Approximation Properties of Multi–Scaling Functions: A Fourier Approach

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Dedicated to Prof. L. Berg on the occasion of his 65th birthday

Abstract

In this paper, we consider approximation properties of a finite set of functions ϕ_ν ($\nu = 0, \ldots, r-1$) which are not necessarily compactly supported, but have a suitable decay rate. Assuming that the function vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ is refinable, we sketch a new way, how to derive necessary and sufficient conditions for the refinement mask in Fourier domain.

1. Introduction

For applications of multi–wavelets in finite element methods, the problem occurs, how to construct refinable vectors $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ ($r \in \mathbb{N}$) of functions with short support, such that algebraic polynomials of degree $< m$ ($m \in \mathbb{N}$) can be exactly reproduced by a linear combination of integer translates of ϕ_ν ($\nu = 0, \ldots, r-1$).
In Heil, Strang and Strela [9] and in Plonka [13], the approximation properties of refinable function vectors $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ were studied in some detail. In particular, new necessary and sufficient conditions for the refinement mask of $\boldsymbol{\phi}$ could be derived. In [13], it could even be shown that the function vector $\boldsymbol{\phi}$ can only provide approximation order m if its refinement mask factorizes in a certain manner. For finding these results, [9] as well as [13] strongly used properties of doubly infinite matrices determined by the matrix coefficients occuring in the refinement equation (in time domain).
Now we want to sketch a way, how the necessary and sufficient conditions for the refinement mask of $\boldsymbol{\phi}$ can completely be derived in the Fourier domain.

As in [13], the functions ϕ_ν are allowed to have a noncompact support if they have a suitable decay rate. The main tool of our new approach is the so called superfunction, which is contained in the span of the integer translates of ϕ_ν ($\nu = 0, \ldots, r-1$) and already provides the same approximation order as $\boldsymbol{\phi}$. The results are applied to some multi–scaling functions ϕ_0, ϕ_1 first considered by Donovan, Geronimo, Hardin and Massopust [6, 7].

2. Notations

Let us introduce some notations. Consider the Hilbert space $L^2 = L^2(\mathbb{R})$ of all square integrable functions on $\mathbb{R}$. The Fourier transform of $f \in L^2(\mathbb{R})$ is defined by $\hat{f} := \int_{-\infty}^{\infty} f(x) e^{-ix\cdot}\ \mathrm{d}x$.

The function vector $\boldsymbol{\phi}$ with elements in $L^2(\mathbb{R})$ is *refinable*, if $\boldsymbol{\phi}$ satisfies a refinement equation of the form

$$\boldsymbol{\phi} = \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, \boldsymbol{\phi}(2\cdot -l) \qquad (\boldsymbol{P}_l \in \mathbb{R}^{r\times r}),$$

or equivalently, if $\boldsymbol{\phi}$ satisfies the Fourier transformed refinement equation

$$\hat{\boldsymbol{\phi}} = \boldsymbol{P}(\cdot/2)\, \hat{\boldsymbol{\phi}}(\cdot/2) \tag{1}$$

with $\hat{\boldsymbol{\phi}} := (\hat{\phi}_\nu)_{\nu=0}^{r-1}$ and with the *refinement mask* (*two–scale symbol*)

$$\boldsymbol{P} = \boldsymbol{P}_{\boldsymbol{\phi}} := \frac{1}{2} \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, e^{-il\cdot}. \tag{2}$$

Note that $\boldsymbol{P}$ is an $(r \times r)$–matrix of 2π–periodic functions. The components ϕ_ν of a refinable function vector $\boldsymbol{\phi}$ are called *multi–scaling functions.*

Let $BV(\mathbb{R})$ be the set of all functions which are of bounded variation over $\mathbb{R}$ and normalized by

$$\lim_{|x|\to\infty} f(x) = 0, \quad f(x) = \frac{1}{2} \lim_{h\to 0}(f(x+h) + f(x-h)) \quad (-\infty < x < \infty).$$

