0)}$. In order to obtain (2.11) we have used some type of conjugation principle. Introducing the function $s_h(x) = x + h$ we can write the operators $S_{G,h}$ and $S_{H,h}$ with the help of compositions of functions as $S_h f(x) = (f \circ s_h)(x)$ and

$$S_{G,h} f(x) = (f \circ G^{-1} \circ s_h \circ G)(x), \qquad S_{H,h} f(x) = H(x)\left(\frac{(f \circ s_h)(x)}{(H \circ s_h)(x)}\right).$$

3. Operators of the form (2.9) are well established as generalized exponential operators in the field of quantum mechanics and are used to solve evolution operator equations in quantum field theory, see [7,8].

4. Besides using the generalized shift operators introduced in (2.6), and (2.7), we can also combine these shift operators to generate further operators. For example, we will consider

$$S_{G,h,-h} f(x) \coloneqq \frac{1}{2}\left(f(G^{-1}(G(x)-h)) + f(G^{-1}(G(x)+h))\right)$$

in Sections 5.3 and 6.1.

5. By Theorem 2.1, the iterated symmetric shift $S^k_{h,-h}$ can be presented as a linear combination of shifts $S_{h\ell,-h\ell}$ for $\ell = 0, \ldots, k$. Applying the symmetric shift operator we will therefore always use these shifts instead of $S^{\ell}_{h,-h}$, $\ell = 0, \ldots, k$. Since the Chebyshev polynomials $T_\ell(z) \coloneqq \cos(\ell \arccos(z))$ satisfy the relation

$$x^k = \frac{1}{2^{k-1}} \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{\ell} T_{k-2\ell}(x) + \delta_{k/2,\lfloor k/2 \rfloor} \frac{1}{2^k} \binom{k}{k/2} T_0(x),$$

it will be advantageous to write the Prony polynomial $P(z)$ as an expansion into Chebyshev polynomials.

6. Besides the symmetric shift, one can consider finite linear combinations of $S_h$ (or of $S_{H,G,h}$). The arising finite difference operators do not directly satisfy a property as in (2.12), but their powers can be written as linear combinations of shift operators.

# 3 Reconstruction of Expansions into Trigonometric and Hyperbolic Functions

First we employ the symmetric shift operator in order to derive a new method to reconstruct expansions into trigonometric functions.

## 3.1 Reconstruction of Cosine Expansions

We want to reconstruct an expansion of $f$ into cosine functions of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x), \tag{3.1}$$

where we need to recover the unknown coefficients $c_j \in \mathbb{R} \setminus \{0\}$ and frequency parameters $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, M$.

**Theorem 3.1.** *Assume that all parameters $\alpha_j$ are in the range $[0, K) \subset \mathbb{R}$ and let $h = \frac{\pi}{K}$. Then, $f$ in (3.1) can be uniquely reconstructed using the $2M$ samples $f(kh)$, $k = 0, \ldots, 2M-1$. More generally, for $x_0 \in \mathbb{R}$ satisfying $\alpha_j x_0 \neq (2k+1)\pi/2$ for $k \in \mathbb{Z}$, the $4M-1$ sample values $f(x_0+hk)$, $k = -2M+1, \ldots, 2M-1$, are sufficient to reconstruct $f$ in (3.1).*

**Proof.** We define the Prony polynomial

$$P(z) := \prod_{j=1}^{M} (z - \cos(h\alpha_j))$$

which can be written as

$$P(z) = \sum_{\ell=0}^{M} p_\ell \, T_\ell(z),$$

where $T_\ell(z) := \cos(\ell \arccos z)$ denotes the Chebyshev polynomial of first kind of degree $\ell$. In particular, $p_M = 2^{-M+1}$ since for $\ell \geq 1$ the leading coefficient of $T_\ell$ is $2^{\ell-1}$. As the first step we compute the coefficients $p_\ell$ of the polynomial $P(z)$ using the sample values. By definition of the Prony polynomial and Theorem 2.1 we have for $f$ in (3.1),

$$\begin{aligned}
&\sum_{\ell=0}^{M} p_\ell \Big( (S_{\ell h, -\ell h}) S_{mh} f(x_0) \Big) \\
= \quad & \frac{1}{2} \sum_{\ell=0}^{M} p_\ell \left( f(x_0 + (m+\ell)h) + f(x_0 + (m-\ell)h) \right) \\
= \quad & \frac{1}{2} \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j \left[ \cos(\alpha_j(x_0 + (m+\ell)h)) + \cos(\alpha_j(x_0 + (m-\ell)h)) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{M} c_j \cos(\alpha_j(x_0+mh)) \sum_{\ell=0}^{M} p_\ell \, \cos(\alpha_j \ell h) \\
&= \sum_{j=1}^{M} c_j \cos(\alpha_j(x_0+mh)) \sum_{\ell=0}^{M} p_\ell T_\ell(\cos(\alpha_j h)) = 0
\end{aligned}
$$

for all $m = 0, \ldots, M-1$. Similarly, it follows that

$$
\sum_{\ell=0}^{M} p_\ell \Big( S_{\ell h, -\ell h}(S_{-mh} f)(x_0) \Big) = \sum_{j=1}^{M} c_j \cos(\alpha_j(x_0 - mh)) \sum_{\ell=0}^{M} p_\ell T_\ell(\cos(\alpha_j h)) = 0
$$

for all $m = 0, \ldots, M-1$. For $x_0 = 0$, we obtain the linear system

$$
\sum_{\ell=0}^{M-1} p_\ell (f((m+\ell)h) + f((m-\ell)h) = -\frac{2}{2^M}(f((m+M)h) + f((m-M)h) \tag{3.2}
$$

for $m = 0, \ldots, M-1$. Since $f$ is an even function, it suffices to know the signal values $f(kh)$, $k = 0, \ldots, 2M-1$ to build this system. The quadratic coefficient matrix has Toeplitz-plus-Hankel structure,

$$
\begin{aligned}
\mathbf{H} &= \Big( (f((m+\ell)h) + f((m-\ell)h) \Big)_{m,\ell=0}^{M-1} \\
&= 2 \Big( \sum_{j=1}^{M} c_j \cos(\alpha_j m h) \cos(\alpha_j \ell h) \Big)_{m,\ell=0}^{M-1} = 2 \, \mathbf{V} \, \mathrm{diag}(c_j)_{j=1}^{M} \, \mathbf{V}^T,
\end{aligned} \tag{3.3}
$$

with the generalized Vandermonde matrix

$$
\mathbf{V} = \Big( T_k(\cos(\alpha_j h)) \Big)_{k=0, j=1}^{M-1, M}. \tag{3.4}
$$

The matrix $\mathbf{V}$ is always invertible, since the terms $\cos(\alpha_j h)$ are nonzero and pairwise distinct by assumption. Therefore, $\mathbf{H}$ is invertible if $c_j \neq 0$ for $j = 1, \ldots, M$. For $x_0 \neq 0$, we need to take all values $S_{\ell h, -\ell h} S_{mh, -mh} f(x_0)$ into account. Here, we apply

$$
\begin{aligned}
&\sum_{\ell=0}^{M-1} p_\ell (f(x_0 + (m+\ell)h) + f(x_0 - (m+\ell)h) + f(x_0 + (m-\ell)h) + f(x_0 - (m-\ell)h)) \\
&= -\frac{2}{2^M}(f(x_0 + (m+M)h) + f(x_0 - (m+M)h) + f(x_0 + (m-M)h) + f(x_0 - (m-M)h)),
\end{aligned} \tag{3.5}
$$

and, similarly as in (3.3), the coefficient matrix factorizes in the form

$$
\begin{aligned}
\mathbf{H} &= \Big( (f(x_0 + (m+\ell)h) + f(x_0 - (m+\ell)h) + f(x_0 + (m-\ell)h) + f(x_0 - (m-\ell)h) \Big)_{\ell,m=0}^{M-1} \\
&= 2 \Big( \sum_{j=1}^{M} c_j \cos(\alpha_j x_0) \cos(\alpha_j m h) \cos(\alpha_j \ell h) \Big)_{\ell,m=0}^{M-1} = 2 \mathbf{V} \mathrm{diag}(c_j \cos(\alpha_j x_0))_{j=1}^{M} \, \mathbf{V}^T.
\end{aligned}
$$

The diagonal matrix $\mathrm{diag}(c_j \cos(\alpha_j x_0))_{j=1}^{M}$ is invertible if we have $c_j \neq 0$ and $\alpha_j x_0 \neq (2k+1)\pi/2$ for all $k \in \mathbb{Z}$ and all $j = 1, \ldots, M$. Having found the coefficients of the Prony polynomial, we can extract the zeros $\cos(h\alpha_j)$, $j = 1, \ldots, M$.
In the second step, we solve the linear system

$$
f(x_0 + hk) = \sum_{j=1}^{M} c_j \, \cos(\alpha_j(x_0 + hk)), \qquad k = 0, \ldots 2M-1
$$

in order to compute $c_j$, $j = 1, \ldots, M$. □

### 3.2 Reconstruction of Sine Expansions

The symmetric shift operator can also be applied for the reconstruction of sparse linear combination of sines of the form

$$f(x) = \sum_{j=1}^{M} c_j \sin(\alpha_j x), \tag{3.6}$$

with unknown coefficients $c_j \in \mathbb{R} \setminus \{0\}$ and $\alpha_j \in \mathbb{R} \setminus \{0\}$. This recovery problem is closely related to the problem in (3.1), but we need to pay attention to some details. For example, $f(0)$ does not give us any information here, since the function $f$ in (3.6) is odd, and a frequency $\alpha_j = 0$ does not occur.