If $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then the Poisson summation formula

$$\sum_{l\in\mathbb{Z}} f(l)\, e^{-iul} = \sum_{j\in\mathbb{Z}} \hat{f}(u + 2\pi j)$$

is satisfied (cf. Butzer and Nessel [3]). By $C(\mathbb{R})$, we denote the set of continuous functions on $\mathbb{R}$. For a measurable function f on $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p\ \mathrm{d}x\right)^{1/p},$$

$$|f|_{m,p} := \|\mathrm{D}^m f\|_p, \qquad \|f\|_{m,p} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_p.$$

Here and in the following, D denotes the differential operator with respect to x D := d/ dx. Let $W_p^m(\mathbb{R})$ be the usual Sobolev space with the norm $\|\cdot\|_{m,p}$. The l^p–norm of a sequence $\boldsymbol{c} := \{c_l\}_{l\in\mathbb{Z}}$ is defined by $\|\boldsymbol{c}\|_{l^p} := (\sum_{l\in\mathbb{Z}} |c_l|^p)^{1/p}$.

For $m \in \mathbb{N}$, let $E_m(\mathbb{R})$ be the space of all functions $f \in C(\mathbb{R})$ with the decay property

$$\sup_{x\in\mathbb{R}}\{|f(x)|\,(1+|x|)^{1+m+\epsilon}\} < \infty \qquad (\epsilon > 0).$$

Let $l^2_{-m} := \{\boldsymbol{c} := (c_k) : \sum_{k=-\infty}^{\infty}(1+|k|^2)^{-m}\,|c_k|^2 < \infty\}$ be a weighted sequence with the corresponding norm

$$\|\boldsymbol{c}\|_{l^2_{-m}} := \left(\sum_{l=-\infty}^{\infty}(1+|l|^2)^{-m}\,|c_l|^2\right)^{1/2}.$$

Considering the functions $\phi_\nu \in E_m(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, we call the set $\mathcal{B}(\boldsymbol{\phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbb{Z},\ \nu = 0,\ldots,r-1\}$ L^2_{-m}*–stable* if there exist constants $0 < A \le B < \infty$ with

$$A\sum_{\nu=0}^{r-1}\|\boldsymbol{c}_\nu\|^2_{l^2_{-m}} \le \|\sum_{\nu=0}^{r-1}\sum_{l\in\mathbb{Z}} c_{\nu,l}\,\phi_\nu(\cdot - l)\|^2_{L^2_{-m}} \le B\sum_{\nu=0}^{r-1}\|\boldsymbol{c}_\nu\|^2_{l^2_{-m}}$$

for any sequences $\boldsymbol{c}_\nu = \{\boldsymbol{c}_{\nu,l}\}_{l\in\mathbb{Z}} \in l^2_{-m}$ $(\nu = 0,\ldots,r-1)$. Here L^2_{-m} denotes the weighted Hilbert space $L^2_{-m} = \{f : \|f\|_{L^2_{-m}} := \|(1+|\cdot|^2)^{-m/2}f\|_2 < \infty\}$. Note that, if the functions ϕ_ν are compactly supported, then the (algebraic) linear independence of the integer translates of ϕ_ν $(\nu = 0,\ldots,r-1)$ yields the L^2_{-m}–stability of $\mathcal{B}(\boldsymbol{\phi})$. For $m = 0$, we obtain the well–known L^2*–stability* (*Riesz stability*).
For $\phi_\nu \in E_m(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, we say that $\boldsymbol{\phi}$ provides *controlled* L^p*–approximation order* m $(1 \le p \le \infty)$, if the following three conditions are satisfied:
For each $f \in W_p^m(\mathbb{R})$ there are sequences $\boldsymbol{c}_\nu^h = \{c_{\nu,l}^h\}_{l\in\mathbb{Z}}$ $(\nu = 0,\ldots,r-1; h > 0)$ such that for a constant c independent of h we have:

(1) $$\|f - h^{-1/p}\sum_{\nu=0}^{r-1}\sum_{l\in\mathbb{Z}} c_{\nu,l}^h\,\phi_\nu(\cdot/h - l)\|_p \le c\,h^m\,|f|_{m,p}.$$

(2) Furthermore,

$$\|\boldsymbol{c}_\nu^h\|_{l^p} \le c\,\|f\|_p \quad (\nu = 0,\ldots,r-1).$$

(3) There is a constant δ independent of h such that for $l \in \mathbb{Z}$

$$\operatorname{dist}(lh, \operatorname{supp} f) > \delta \quad \Rightarrow \quad c_{\nu,l}^h = 0 \quad (\nu = 0,\ldots,r-1).$$