**Theorem 3.2.** *Assume that all parameters $\alpha_j$ are in the range $(0, K)$ and let $h = \frac{\pi}{K}$. Then, $f$ in (3.6) can be reconstructed using the $4M-1$ sample values $f(x_0 + hk)$, $k = -2M+1, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ satisfies $\sin(\alpha_j x_0) \neq 0$ for $j = 1, \ldots, M$. In particular, the $2M$ samples $f(kh)$, $k = 1, \ldots, 2M$ are sufficient to reconstruct $f$ in (3.6).*

**Proof.** We proceed similarly as in the last proof. We define the Prony polynomial

$$P(z) := \prod_{j=1}^{M} (z - \cos(h\alpha_j)) = \sum_{\ell=0}^{M} p_\ell T_\ell(z)$$

with coefficients $p_\ell$ in the Chebyshev expansion and $p_M = 2^{-M+1}$. In order to compute the coefficients of $P(z)$, we obtain a linear system as in (3.5), this time with the coefficient matrix

$$\begin{aligned}
\mathbf{H} &= \Big( (f(x_0 + (m+\ell)h) + f(x_0 - (m+\ell)h) + f(x_0 + (m-\ell)h) + f(x_0 - (m-\ell)h)) \Big)_{\ell,m=0}^{M-1} \\
&= 4 \Big( \sum_{j=1}^{M} c_j \sin(\alpha_j x_0) \cos(\alpha_j m h) \cos(\alpha_j \ell h) \Big)_{\ell,m=0}^{M-1} \\
&= 4\, \mathbf{V}\, \mathrm{diag}(c_j \sin(\alpha_j x_0))_{j=1}^{M}\, \mathbf{V}^T
\end{aligned}$$

with $\mathbf{V}$ as in (3.4). Invertibility follows if $\sin(\alpha_j x_0) \neq 0$. This is satisfied for $x_0 = \frac{\pi}{K} = h$. Thus, the function values $f(x_0 + h\ell) = f(h(\ell+1))$, $\ell = 0, \ldots, 2M-1$, are already sufficient for the reconstruction, since we have

$$f(x_0 - h\ell) = f(h(1-\ell)) = \begin{cases} 0 & \ell = 1, \\ -f(h(\ell-1)) & \ell \geq 2. \end{cases}$$

After having found $P(z)$ we obtain the zeros $\cos(h\alpha_j)$ and can compute the coefficients $c_j$ in (3.6) by solving a linear system using the sample values. □

**Remark 3.3.** 1. By Theorem 2.2, the symmetric shift operator also possesses the eigenfunctions $\sinh(\alpha x)$ and $\cosh(\alpha x)$ with $\alpha \in \mathbb{R}$. Therefore, sparse expansions of functions into hyperbolic functions of the form

$$f(x) = \sum_{j=1}^{M} c_j \cosh(\alpha_j x), \tag{3.7}$$

and

$$f(x) = \sum_{j=1}^{M} c_j \sinh(\alpha_j x), \tag{3.8}$$

can be reconstructed using at most $4M-1$ consecutive sample values $f(x_0+kh)$. Taking the samples $f(h\ell)$, $\ell=0,\ldots,2M-1$ for (3.7) or $f(h\ell)$, $\ell=1,\ldots,2M$ for (3.8) is also sufficient for reconstructing these expansions.

2. Obviously, the considered expansions can also be studied by transferring trigonometric and hyperbolic functions into sums of exponentials. But in this case, the number of terms in the sparse sums is doubled from $M$ to $2M$.

3. Using the Laplace transform with $\mathcal{L}(\cos(\alpha\cdot))(s)=\frac{s}{s^2+\alpha^2}$ and $\mathcal{L}(\sin(\alpha\cdot))(s)=\frac{\alpha}{s^2+\alpha^2}$, we can also reconstruct signals of the form

$$f(s)=\sum_{j=1}^{M}\frac{c_j s}{s^2+\alpha_j^2} \qquad \text{or} \qquad f(s)=\sum_{j=1}^{M}\frac{c_j\alpha_j}{s^2+\alpha_j^2}.$$

Analogously, models arising for the Laplace transform of cosh and sinh can be recovered.

# 4 Reconstruction of Expansions Using the Operator $S_{H,h}$

In this section, we study signal models that arise using the generalized shift operator

$$S_{H,h}f(x)=\frac{H(x)}{H(x+h)}\,f(x+h)$$

in (2.8). The operator $S_{H,h}$ can be applied to general expansions of the form

$$f(x)=\sum_{j=1}^{M}c_j\,H(x)\,\mathrm{e}^{x\alpha_j} \tag{4.1}$$

with unknown parameters $c_j\in\mathbb{C}\setminus\{0\}$ and pairwise different $\alpha_j\in\mathbb{C}$ with $\operatorname{Im}\alpha_j\in[-\pi,\pi)$. We assume here that $H(x)$ is a known and well-defined function and that $1/H(x)$ is bounded almost everywhere in the sampling interval. Then, obviously

$$(S_{H,h}(H(\cdot)\,\mathrm{e}^{\alpha_j\cdot}))(x)=\left(\frac{H(x)}{H(h+x)}\right)H(x+h)\,\mathrm{e}^{\alpha_j(x+h)}=\mathrm{e}^{\alpha_j h}\,H(x)\,\mathrm{e}^{\alpha_j x}.$$

We can also define $\tilde{f}(x):=f(x)/H(x)$, which is an exponential sum as considered in Section 2.1.

## 4.1 Reconstruction of Expansions into Gaussian chirps

Let $g(x):=\mathrm{e}^{-\beta x^2}$ for some given $\beta\in\mathbb{C}\setminus\{0\}$. For real $\beta>0$, the shifts $g(x-\alpha_j)$ are known as shifted Gaussians, for complex $\beta$ they are also called shifted *Gaussian chirps*. We want to reconstruct an expansion of $f$ of the form

$$f(x)=\sum_{j=1}^{M}c_j\,g(x-\alpha_j)=\sum_{j=1}^{M}c_j\,\mathrm{e}^{-\beta(x-\alpha_j)^2}, \tag{4.2}$$

and need to recover the $2M$ coefficients $c_j\in\mathbb{C}\setminus\{0\}$ and pairwise different $\alpha_j\in\mathbb{R}$, $j=1,\ldots,M$. We define $H(x):=g(x)=\mathrm{e}^{-\beta x^2}$ such that

$$\frac{H(x)}{H(x+h)}=\mathrm{e}^{\beta h(2x+h)}.$$

Then, the functions $\mathrm{e}^{-\beta(\cdot-\alpha_j)^2}$ are eigenfunctions of $S_{H,h}$ for all $\alpha_j \in \mathbb{R}$, since

$$(S_{H,h}\,\mathrm{e}^{-\beta(\cdot-\alpha_j)^2})(x) = \mathrm{e}^{\beta h(2x+h)}\,\mathrm{e}^{-\beta(x+h-\alpha_j)^2} = \mathrm{e}^{2\beta\alpha_j h}\,\mathrm{e}^{-\beta(x-\alpha_j)^2}.$$

**Theorem 4.1.** *Let $\beta \in \mathbb{C}\setminus\{0\}$ be given. If $\operatorname{Re}\beta = 0$, we assume that $\alpha_j \in (-T,T)$ for $j = 1,\dots,M$ for some given $T$ and choose $0 < h \le \frac{\pi}{2|\operatorname{Im}\beta|\,T}$. If $\operatorname{Re}\beta \neq 0$, the stepsize $h \in \mathbb{R}\setminus\{0\}$ can be taken arbitrarily. Then, $f$ in (4.2) can be reconstructed using the $2M$ sample values $f(x_0+hk)$, $k = 0,\dots,2M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number.*

**Proof.** The signal $f(x)$ in (4.2) can be rewritten as

$$f(x) = \sum_{j=1}^{M}(c_j\,\mathrm{e}^{-\beta\alpha_j^2})\,\mathrm{e}^{-\beta x^2}\,\mathrm{e}^{2\beta x\alpha_j} = \sum_{j=1}^{M}\tilde{c}_j\,H(x)\,\mathrm{e}^{\tilde{\alpha}_j x}$$

with $\tilde{c}_j = c_j\mathrm{e}^{-\beta\alpha_j^2}$ and $\tilde{\alpha}_j = 2\,\beta\,\alpha_j$. We reconstruct

$$\tilde{f}(x) = \mathrm{e}^{\beta x^2} f(x) = \sum_{j=1}^{M}\tilde{c}_j\,\mathrm{e}^{\tilde{\alpha}_j x}$$

applying the Prony method as described in Section 2.1 and using the function values

$$\tilde{f}(x_0+hk) = \mathrm{e}^{-\beta(x_0+hk)^2}\,f(x_0+hk) = \mathrm{e}^{-(\beta x_0)^2}\,S_{H,hk}f(x_0).$$

The arising Hankel matrix

$$\mathbf{H} = \Big(\tilde{f}(x_0+h(k+m))\Big)_{k,m=0}^{M-1} = \mathrm{e}^{-(\beta x_0)^2}\,\Big(S_{H,h(k+m)}f(x_0)\Big)_{k,m=0}^{M-1}$$

is used to recover the parameters $\tilde{\alpha}_j$. It is invertible since $c_j \neq 0$ implies that $\tilde{c}_j = c_j\mathrm{e}^{-\beta\alpha_j^2} \neq 0$, and the $\tilde{\alpha}_j = 2\beta\alpha_j$ are pairwise different since $\beta \neq 0$ and $\alpha_j$ are pairwise different. In the last step we extract $c_j = \mathrm{e}^{\beta\alpha_j^2}c_j$ and $\alpha_j = \frac{\tilde{\alpha}_j}{2\beta}$. □

**Remark 4.2.** 1. The recovery of sums of Gaussians has also been considered in [22], [14], and in a short note in the multivariate case, see [16], but without using the property that the Gaussian is an eigenfunction of a suitable generalized shift operator. In [22], the representation (4.2) has been used to recover a stream of Diracs from measurements that are obtained by applying a Gaussian sampling kernel.

Although the calculations are similar, our approach via the eigenfunction formulation gives more inside and the possibility for further application beyond the Gaussian case.

2. The model (4.2) is also of the form (1.3) and can therefore be reconstructed using equidistant Fourier values, as it has been done e.g. in [19].

## 4.2 Reconstruction of Expansions into Modulated Gaussians

Similarly as in the previous subsection, we can consider modulated Gaussians which are also called Gabor chirps. We now want to recover an expansion of $f$ into modulated Gaussians of the form

$$f(x) = \sum_{j=1}^{M} c_j\,\mathrm{e}^{2\pi\mathrm{i}x\alpha_j}\,g(x-s_j), \tag{4.3}$$

where $g(x) := \mathrm{e}^{-\beta x^2}$ is the Gaussian window with known real $\beta > 0$ and where we have to recover the parameters $c_j \in \mathbb{R} \setminus \{0\}$, $\alpha_j \in [0,1)$, and the shifts $s_j \in \mathbb{R}$, $j = 1, \ldots, M$. Further, we assume that the complex numbers $\beta s_j + \pi \mathrm{i} \alpha_j$ are pairwise different. Again we use the shift operator $S_{H,h}$ in (2.8) with $H(x) = \mathrm{e}^{-\beta x^2}$. Then, $\mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j) = \mathrm{e}^{2\pi \mathrm{i} x \alpha_j}\, \mathrm{e}^{-\beta(x-s_j)^2}$ are eigenfunctions of $S_{H,h}$,

$$\begin{aligned}(S_{H,h}\, \mathrm{e}^{2\pi \mathrm{i}\alpha_j \cdot - \beta(\cdot - s_j)^2})(x_0) &= \mathrm{e}^{\beta h(2x_0+h)}\, \mathrm{e}^{2\pi \mathrm{i}(x_0+h)\alpha_j}\, \mathrm{e}^{-\beta(x_0+h-s_j)^2} \\ &= \mathrm{e}^{2h(\beta s_j + \pi \mathrm{i} \alpha_j)}\, \mathrm{e}^{2\pi \mathrm{i} x_0 \alpha_j - \beta(x_0 - s_j)^2}.\end{aligned}$$

We observe that the eigenvalues of $_{H,h}$ are complex where the imaginary part covers the modulation parameters $\alpha_j$ and the real part the shift parameters $s_j$.

**Theorem 4.3.** *Let $\beta > 0$ be given. Assume that all parameters $\alpha_j$ are in the range $(-K, K)$ for $j = 1, \ldots, M$ and let $0 < h \leq \frac{1}{2K}$. Then, $f$ in (4.3) can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number.*

**Proof.** Again we can transfer the function (4.3) into the form (4.1),

$$\begin{aligned} f(x) &= \sum_{j=1}^{M} c_j\, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} \mathrm{e}^{-\beta(x-s_j)^2} \\ &= \sum_{j=1}^{M} (c_j\, \mathrm{e}^{-\beta s_j^2})\, e^{-\beta x^2}\, \mathrm{e}^{x(2\pi \mathrm{i}\alpha_j + 2\beta s_j)} = \sum_{j=1}^{M} \tilde{c}_j\, H(x)\, \mathrm{e}^{x\tilde{\alpha}_j}\end{aligned}$$

with $\tilde{c}_j = c_j\, \mathrm{e}^{-\beta s_j^2}$ and $\tilde{\alpha}_j = 2(\pi \mathrm{i} \alpha_j + \beta s_j)$. Now we reconstruct $\tilde{f}(x) = \mathrm{e}^{\beta x^2} f(x) = \sum_{j=1}^{M} \tilde{c}_j\, \mathrm{e}^{\tilde{\alpha}_j x}$ applying the Prony method described in Section 2.1 and using the function values

$$\tilde{f}(x_0 + hk) = \mathrm{e}^{-(\beta x_0)^2}\, S_{H,hk} f(x_0) = \mathrm{e}^{\beta(x_0+hk)^2}\, f(x_0 + hk).$$

The arising Hankel matrix

$$\mathbf{H} = \left(\tilde{f}(x_0 + h(k+m))\right)_{k,m=0}^{M-1} = \mathrm{e}^{-(\beta x_0)^2} \left(S_{H,h(k+m)} f(x_0)\right)_{k,m=0}^{M-1}$$

used to recover the parameters $\tilde{\alpha}_j$ is invertible since $c_j \neq 0$ implies that $\tilde{c}_j = c_j \mathrm{e}^{-\beta s_j^2} \neq 0$ and the $\tilde{\alpha}_j$ are pairwise different since $\beta \neq 0$ and $\pi \mathrm{i} \alpha_j + \beta s_j$ have been assumed to be pairwise different in this model. In the last step we extract $c_j = \mathrm{e}^{\beta s_j^2} \tilde{c}_j$, and $\alpha_j = \frac{\operatorname{Im} \tilde{\alpha}_j}{2\pi}$ and $s_j = \frac{\operatorname{Re} \tilde{\alpha}_j}{2\beta}$. □

## 4.3 Reconstruction of Generalized Exponential Sums

We want to reconstruct expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j\, (x\alpha_j)^r\, \mathrm{e}^{x\alpha_j}, \tag{4.4}$$

where $r \in \mathbb{R}$ is known and where we have to recover $c_j$, $\alpha_j \in \mathbb{C} \setminus \{0\}$. We assume further that the $\alpha_j$ are pairwise different. Here we employ the shift operator $S_{H,h}$ with $H(x) = x^r$. The expansion in (4.4) is equivalent to

$$f(x) = \sum_{j=1}^{M} \tilde{c}_j\, x^r\, \mathrm{e}^{x\alpha_j} = \sum_{j=1}^{M} \tilde{c}_j\, H(x)\, \mathrm{e}^{x\alpha_j}$$

if we take $\tilde{c}_j = c_j \alpha_j^r$.