This notation of controlled L^p–approximation order, first introduced in Jia and Lei [11], is a generalization of the well–known definition of approximation order for compactly supported functions. In [11], the strong connection of controlled approximation order provided by $\boldsymbol{\phi}$ and the Strang–Fix conditions for $\boldsymbol{\phi}$ was shown. Note

that, instead of using the definition of Jia and Lei [11], we also could take the definition of local approximation order by Halton and Light [8]. For our considerations the equivalence to the Strang–Fix conditions is important.
The theory of closed shift–invariant subspaces of $L^2(\mathbb{R})$, spanned by integer translates of a finite set of functions has been extensively studied (cf. e.g. de Boor, DeVore and Ron [1, 2]; Jia [10]). In particular, it has been shown that the approximation order provided by a vector $\boldsymbol{\phi}$ can already be realized by a finite linear combination

$$f = \sum_{\nu=0}^{r-1} \sum_{l \in \mathbb{Z}} a_{\nu l}\, \phi_\nu(\cdot - l). \qquad (a_{\nu l} \in \mathbb{R}).$$

We call f *superfunction* of $\boldsymbol{\phi}$.

3. Approximation by refinable function vectors

In this section we shall give a new approach to necessary and sufficient conditions for the refinement mask of a refinable vector $\boldsymbol{\phi}$ ensuring controlled L^p–approximation order m. In particular, we show, how a superfunction f of $\boldsymbol{\phi}$ (providing the same approximation order as $\boldsymbol{\phi}$) can be constructed by the coefficients which occur in the linear combinations of ϕ_ν reproducing the monomials.
In the following, let $r \in \mathbb{N}$ and $m \in \mathbb{N}$ be fixed. First we want to recall the result in [13] dealing with the connection between controlled L^p–approximation order, reproduction of polynomials and Strang–Fix conditions.

Theorem 1 (cf. [13]) *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a vector of functions $\phi_\nu \in E_m(\mathbb{R}) \cap BV(\mathbb{R})$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ be L^2_{-m}–stable. Then the following conditions are equivalent:*
(a) *The function vector $\boldsymbol{\phi}$ provides controlled approximation order m $(m \in \mathbb{N})$.*
(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced by integer translates of ϕ_ν,i.e., there are vectors $\boldsymbol{y}_l^n \in \mathbb{R}^r$ $(l \in \mathbb{Z};\ n = 0, \ldots, m-1)$ such that the series $\sum_{l \in \mathbb{Z}} (\boldsymbol{y}_l^n)^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot - l)$ are absolutely and uniformly convergent on any compact interval of $\mathbb{R}$ and*

$$\sum_{l \in \mathbb{Z}} (\boldsymbol{y}_l^n)^T\, \boldsymbol{\phi}(x - l) = x^n \qquad (x \in \mathbb{R};\ n = 0, \ldots, m-1).$$

(c) *The function vector $\boldsymbol{\phi}$ satisfies the Strang–Fix conditions of order m, i.e., there is a finitely supported sequence of vectors $\{\boldsymbol{a}_l\}_{l \in \mathbb{Z}}$, such that*

$$f := \sum_{l \in \mathbb{Z}} \boldsymbol{a}_l^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot - l)$$

satisfies

$$\hat{f}(0) \neq 0; \quad \mathrm{D}^n \hat{f}(2\pi l) = 0 \quad (l \in \mathbb{Z} \setminus \{0\}; n = 0, \ldots, m-1).$$

The equivalence of (a) and (c) is already shown in Jia and Lei [11], Theorem 1.1. Further, (b) follows from (c) by [11], Corollary 2.3. For showing that (b) yields (c), in [13] the function

$$f := \sum_{k=0}^{m-1} \boldsymbol{a}_k^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot + k), \tag{3}$$

is introduced. Here, the coefficient vectors $\boldsymbol{a}_k$ are determined by