**Theorem 4.4.** *Let $r \in \mathbb{R}$ be known. Further, let $h \in \mathbb{R} \setminus \{0\}$ be such that $\operatorname{Im} \alpha_j \in [-\pi/h, \pi/h)$. Then, $f$ in (4.4) can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ is chosen such that $x_0 + hk \neq 0$ for $k = 0, \ldots, 2M-1$.*

**Proof.** From the assumption it follows that

$$\tilde{f}(x) = \frac{f(x)}{H(x)} = x^{-r} f(x) = \sum_{j=1}^{M} \widetilde{c}_j \, \mathrm{e}^{x\alpha_j}$$

is well defined for $x = x_0 + hk$, $k = 0, \ldots, 2M-1$, and we can reconstruct $\alpha_j$ and $\widetilde{c}_j$ as in Section 2.1. The corresponding Hankel matrix to reconstruct $\alpha_j$,

$$\mathbf{H} = x_0^r \left( S_{H,h(k+\ell)} f(x_0) \right)_{k,\ell=0}^{M-1} = \left( (x_0 + h(k+\ell))^{-r} f(x_0 + h(k+\ell)) \right)_{k,\ell=0}^{M-1},$$

is invertible since the coefficients $\tilde{c}_j$ do not vanish and the $\alpha_j$ are pairwise different by assumption. Finally we obtain $c_j = \alpha_j^{-r} \, \tilde{c}_j$. □

# 5 Reconstruction of Expansions Using the Operator $S_{G,h}$

Now we consider the generalized shift operator $S_{G,h}$ in (2.9) with

$$S_{G,h} f(x) = f(G^{-1}(G(x) + h)).$$

This operator gives us a lot of freedom to generate generalized shifts.

## 5.1 Reconstruction of Expansions into Monomials

If we consider for example $G(x) = \ln x$ and $G^{-1}(x) = \mathrm{e}^x$, we obtain

$$S_{\ln,h} f(x) = f(\mathrm{e}^{(\ln x)+h}) = f(x\mathrm{e}^h) = f(xa)$$

with $a := \mathrm{e}^h > 0$. In other words, $S_{\ln,h}$ is the dilation operator with the dilation factor $a = \mathrm{e}^h$. In particular, all functions of the form $x^{p_k}$ with $p_k \in \mathbb{C}$ are eigenfunctions of this operator,

$$S_{\ln,h}((\cdot)^{p_k})(x) = (\mathrm{e}^{(\ln x)+h})^{p_k} = x^{p_k} \, \mathrm{e}^{hp_k} = x^{p_k} \, a^{p_k}.$$

If the $p_k$ are pairwise distinct and $\operatorname{Im} p_k \in [-\pi/h, \pi/h)$, then the eigenvalues $\mathrm{e}^{hp_k}$ are pairwise distinct. We now want to recover an expansion of $f$ into monomials

$$f(x) = \sum_{j=1}^{M} c_j \, x^{p_j}, \tag{5.1}$$

where $c_j \in \mathbb{C}$ and $p_j \in \mathbb{C}$ with $\operatorname{Im} p_j \in [-\pi/h, \, \pi/h)$ for all $j = 1, \ldots, M$.

**Theorem 5.1.** *Let $h \in \mathbb{C} \setminus \{0\}$ and $a := \mathrm{e}^h \notin \{\mathrm{e}^{2\pi \mathrm{i} k/N} : \, k \in \mathbb{Z}\}$ for all $N < 2M$ be taken such that $a^k$, $k = 0, \ldots, 2M-1$ are pairwise distinct values. Then, $f$ in (5.1) can be reconstructed by the $2M$ sample values $f(a^k x_0)$, where $x_0 \in \mathbb{C} \setminus \{0\}$ can be chosen arbitrarily.*

**Proof.** We define

$$P(z) := \prod_{k=1}^{M} (z - a^{p_j}) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

and observe from (5.1) that

$$\sum_{\ell=0}^{M} p_\ell f(a^{\ell+m} x_0) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j a^{(\ell+m)p_j} x_0^{p_j} = \sum_{j=1}^{M} c_j (a^m x_0)^{p_j} \sum_{\ell=0}^{M} p_\ell \, a^{p_j \ell} = 0$$

for $m = 0, \ldots, M-1$, leading to the system

$$\sum_{\ell=0}^{M-1} f(a^{\ell+m} x_0) p_\ell = -f(a^{M+m} x_0), \qquad m = 0, \ldots, M-1.$$

The invertibility of the Hankel matrix $\mathbf{H} = (f(a^{\ell+m} x_0))_{\ell,m=0}^{M-1}$ follows from

$$\mathbf{H} = \mathbf{V} \, \mathrm{diag}\, (c_j x_0^{p_j})_{j=1}^{M} \, \mathbf{V}^T$$

with the Vandermonde matrix $\mathbf{V} = (a^{p_j k})_{k=0,j=1}^{M-1,M}$ generated by the $N$ pairwise different knots $a^{p_j}$, $j = 1, \ldots, M$. Once the coefficients of the Prony polynomial $P(z)$ are found, we compute $a^{p_j}$ as the zeros of $P(z)$, extract $p_j$, and finally compute $c_j$ in (5.1) by solving the system

$$f(x_0 a^k) = \sum_{j=1}^{M} c_j (x_0 a^k)^{p_j}, \qquad k = 0, \ldots, 2M-1.$$

□

**Remark 5.2.** This example was already discussed in [15] using the dilation operator $D_a$ with $D_a f(x) := f(ax)$. Moreover, using the substitution $x = \mathrm{e}^x$, the model (5.1) can be transferred to the original model (1.1).

If the parameters $p_j$ are positive integers, then $f$ in (5.1) is a sparse polynomial, and its reconstruction can be performed using the Ben-Or and Tiwari Algorithm, see e.g. [1, 13].

## 5.2 Expansions into Gaussians Chirps with Different Scaling

Let now $G(x) = x^2$ and $G^{-1}(x) = \sqrt{x}$ for $x \geq 0$. We consider the corresponding generalized shift operator

$$(S_{x^2,h} f)(x) = f\left(\sqrt{x^2 + h}\right)$$

and observe that the Gaussian chirps $\mathrm{e}^{\alpha x^2}$ with $\alpha \in \mathbb{C}$ are eigenfunctions of this operator,

$$(S_{x^2,h} \, \mathrm{e}^{\alpha(\cdot)^2})(x) = \mathrm{e}^{\alpha(\sqrt{x^2+h}^2)} = \mathrm{e}^{\alpha h} \mathrm{e}^{\alpha x^2}.$$

The parameter $\alpha$ can be uniquely recovered from the eigenvalues $\mathrm{e}^{\alpha h}$ if $\mathrm{Im}\, \alpha h \in [-\pi, \pi)$, i.e. if $\mathrm{Im}\, \alpha \in [-\pi/h, \pi/h)$. We now consider the reconstruction of expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x^2} \tag{5.2}$$

where we need to recover $c_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$.

**Theorem 5.3.** *Let* $\operatorname{Im} \alpha_j \in (-K, K)$ *for some* $K > 0$ *for all* $j = 1, \ldots, M$. *We choose* $h := 1/K$. *Then, the expansion in* (5.2) *can be uniquely recovered from the samples* $f\left(\sqrt{x_0^2 + kh}\right)$, $k = 0, \ldots, 2M-1$, *where* $x_0 \in \mathbb{R}$ *can be chosen arbitrarily.*

**Proof.** We define

$$P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{\alpha_j h}) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

and observe for the monomial coefficients $p_\ell$ of $P(z)$,

$$\begin{aligned} \sum_{\ell=0}^{M} p_\ell f\left(\sqrt{x_0^2 + (m+\ell)h}\right) &= \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j (x_0^2 + (m+\ell)h)} \\ &= \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j (x_0^2 + mh)} \sum_{\ell=0}^{M} p_\ell \, \mathrm{e}^{\alpha_j h \ell} = 0 \end{aligned}$$

for all $m = 0, \ldots, M-1$. Therefore, this system can be used to compute the coefficients $p_\ell$ (using $p_M = 1$), since the corresponding matrix $\left(f\left(\sqrt{x_0^2 + (m+\ell)h}\right)\right)_{\ell,m=0}^{M-1}$ can be factorized in the form

$$\left(f\left(\sqrt{x_0^2 + (m+\ell)h}\right)\right)_{\ell,m=0}^{M-1} = \mathbf{V} \operatorname{diag}(c_j \mathrm{e}^{\alpha_j x_0^2})_{j=1}^{M} \, \mathbf{V}^T$$

with the Vandermonde matrix $\mathbf{V} = (\mathrm{e}^{h\alpha_j k})_{k=0,j=1}^{M-1,M}$ generated by the pairwise different knots $\mathrm{e}^{h\alpha_j}$, $j = 1, \ldots, M$. Having found $P(z)$, we can extract its zeros, recover $p_j$ and finally also $c_j$ by solving a linear system. □

**Remark 5.4.** 1. Similarly, using $G(x) = x^p$ and $G^{-1}(x) = \sqrt[p]{x}$ with given $p > 0$ and for $x \geq 0$ we can recover expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x^p}$$

by reconstructing $c_j$, $\alpha_j \in \mathbb{C}$ from the samples $f\left(\sqrt[p]{x_0^p + hk}\right)$, $k = 0, \ldots, 2M-1$ for some $x_0 \geq 0$, $h > 0$.

2. Taking e.g. $G(x) = \cos(x)$ for $x \in [0, \pi]$ and $G^{-1}(x) = \arccos(x)$ we obtain the operator

$$S_{\cos,h} f(x) = f(\arccos(\cos(x) + h))$$

which has the eigenfunctions $\mathrm{e}^{\alpha_k \cos(x)}$ with

$$(S_{\cos,h} \mathrm{e}^{\alpha_k \cos(\cdot)})(x) = \mathrm{e}^{\alpha_k \cos(\arccos(\cos(x)+h))} = \mathrm{e}^{\alpha_k (\cos(x)+h)}.$$

In this way, we can also recover expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cos(x)}$$

with parameters $c_j, \alpha_j \in \mathbb{C}$ using the samples $f(\arccos(\cos(x_0)+kh))$, $k = 0, \ldots 2M-1$ with suitably chosen $h$ and $x_0$ such that $\cos(x_0) + hk \in [-1, 1]$.

### 5.3 Sparse Expansions into Chebyshev Polynomials

Let now $x \in [-1,1]$ and $G(x) = \arccos(x)$. Then, $G$ is monotonous in $[-1,1]$ and $G^{-1}(y) = \cos(y)$ for $y \in [0,\pi]$. We apply a combination of the symmetric shift operator $S_{h,-h}$ and $S_{G,h}$,

$$(S_{G,h,-h}f)(x) := \frac{1}{2}\left(f(\cos(\arccos(x)+h)) + f(\cos(\arccos(x)-h))\right),$$

which is also called Chebyshev shift operator, see also [15,20]. Then, the Chebyshev polynomials $T_k(x) = \cos(k\arccos(x))$ of degree $k \geq 0$ are eigenfunctions of this operator,

$$\begin{aligned}(S_{G,h,-h}T_k)(x) &= \frac{1}{2}\left(T_k(\cos(\arccos(x)+h)) + T_k(\cos(\arccos(x)-h))\right) \\ &= \frac{1}{2}\left(\cos(k(\arccos(x)+h)) + \cos(k(\arccos(x)-h))\right) \\ &= \cos(kh)\, \cos(k\arccos(x)) = \cos(kh)\, T_k(x).\end{aligned}$$

As in Section 3, the symmetric shift operator is appropriate here since the symmetric shifts of a cosine result in a product of cosines, where one factor only depends on $h$. We want to reconstruct a sparse Chebyshev expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j\, T_{n_j}(x), \tag{5.3}$$

where we need to recover the unknown indices $0 \leq n_1 < n_2 < \ldots < n_M$ and the coefficients $c_j \in \mathbb{R}$. We assume that an upper bound $K$ of the degree $n_M$ of the polynomial in (5.3) is a priori known.

**Theorem 5.5.** *Let $K$ be a bound of the degree of the polynomial $f$ in (5.3) and let $0 < h \leq \frac{\pi}{K}$. Then, the Chebyshev expansion in (5.3) can be uniquely recovered from the samples $f(\cos(kh))$, $k = 0, \ldots, 2M-1$.*

**Proof.** Let $P(z) = \prod_{j=1}^{M}(z - \cos(n_j h)) = \sum_{\ell=0}^{M} p_\ell T_\ell(z)$, where $p_\ell$ are the coefficients of $P(z)$ in the Chebyshev expansion, and $p_M = 2^{-M+1}$. Then, since the cosine is even, we obtain

$$\begin{aligned}&\sum_{\ell=0}^{M} p_\ell \left(S_{G,(m+\ell)h,-(m+\ell)h} f(\cos(0)) + S_{G,(m-\ell)h,-(m-\ell)h} f(\cos(0))\right) \\ =\ &\sum_{\ell=0}^{M} p_\ell \left(f(\cos(m+\ell)h) + f(\cos(m-\ell)h)\right) \\ =\ &\sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j \left(T_{n_j}(\cos(m+\ell)h) + T_{n_j}(\cos(m-\ell)h)\right) \\ =\ &\sum_{j=1}^{M} c_j \sum_{\ell=0}^{M} p_\ell (2\, \cos(n_j m\ell)\, \cos(n_j \ell h)) \\ =\ &2\sum_{j=1}^{M} c_j \cos(n_j mh) \sum_{\ell=0}^{M} p_\ell\, T_\ell\, \cos(n_j h) = 0.\end{aligned}$$

This observation leads to the linear system

$$\left(f(\cos(m+\ell)h)+f(\cos(m-\ell)h)\right)_{\ell,m=0}^{M-1}\,\mathbf{p}=-2^{-M+1}\,\left(f(\cos(m+M)h)\right)_{m=0}^{M-1}$$

to evaluate the vector $\mathbf{p}=(p_0,\ldots,p_{M-1})^T$ of Prony polynomial coefficients. It can be simply shown that the coefficient matrix of this system is invertible, provided that $\cos(n_jh)$ are pairwise distinct. Once we know $P(z)$ it is simple to recover the degrees $n_j$ of active Chebyshev polynomials and afterwards the coefficients $c_j$ in (5.3). □

**Remark 5.6.** 1. Similarly as in Sections 3.1 and 3.2, Theorem 5.5 can be generalized by taking samples $f(\cos(x_0+kh))$, $k=-2M+1,\ldots,2M-1$, and the choice of $x_0$ governs the number of needed function values taking into account that cosine is even.

2. A numerical algorithm for the reconstruction of sparse expansions into Chebyshev polynomials can be already found in [20]. The approach can also be transferred to Chebyshev polynomials of second, third and fourth kind, see [20]. However, in [20] the connection to shift operators with Chebyshev polynomials as eigenfunctions has not been explicitly used.

# 6 Reconstruction of Non-stationary Signals

Within the last years, more efforts have been made to reconstruct non-stationary signals of the form

$$f(x)=\sum_{j=1}^{M}c_j(x)\,\cos(\phi_j(x)).$$

The empirical mode decomposition method described in [11,12] is a heuristic iterative method to decompose a given non-stationary signal into certain signal components. Still, this algorithm does not always provide the wanted signal components in a suitable way. If there is more a priori information on the envelope functions $c_j(x)$ and the phase functions $\phi_j(x)$ available we may be able to exploit it in a direct way. Using the Prony method with generalized shifts we will consider some models of non-stationary signals. In particular, we will study constant envelope functions and polynomial phase functions with a priori known polynomial structure.

## 6.1 Non-stationary Signals with Phase Functions $\phi_j(x)=\alpha_jx^p+\beta_j$

We want to recover signals of the form

$$f(x)=\sum_{j=1}^{M}c_j\,\cos(\alpha_jx^p+\beta_j), \tag{6.1}$$

where the odd integer $p>0$ is a priori known, and where the coefficients $c_j,\alpha_j\in\mathbb{R}$ and $\beta_j\in[-\pi,\pi)$ need to be recovered. We assume here that the $\alpha_j$ are pairwise different and nonnegative. This last assumption is not a restriction since $\cos(x)$ is an even function.