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{m-1}) := (\boldsymbol{y}_0^0, \ldots, \boldsymbol{y}_0^{m-1})\, \boldsymbol{V}^{-1}$$

with the Vandermonde matrix $\boldsymbol{V} := (k^n)_{k,n=0}^{m-1}$. Hence we have

$$\boldsymbol{y}_0^n = \sum_{k=0}^{m-1} k^n\, \boldsymbol{a}_k \qquad (n = 0, \ldots, m-1). \tag{4}$$

By Fourier transform of (3) we obtain

$$\hat{f}(u) = \boldsymbol{A}(u)^{\mathrm{T}}\, \hat{\boldsymbol{\phi}}(u)$$

with

$$\boldsymbol{A}(u) := \sum_{k=0}^{m-1} \boldsymbol{a}_k\, e^{iuk}. \tag{5}$$

That means, $\boldsymbol{A}(u)$ is an $(r \times r)$–matrix of trigonometric polynomials. Observe that by (4)

$$(\mathrm{D}^n \boldsymbol{A})(0) = \sum_{k=0}^{m-1} (ik)^n\, \boldsymbol{a}_k = i^n\, \boldsymbol{y}_0^n \quad (n = 0, \ldots, m-1).$$

Using the Poisson summation formula it can be shown that f satisfies the conditions

$$(\mathrm{D}^\mu \hat{f})(2\pi l) = \delta_{0,l}\, \delta_{0,\mu} \quad (l \in \mathbb{Z};\ \mu = 0, \ldots, m-1)$$

and hence the Strang–Fix conditions of order m (cf. [13]). Observe that f in (3) is a superfunction of $\boldsymbol{\phi}$.

In the new proof for the following theorem, this superfunction will be the main tool.

Theorem 2 *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of functions $\phi_\nu \in E_m(\mathbb{R}) \cap BV(\mathbb{R})$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ be L^2_{-m}–stable. Then the function vector $\boldsymbol{\phi}$ provides L^p–controlled approximation order m if and only if the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ in (2) satisfies the following conditions:*
There are vectors $\boldsymbol{y}_0^k \in \mathbb{R}^r$; $\boldsymbol{y}_0^0 \neq \mathbf{0}$ $(k = 0, \ldots, m-1)$ such that for $n = 0, \ldots, m-1$ we have

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k} \boldsymbol{P})(0) \;=\; 2^{-n}\, (\boldsymbol{y}_0^n)^{\mathrm{T}}, \tag{6}$$

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k} \boldsymbol{P})(\pi) \;=\; \mathbf{0}^{\mathrm{T}}, \tag{7}$$

where $\mathbf{0}$ denotes the zero vector.

Proof: Note that the conditions (6)–(7) can also be written in the form

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^n\boldsymbol{A})^T(0) \quad (n=0,\ldots,m-1), \tag{8}$$
$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T \quad (n=0,\ldots,m-1), \tag{9}$$

where $\boldsymbol{A}$, defined in (5), is the symbol of a superfunction f of $\boldsymbol{\phi}$ in (3). From Theorem 1 we know that $\boldsymbol{\phi}$ provides controlled approximation order m if and only if f satisfies the Strang–Fix conditions of order m. Hence we only have to prove:
The relations (8)–(9) are satisfied if and only if f satisfies the Strang–Fix conditions of order m, i.e.,

$$(\mathrm{D}^n\hat{f})(2\pi l) = c_n\,\delta_{0,l} \quad (n=0,\ldots,m-1) \tag{10}$$

with constants $c_n \in \mathbb{R}$ and $c_0 \neq 0$.
1. We show that the relations (8)–(9) are satisfied if we have (10).
Note that by (1)

$$\hat{f}(2u) = \boldsymbol{A}^T(2u)\,\hat{\boldsymbol{\phi}}(2u) = \boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)\hat{\boldsymbol{\phi}}(u).$$

Taking the derivatives, it follows on the one hand

$$\begin{aligned}(\mathrm{D}^n\hat{f})(u) &= \mathrm{D}^n[\boldsymbol{A}^T(u)\,\hat{\boldsymbol{\phi}}(u)]\\ &= \sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^k\boldsymbol{A}^T)(u)\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(u)\end{aligned}$$

and on the other hand

$$\begin{aligned}2^n\,(\mathrm{D}^n\hat{f})(2u) &= \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)\,\hat{\boldsymbol{\phi}}(u)]\\ &= \sum_{k=0}^{n}\binom{n}{k}\mathrm{D}^k[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(u).\end{aligned}$$