First, we construct a generalized shift operator that possesses the eigenfunctions $\cos(\alpha_jx^p+\beta_j)$. We employ the operator $S_{x^p,h,-h}\colon C(\mathbb{R})\to C(\mathbb{R})$ with $h>0$, which is a combination of the symmetric shift operator $S_{h,-h}$ and the operator $S_{G,h}$ with $G(x)=x^p=\operatorname{sgn}(x)|x|^p$ and $G^{-1}(x)=\operatorname{sgn}(x)\sqrt[p]{|x|}$, given by

$$S_{x^p,h,-h}f(x):=\frac{1}{2}\left(f\left(\operatorname{sgn}(x^p+h)\sqrt[p]{|x^p+h|}\right)+f\left(\operatorname{sgn}(x^p-h)\sqrt[p]{|x^p-h|}\right)\right).$$

Here $\operatorname{sgn}(x)$ denotes the sign of $x$ and is 1 for $x > 0$, $-1$ for $x < 0$, and 0 for $x = 0$. We find

$$\begin{aligned} S_{x^p,h,-h} \cos(\alpha_j x^p + \beta_j) &= \frac{1}{2}\cos\Big(\alpha_j \Big(\operatorname{sgn}(x^p+h)\sqrt[p]{|x^p+h|}\Big)^p + \beta_j\Big) \\ &\quad + \frac{1}{2}\cos\Big(\alpha_j \Big(\operatorname{sgn}(x^p-h)\sqrt[p]{|x^p-h|}\Big)^p + \beta_j\Big) \\ &= \frac{1}{2}\Big(\cos(\alpha_j(x^p+h)+\beta_j) + \cos(\alpha_j(x^p-h)+\beta_j)\Big) \\ &= \cos(\alpha_j x^p + \beta_j)\,\cos(\alpha_j h). \end{aligned}$$

The eigenvalues $\cos(\alpha_j h)$ and $\cos(\alpha_k h)$ are pairwise different for $\alpha_j \neq \alpha_k$ if $\alpha_j, \alpha_k \in [0, \pi/h]$. We conclude

**Theorem 6.1.** *Let $f$ be of the form (6.1) with known odd integer $p > 0$, with unknown parameters $c_j \in \mathbb{R}$, $\beta_j \in [0, 2\pi)$, and pairwise different $\alpha_j \in [0, K)$ for some $K > 0$ for all $j = 1, \ldots, M$. Let $h := \pi/K$.*

1. *If the parameters $\beta_j$ do not appear in the model (6.1), then $f$ can be uniquely recovered from its signal values $f\Big(\operatorname{sgn}(x_0+hk)\sqrt[p]{|x_0+hk|}\Big)$, $k = -2M+1, \ldots, 2M-1$, where $x_0 \geq 0$ only needs to satisfy $\cos(\alpha_j x_0) \neq 0$ for $j = 1, \ldots, M$. Taking $x_0 = 0$, the function values $f\Big(\sqrt[p]{hk}\Big)$, $k = 0, \ldots, 2M-1$ are already sufficient for the reconstruction of the parameters $\alpha_j, c_j$, $j = 1, \ldots, M$.*
2. *If the nonzero parameters $\beta_j$ appear in (6.1), then the parameters $\alpha_j$, $j = 1, \ldots, M$, can be recovered from signal values $f\Big(\operatorname{sgn}(x_0+hk)\sqrt[p]{|x_0+hk|}\Big)$, $k = -2M+1, \ldots, 2M-1$, in a first step. Using in a second step additionally the signal values $f\big(\operatorname{sgn}(x_0+hk-\pi/(2\alpha_j))\sqrt[p]{|x_0+hk-\pi/(2\alpha_j)|}\big)$ for $k = -M+1, \ldots, M-1$, the parameters $c_j$ and $\beta_j$ can be reconstructed. The value $x_0$ needs to be chosen such that $\cos(\alpha_j x_0 + \beta_j) \neq 0$ for $j = 1, \ldots, M$.*

**Proof.** We consider the Prony polynomial of the form

$$P(z) := \prod_{j=1}^{M}(z - \cos(\alpha_j h)) = \sum_{\ell=0}^{M} p_\ell\, T_\ell(z),$$

where $p_\ell$ denote the coefficients in the representation of $P(z)$ using the Chebyshev polynomials $T_\ell(z) = \cos(\ell \arccos z)$ with $p_M = 2^{-M+1}$. Then, we observe that

$$\begin{aligned} &\sum_{\ell=0}^{M} p_\ell \Big(S_{x^p,h(m+\ell),-h(m+\ell)} f(\sqrt[p]{x_0}) + S_{x^p,h(m-\ell),-h(m-\ell)} f(\sqrt[p]{x_0})\Big) \\ = &\sum_{\ell=0}^{M-1} p_\ell \frac{1}{2}\Big(f(\sqrt[p]{x_0+h(m+\ell)}) + f(\operatorname{sgn}(x_0-h(m+\ell))\sqrt[p]{|x_0-h(m+\ell)|}) \\ &\quad + f(\operatorname{sgn}(x_0+h(m-\ell))\sqrt[p]{|x_0+h(m-\ell)|}) \\ &\quad + f(\operatorname{sgn}(x_0-h(m-\ell))\sqrt[p]{|x_0-h(m-\ell)|})\Big) \\ = &\sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j \frac{1}{2}\Big(\cos(\alpha_j(x_0+h(m+\ell))+\beta_j) + \cos(\alpha_j(x_0-h(m+\ell))+\beta_j) \\ &\quad + \cos(\alpha_j(x_0+h(m-\ell))+\beta_j) + \cos(\alpha_j(x_0-h(m-\ell))+\beta_j)\Big) \end{aligned}$$

$$\begin{aligned} &= 2\sum_{j=1}^{M} c_j \cos(\alpha_j x_0 + \beta_j)\cos(\alpha_j h m)\sum_{\ell=0}^{M} p_\ell \cos(\alpha_j h \ell) \\ &= 2\sum_{j=1}^{M} c_j \cos(\alpha_j x_0 + \beta_j)\cos(\alpha_j h m)\sum_{\ell=0}^{M} p_\ell T_\ell(\cos(\alpha_j h)) = 0 \end{aligned}$$

for all $m = 0, \ldots, M-1$. We obtain the linear system

$$\mathbf{H}\mathbf{p} = -2^{-M+1}\, \mathbf{f}_M$$

with $\mathbf{p} = (p_0, \ldots, p_{M-1})^T$ and with

$$\begin{aligned} \mathbf{H} = \Big(& f\big(\sqrt[p]{x_0+h(m+\ell)}\big) + f\big(\operatorname{sgn}(x_0-h(m+\ell))\sqrt[p]{|x_0-h(m+\ell)|}\big) \\ &+ f\big(\operatorname{sgn}(x_0+h(m-\ell))\sqrt[p]{|x_0+h(m-\ell)|}\big) + f\big(\operatorname{sgn}(|x_0-h(m-\ell)|)\sqrt[p]{|x_0-h(m-\ell)|}\big)\Big)_{m,\ell=0}^{M-1}, \\ \mathbf{f}_M = \Big(& f\big(\sqrt[p]{x_0+h(m+M)}\big) + f\big(\operatorname{sgn}(x_0-h(m+M))\sqrt[p]{|x_0-h(m+M)|}\big) \\ &+ f\big(\operatorname{sgn}(x_0+h(m-M))\sqrt[p]{|x_0+h(m-M)|}\big) + f\big(\operatorname{sgn}(x_0-h(m-M))\sqrt[p]{|x_0-h(m-M)|}\big)\Big)_{m=0}^{M-1}, \end{aligned}$$

to compute the coefficients $p_\ell$ of the Prony polynomial in Chebyshev representation. The coefficient matrix of this system has the form

$$\mathbf{H} = \mathbf{V}\,\mathrm{diag}(c_j \cos(\alpha_j x_0 + \beta_j))_{j=1}^{M}\,\mathbf{V}^T \tag{6.2}$$

with the generalized Vandermonde matrix $\mathbf{V} = (\cos(\alpha_j h \ell))_{\ell,j=0}^{M-1} = (T_\ell(\cos(\alpha_j h)))_{\ell,j=0}^{M-1}$ as in (3.4). These Vandermonde matrices are invertible if the values $\cos(\alpha_j h)$ are pairwise different, which is ensured by the choice of $h$. The invertibility of the diagonal matrix is ensured if $c_j \neq 0$ and if $\alpha_j x_0 + \beta_j$ is not of the form $\pi(k+1/2)$ for some $k \neq 0$. In particular, for vanishing $\beta_j$ we can simply use $x_0 = 0$ and need only the function values $f(\ell h)$, $\ell = 0, \ldots 2M-1$ to recover $\alpha_j$, since $f$ in (6.1) is an even function.