2. Let us first show that the conditions (9) are satisfied. For all $l \in \mathbb{Z}$ we find by (10) that

$$0 = \hat{f}(4\pi l + 2\pi) = \boldsymbol{A}^T(4\pi l + 2\pi)\,\boldsymbol{P}(2\pi l + \pi)\,\hat{\boldsymbol{\phi}}(2\pi l + \pi) = \boldsymbol{A}^T(0)\,\boldsymbol{P}(\pi)\,\hat{\boldsymbol{\phi}}(2\pi l + \pi).$$

Hence, linear independence of the sequences $\{\hat{\boldsymbol{\phi}}_\nu(\pi + 2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0,\ldots,r-1$ gives

$$\boldsymbol{A}^T(0)\,\boldsymbol{P}(\pi) = \boldsymbol{0}^T.$$

Note that the linear independence of the sequences $\{\hat{\boldsymbol{\phi}}_\nu(u + 2\pi l)\}_{l\in\mathbb{Z}}$ for all $u \in \mathbb{R}$ and for $\nu = 0,\ldots,r-1$ is satisfied if and only if the integer translates of ϕ_ν form a L^2–stable basis of their closed span (cf. Jia and Micchelli [12]). This was the first step of the induction proof.

Let now $\mathrm{D}^{\mu}[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T$ be satisfied for $\mu = 0, \ldots, n-1$ ($n < m$), and observe that by assumption (10) $(\mathrm{D}^n \hat{f})(4\pi l + 2\pi) = 0$ for all $l \in \mathbb{Z}$. Then, by the linear independence of $\{\hat{\boldsymbol{\phi}}_{\nu}(\pi + 2\pi l)\}_{l \in \mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ and by

$$\begin{aligned} 0 &= 2^n\,(\mathrm{D}^n \hat{f})(4\pi l + 2\pi) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi}\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(\pi + 2\pi l) \\ &= \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi}\,\hat{\boldsymbol{\phi}}(\pi + 2\pi l) \end{aligned}$$

it follows that

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T.$$

Thus, the relations (9) are satisfied.
3. Now we show that (10) yields (8). Let $u = 2\pi l$ ($l \in \mathbb{Z}$). Then we have on the one hand by the Strang–Fix conditions

$$\hat{f}(4\pi l) = \boldsymbol{A}^T(0)\,\boldsymbol{P}(0)\,\hat{\boldsymbol{\phi}}(2\pi l) = c_0\,\delta_{0,l}$$

and on the other hand

$$\hat{f}(2\pi l) = \boldsymbol{A}^T(0)\,\hat{\boldsymbol{\phi}}(2\pi l) = c_0\,\delta_{0,l}.$$

By linear independence of $\{\hat{\boldsymbol{\phi}}_{\nu}(2\pi l)\}_{l \in \mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ we obtain

$$\boldsymbol{A}^T(0)\,\boldsymbol{P}(0) = \boldsymbol{A}^T(0).$$

Again, we proceed by induction. Let now $\mathrm{D}^{\mu}[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^{\mu}\boldsymbol{A}^T)(0)$ be satisfied for $\mu = 0, \ldots, n-1$ ($n < m$) and observe that by assumption $(\mathrm{D}^n \hat{f})(2\pi l) = c_n\,\delta_{0,l}$ ($l \in \mathbb{Z}$, $c_0 \neq 0$). Then we find for all $l \in \mathbb{Z}$

$$\begin{aligned} 2^n(\mathrm{D}^n \hat{f})(4\pi l) &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}(2u)^T\,\boldsymbol{P}(u)]|_{u=0}\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(0)\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &\quad + \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0}\,\hat{\boldsymbol{\phi}}(2\pi l) = c_n\,\delta_{0,l} \end{aligned}$$

On the other hand, for $l \in \mathbb{Z}$

$$(\mathrm{D}^n \hat{f})(2\pi l) = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(0)\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) = c_n\,\delta_{0,l}.$$

Hence, a comparison yields

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0}\,\hat{\boldsymbol{\phi}}(2\pi l) = (\mathrm{D}^n \boldsymbol{A}^T)(0)\,\hat{\boldsymbol{\phi}}(2\pi l)$$

By linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0,\ldots,r-1$ we obtain

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^n\boldsymbol{A}^T)(0).$$

Now the proof by induction is complete.
4. We are going to prove the reverse direction. Assume that the relations (8)–(9) are satisfied. We show that then the conditions $(\mathrm{D}^n\hat{f})(2\pi l) = c_n\,\delta_{0,l}$ $(n = 0,\ldots,m-1)$ hold, where $c_0 \neq 0$.
For the μ-th derivative of $\hat{f}$ we find

$$\begin{aligned} 2^\mu\,(\mathrm{D}^\mu\hat{f})(4\pi l) &= \sum_{k=0}^{\mu}\binom{\mu}{k}\,\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0}\,(\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(2\pi l)\\ &= \sum_{k=0}^{\mu}\binom{\mu}{k}\,(\mathrm{D}^\mu\boldsymbol{A})(0)\,(\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(2\pi l)\\ &= (\mathrm{D}^\mu\hat{f})(2\pi l)\end{aligned}$$

and

$$\begin{aligned} 2^\mu\,(\mathrm{D}^\mu\hat{f})(4\pi l+2\pi) &= \sum_{k=0}^{\mu}\binom{\mu}{k}\,\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi}\,(\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(2\pi l+\pi)\\ &= 0.\end{aligned}$$

Thus, we indeed obtain $(\mathrm{D}^n\hat{f})(2\pi l) = c_n\,\delta_{0,l}$. It only remains to show that $c_0 \neq 0$. By Poisson summation formula and using the L^2–stability of $\boldsymbol{\phi}$ we have

$$\hat{\phi}(0) = \boldsymbol{A}^T(0)\,\hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^T\,\hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^T\sum_{l\in\mathbb{Z}}\boldsymbol{\phi}(\cdot - l) \neq 0.$$

Hence f satisfies the Strang–Fix conditions of order m. ■

Remark 3 *For proving the second direction in Theorem 2 we do not need any stability condition if we assume that $(\boldsymbol{y}_0^0)^T\,\hat{\boldsymbol{\phi}}(0) \neq 0$. Since $\boldsymbol{y}_0^0$ and $\hat{\boldsymbol{\phi}}(0)$ are a left and a right eigenvector of $\boldsymbol{P}(0)$, respectively, this assumption is satisfied if the eigenvalue 1 of $\boldsymbol{P}(0)$ is simple.*

4. The GHM–multi–scaling functions

We consider the example of a vector of two multi–scaling functions $\boldsymbol{\phi} := (\phi_0,, \phi_1)^T$ treated in Donovan, Geronimo, Hardin and Massopust ([6, 7]). In the special case $s = s_0 = s_1$ (with $s \in [-1, 1]$) of their construction, the refinement equation of $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi}(x) = \boldsymbol{P}_0\,\boldsymbol{\phi}(2x) + \boldsymbol{P}_1\,\boldsymbol{\phi}(2x-1) + \boldsymbol{P}_2\,\boldsymbol{\phi}(2x-2) + \boldsymbol{P}_3\,\boldsymbol{\phi}(2x-3), \qquad (11)$$

where

$$\boldsymbol{P}_0 := \begin{pmatrix} -\frac{s^2-4s-3}{2\,(s+2)} & 1 \\ -\frac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & \frac{3s^2+s-1}{2\,(s+2)} \end{pmatrix}, \quad \boldsymbol{P}_1 := \begin{pmatrix} -\frac{s^2-4s-3}{2\,(s+2)} & 0 \\ -\frac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & 1 \end{pmatrix},$$

$$\boldsymbol{P}_2 := \begin{pmatrix} 0 & 0 \\ -\frac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & \frac{3s^2+s-1}{2\,(s+2)} \end{pmatrix}, \quad \boldsymbol{P}_3 := \begin{pmatrix} 0 & 0 \\ -\frac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & 0 \end{pmatrix}.$$

For the refinement mask $\boldsymbol{P}$ we have

$$\boldsymbol{P}(u) := \frac{1}{2}\,(\boldsymbol{P}_0 + \boldsymbol{P}_1\,e^{-iu} + \boldsymbol{P}_2\,e^{-2iu} + \boldsymbol{P}_3\,e^{-3iu}).$$