Having found the parameters $\alpha_j$ by the described procedure, in the case of vanishing $\beta_j$ we can simply obtain the values $c_j \cos(\alpha_j x_0)$ by solving the linear system

$$\begin{aligned} &\sum_{j=1}^{M} c_j (\cos(\alpha_j(x_0+\ell h)) + \cos(\alpha_j(x_0-\ell h)) \\ &= 2\sum_{j=1}^{M} c_j \cos(\alpha_j x_0)\cos(\alpha_j \ell h) = f(\sqrt[p]{x_0+\ell h}) + f(\operatorname{sgn}(x_0-\ell h)\sqrt[p]{|x_0-\ell h|}) \end{aligned}$$

for $\ell = 0, \ldots, M-1$, where the coefficient matrix is the same generalized Vandermonde matrix $\mathbf{V}$ as in (6.2).

If the model contains nonvanishing parameters $\beta_j$, then we have to solve the system

$$\begin{aligned} &\sum_{j=1}^{M} c_j (\cos(\alpha_j(x_0+\ell h)+\beta_j) + \cos(\alpha_j(x_0-\ell h)+\beta_j)) \\ &= 2\sum_{j=1}^{M} c_j \cos(\alpha_j x_0+\beta_j)\cos(\alpha_j \ell h) = f(\sqrt[p]{x_0+\ell h}) + f(\operatorname{sgn}(x_0-\ell h)\sqrt[p]{|x_0-\ell h|}) \end{aligned}$$

to obtain $d_j \coloneqq c_j \cos(\alpha_j x_0 + \beta_j)$. In addition, we have to solve

$$\sum_{j=1}^{M} c_j \Big(\cos(\alpha_j(x_0+\ell h) - \frac{\pi}{2} + \beta_j) + \cos(\alpha_j(x_0-\ell h) - \frac{\pi}{2} + \beta_j)\Big)$$

$$= 2\sum_{j=1}^{M} c_j \cos(\alpha_j x_0 - \frac{\pi}{2} + \beta_j)\cos(\alpha_j \ell h)$$

$$= f\Big(\mathrm{sgn}(x_0 - \tfrac{\pi}{2\alpha_j} + \ell h)\sqrt[p]{|x_0 - \tfrac{\pi}{2\alpha_j} + \ell h|}\Big) + f\Big(\mathrm{sgn}(x_0 - \tfrac{\pi}{2\alpha_j} - \ell h)\sqrt[p]{|x_0 - \tfrac{\pi}{2\alpha_j} - \ell h|}\Big)$$

for $\ell = 0, \ldots, M-1$, to find $\tilde{d}_j := c_j \cos(\alpha_j x_0 - \frac{\pi}{2} + \beta_j) = \sin(\alpha_j x_0 + \beta_j)$ for $j = 1, \ldots, M$. Thus, we conclude

$$c_j = \sqrt{d_j^2 + \tilde{d}_j^2}, \qquad \beta_j = \arg(d_j + \mathrm{i}\tilde{d}_j) - \alpha_j x_0 \bmod 2\pi.$$

□

## 6.2 Non-stationary Signals with Quadratic Phase Functions

Finally, we consider signals of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(x^2 + \alpha_j x + \beta_j) \tag{6.3}$$

with parameters $c_j \in \mathbb{R}$, $\alpha_j \in (-T, T)$ for some $T > 0$, and $\beta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $j = 1, \ldots, M$. This model can be rewritten as

$$\begin{aligned}
f(x) &= \sum_{j=1}^{M} (c_j \cos(\beta_j))\cos(x^2 + \alpha_j x) - (c_j \sin(\beta_j))\sin(x^2 + \alpha_j x) \\
&= \sum_{j=1}^{M} c_{j,1}\cos(x^2 + \alpha_j x) - c_{j,2}\sin(x^2 + \alpha_j x) \\
&= \sum_{j=1}^{M} \Big(\frac{c_{j,1} + \mathrm{i}\, c_{j,2}}{2}\Big) \mathrm{e}^{\mathrm{i}(x^2 + \alpha_j x)} + \Big(\frac{c_{j,1} - \mathrm{i}\, c_{j,2}}{2}\Big) \mathrm{e}^{-\mathrm{i}(x^2 + \alpha_j x)} \\
&= \sum_{j=1}^{M} b_j\, \mathrm{e}^{\mathrm{i}((x+\alpha_j/2)^2 - \alpha_j^2/4)} + \overline{b}_j\, \mathrm{e}^{-\mathrm{i}((x+\alpha_j/2)^2 - \alpha_j^2/4)} \\
&= 2\,\mathrm{Re} \sum_{j=1}^{M} b_j \mathrm{e}^{-\mathrm{i}\alpha_j^2/4}\, \mathrm{e}^{\mathrm{i}(x+\alpha_j/2)^2} \\
&= 2\,\mathrm{Re} \sum_{j=1}^{M} d_j \mathrm{e}^{\mathrm{i}(x+\alpha_j/2)^2},
\end{aligned} \tag{6.4}$$

where we have used the substitutions $c_{j,1} := c_j \cos(\beta_j)$, $c_{j,2} := c_j \sin(\beta_j)$, $b_j := \Big(\frac{c_{j,1} + \mathrm{i} c_{j,2}}{2}\Big)$, and $d_j := b_j \mathrm{e}^{-\mathrm{i}\alpha_j^2/4}$. Similarly, we observe that

$$\begin{aligned}
\tilde{f}(x) &= \sum_{j=1}^{M} c_j \sin(x^2 + \alpha_j x + \beta_j) = \sum_{j=1}^{M} c_j \cos(x^2 + \alpha_j x + \beta_j - \frac{\pi}{2}) \\
&= 2\,\mathrm{Im} \sum_{j=1}^{M} d_j \mathrm{e}^{\mathrm{i}(x+\alpha_j/2)^2}
\end{aligned}$$

with $d_j = b_j \mathrm{e}^{-\mathrm{i}\alpha_j^2/4}$. The model is therefore closely related to the model in (4.2) (with $\beta = -\mathrm{i}$). We conclude

**Theorem 6.2.** *Assume that* $\beta_j \in [-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $\alpha_j \in (-T, T)$, $j = 1, \ldots, M$, *for some* $T > 0$ *and let* $0 < h \le \frac{\pi}{T}$. *Then,* $f$ *in* (6.3) *can be reconstructed using the* $2M$ *sample values* $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$ *and the* $2M$ *sample values* $\tilde{f}(x_0 + hk)$, *where* $x_0 \in \mathbb{R}$ *is an arbitrary real number.*

**Proof.** Considering the function $h(x) = f(x) + \mathrm{i}\,\tilde{f}(x)$, we can apply Theorem 4.1 to recover the parameters $d_j$ and $\alpha_j$ for $j = 1, \ldots, M$. The original parameters $c_j$ and $\beta_j$, $j = 1, \ldots, M$, are then obtained using the relations

$$b_j = d_j \mathrm{e}^{\mathrm{i}\alpha_j^2/4},\ c_{j,1} = 2\mathrm{Re}\, b_j,\ c_{j,2} = 2\mathrm{Im}\, b_j,\ |c_j| = 2|b_j|,\ \beta_j = \arg(b_j),\ \mathrm{sgn}\, c_j = \mathrm{sgn}\, c_{j,1}.$$

□

## 7 Numerical examples

In this section we want to illustrate the recovery method for non-stationary signals with some examples.

**Example 7.1.** We start with considering the recovery of an expansion of $f$ into Gaussian chirps,

$$f(x) = \sum_{j=1}^{M} c_j\, g(x - \alpha_j) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{-\beta(x-\alpha_j)^2},$$

with $M = 5$, $g(x) = \mathrm{e}^{\mathrm{i}x^2}$, i.e., $\beta = -\mathrm{i}$, and with complex coefficients $c_j$ and real shifts $\alpha_j$ given in Table 1. The coefficients have been obtained by applying a uniform random choice from the intervals $(-5, 5) + \mathrm{i}(-2, 2)$ for $c_j$ and from $(-\pi, \pi)$ for $\alpha_j$. For reconstruction, we have used the 10 signal values $f(j)$, $j = -1, \ldots, 8$, indicated by $*$ in Figure 1 (left). The maximal error for recovering the parameters is given by

$$\max_j |c_j - \tilde{c}_j| = 1.5 \cdot 10^{-10}, \qquad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.5 \cdot 10^{-12},$$

where $\tilde{c}_j$ and $\tilde{\alpha}_j$ denote the computed parameters.

| | $j=1$ | $j=2$ | $j=3$ | $j=4$ | $j=5$ |
|---|---|---|---|---|---|
| $\mathrm{Re}\, c_j$ | −2.37854 | −4.55545 | 2.54933 | −2.57214 | −0.57597 |
| $\mathrm{Im}\, c_j$ | 0.75118 | −0.56308 | 0.94536 | 0.42117 | 0.73366 |
| $\alpha_j$ | 0.64103 | −0.18125 | −1.50929 | −0.53137 | −0.23778 |

**Table 1** *Coefficients* $c_j \in \mathbb{C}$ *and* $\alpha_j \in \mathbb{R}$ *for the expansion of* $f$ *into Gaussian chirps in Example 7.1.*

**Example 7.2.** Next, we consider the recovery of an expansion into modulated Gaussians of the form

$$f(x) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{2\pi\mathrm{i}x\alpha_j} g(x - s_j),$$

with $g(x) = \mathrm{e}^{-x^2/2}$, $M = 6$, real coefficients $c_j$, $\alpha_j$ and $s_j$ as given in Table 2. The coefficients have been obtained by applying a uniform random sampling from the intervals $(-10, 10)$ for $c_j$, from $(-5, 5)$ for $s_j$ and from $(0, 1)$ for $\alpha_j$. For the

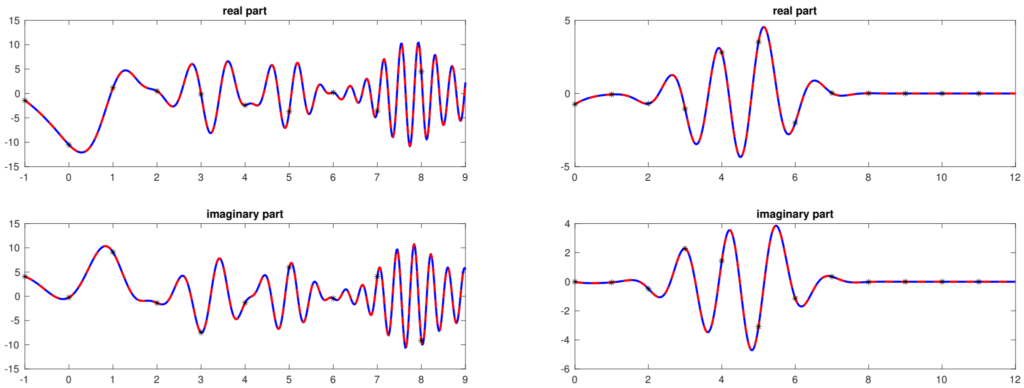


**Figure 1** *Left: Real and imaginary part of the signal $f(x)$ consisting of Gaussian chirps given in Example 7.1. Right: Real and imaginary part of the expansion into modulated Gaussians considered in Example 7.2. Stars indicate the used signal values.*

reconstruction we have used the 12 signal values $f(\ell)$, $\ell = 0, \ldots, 11$ indicated by $*$ in Figure 1 (right). In this example we obtain the errors

$$\max_j |c_j - \tilde{c}_j| = 1.3 \cdot 10^{-6}, \quad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.3 \cdot 10^{-7}, \quad \max_j |s_j - \tilde{s}_j| = 3.1 \cdot 10^{-6},$$

where $\tilde{c}_j$, $\tilde{\alpha}_j$, $\tilde{s}_j$ denote the parameters computed by the numerical procedure.

| | $j=1$ | $j=2$ | $j=3$ | $j=4$ | $j=5$ | $j=6$ |
|---|---|---|---|---|---|---|
| $c_j$ | 0.0777 | 2.9361 | −3.8450 | −7.2255 | −0.4885 | −2.7508 |
| $s_j$ | −1.9918 | −4.3941 | 4.8090 | −2.1337 | 3.0082 | 3.9611 |
| $\alpha_j$ | 0.7881 | 0.7802 | 0.6685 | 0.1335 | 0.0215 | 0.5598 |

**Table 2** *Coefficients $c_j$, $\alpha_j$, $s_j \in \mathbb{R}$ for the expansion into modulated Gaussians in Example 7.2.*

**Example 7.3.** Finally, we consider two examples for the model with quadratic phase function

$$f(x) = \sum_{j=1}^{M} c_j \cos(x^2 + \alpha_j x + \beta_j)$$

in (6.3). In Figure 2 (left), we display a signal with $M = 3$ components with corresponding parameters given in Table 3. For reconstruction, we have used the signal values $f(\ell)$, $\ell = 0, \ldots, 5$. In Figure 2 (right), we give a second example with coefficients given in Table 4. Here, $M = 6$, and we have used the signal values $f(-1 + \frac{5\ell}{12})$, $\ell = 0, \ldots, 11$, for reconstruction. The coefficients have been obtained by applying a uniform random sampling from the intervals $(-1, 5)$ for $c_j$ in the first and from $(0, 5)$ in the second example, from $(-\pi, \pi)$ for $\alpha_j$ and from $(-\pi/2, \pi/2)$ for $\beta_j$ (for both examples). The reconstruction errors in the first example with $M = 3$ terms are

$$\max_j |c_j - \tilde{c}_j| = 1.3 \cdot 10^{-8}, \quad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.3 \cdot 10^{-11}, \quad \max_j |\beta_j - \tilde{\beta}_j| = 1.7 \cdot 10^{-9}.$$

For the second example with $M = 6$ we obtain

$$\max_j |c_j - \tilde{c}_j| = 7.7 \cdot 10^{-5}, \quad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.6 \cdot 10^{-6}, \quad \max_j |\beta_j - \tilde{\beta}_j| = 5.5 \cdot 10^{-5}.$$

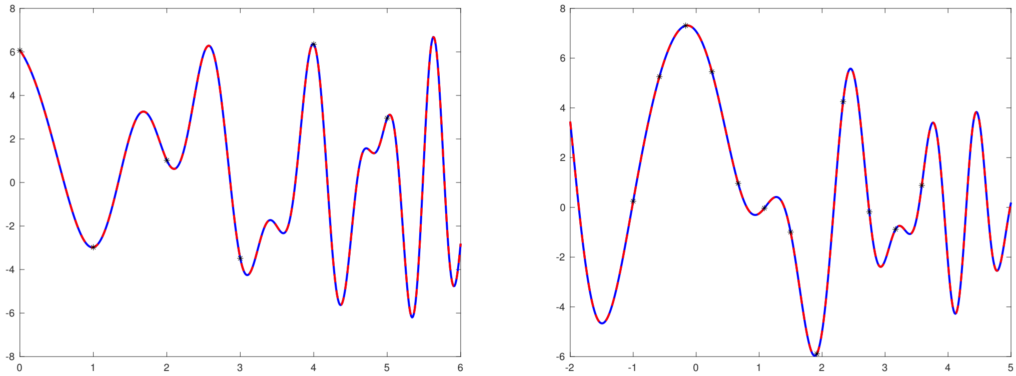

**Figure 2** *Left: Non-stationary signal f(x) with quadratic phase function with parameters given in Table 3. Right: Non-stationary signal f(x) with quadratic phase function with parameters given in Table 4. Stars indicate the used signal values.*

| | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|
| $c_j$ | $-0.1835$ | $4.2157$ | $2.478$ |
| $\alpha_j$ | $0.3132$ | $2.2308$ | $2.2181$ |
| $\beta_j$ | $0.3834$ | $-0.4682$ | $0.0416$ |

**Table 3** *Coefficients $c_j$, $\alpha_j, \beta_j \in \mathbb{R}$ for the non-stationary signal in Figure 2 (left).*

| | $j=1$ | $j=2$ | $j=3$ | $j=4$ | $j=5$ | $j=6$ |
|---|---|---|---|---|---|---|
| $c_j$ | $3.8940$ | $2.117$ | $0.4541$ | $1.3323$ | $0.7682$ | $1.4050$ |
| $\alpha_j$ | $-0.3764$ | $0.1705$ | $-0.2675$ | $2.3585$ | $0.1134$ | $2.7873$ |
| $\beta_j$ | $0.4326$ | $1.4378$ | $-0.8145$ | $0.5533$ | $-0.6626$ | $0.5397$ |

**Table 4** *Coefficients $c_j$, $\alpha_j$, $s_j \in \mathbb{R}$ for the non-stationary signal in Figure 2 (right).*

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