Applying the result of Theorem 2 we can show that $\boldsymbol{\phi}$ provides the controlled L^p–approximation order $m = 2$:
Observing that

$$\boldsymbol{P}(0) = \begin{pmatrix} \frac{-s^2+4s+3}{2(s+2)} & \frac{1}{2} \\ \frac{-3(s-1)^3(s+1)}{2(s+2)^2} & \frac{3s^2+2s+1}{2(s+2)} \end{pmatrix}, \quad (\mathrm{D}\boldsymbol{P})(0) = i \begin{pmatrix} \frac{s^2-4s-3}{4(s+2)} & 0 \\ \frac{9(s-1)^3(s+1)}{4(s+2)^2} & \frac{-(3s^2+2s+1)}{2(s+2)} \end{pmatrix},$$

and

$$\boldsymbol{P}(\pi) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{3(s^2-1)}{2(s+2)} \end{pmatrix}, \quad (\mathrm{D}\boldsymbol{P})(\pi) = i \begin{pmatrix} \frac{-s^2+4s+3}{4(s+2)} & 0 \\ \frac{3(s^2-1)(-s^2+4s+3)}{4(s+2)^2} & \frac{-3(s^2-1)}{2(s+2)} \end{pmatrix},$$

we find with

$$\boldsymbol{y}_0^0 = \left(\frac{-3(s^2-1)}{s+2}, 1\right), \qquad \boldsymbol{y}_0^1 = \left(\frac{-3(s^2-1)}{2(s+2)}, 1\right)$$

the relations

$$(\boldsymbol{y}_0^0)^T\,\boldsymbol{P}(0) = (\boldsymbol{y}_0^0)^T, \quad (\boldsymbol{y}_0^0)^T\,\boldsymbol{P}(\pi) = \boldsymbol{0}^T$$

and

$$\begin{aligned} (2i)^{-1}\,(\boldsymbol{y}_0^0)^T\,(\mathrm{D}\boldsymbol{P})(0) + (\boldsymbol{y}_0^1)^T\,\boldsymbol{P}(0) &= 2^{-1}\,(\boldsymbol{y}_0^1)^T, \\ (2i)^{-1}\,(\boldsymbol{y}_0^0)^T\,(\mathrm{D}\boldsymbol{P})(\pi) + (\boldsymbol{y}_0^1)^T\,\boldsymbol{P}(\pi) &= \boldsymbol{0}^T. \end{aligned}$$

Hence, (8)–(9) are satisfied for $m = 2$. Knowing $\boldsymbol{y}_0^0$ and $\boldsymbol{y}_0^1$, we can construct a superfunction f of $\boldsymbol{\phi}$ (as defined in (3)) by

$$f(x) = (\boldsymbol{y}_0^0 - \boldsymbol{y}_0^1)^T\,\boldsymbol{\phi}(x) + (\boldsymbol{y}_0^1)^T\,\boldsymbol{\phi}(x+1)$$

obtaining

$$f(x) = \frac{3(1-s^2)}{2(s+2)}(\phi_0(x) + \phi_0(x+1)) + \phi_1(x+1).$$

Application of the refinement equation (11) on the right hand side yields

$$\begin{aligned} f(x) &= \frac{9(1-s^2)}{4(s+2)}(\phi_0(2x)+\phi_0(2x+1)) + \frac{3(1-s^2)}{4(s+2)}(\phi_0(2x-1)+\phi_0(2x+2)) \\ &\quad + \frac{1}{2}(\phi_1(2x+2)+\phi_1(2x)) + \phi_1(2x+1) \\ &= \frac{1}{2}f(2x-1) + f(2x) + \frac{1}{2}f(2x+1). \end{aligned}$$

That means, f itself satisfies the refinement equation of the hat–function $h(x) := \max\{(1-|x|), 0\}$. Hence, taking a proper normalization constant, the superfunction f coincides with the hat function h. Indeed, in [6] the approximation order 2 provided by $\boldsymbol{\phi}$ was derived by showing that the hat–function h lies in the span of the integer translates of ϕ_0, ϕ_1.

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