Regularity of Refinable Function Vectors

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We study the existence and regularity of compactly supported solutions $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ of vector refinement equations. The space spanned by the translates of ϕ_ν can only provide approximation order if the refinement mask $\boldsymbol{P}$ has certain particular factorization properties. We show, how the factorization of $\boldsymbol{P}$ can lead to decay of $|\hat{\boldsymbol{\phi}}_\nu(u)|$ as $|u| \to \infty$. The results on decay are used in order to prove uniqueness of solutions and convergence of the cascade algorithm.

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1. Introduction

In this paper we shall discuss the smoothness of refinable function vectors. These are solutions to functional equations of the type

$$\boldsymbol{\phi}(x) = \sum_{n=0}^{N} \boldsymbol{P}_n \, \boldsymbol{\phi}(2x-n) \ , \tag{1.1}$$

where the "coefficients" $\boldsymbol{P}_n$ are $r \times r$ matrices ($r \in \mathbb{N}, r \geq 1$), and where $\boldsymbol{\phi} := (\phi_0, \ldots, \phi_{r-1})^T$ is an r–dimensional function vector. Equations of type (1.1) are natural generalizations of the refinement equations studied in e.g. Cavaretta, Dahmen, and Micchelli (1991), where $r = 1$; therefore we shall call them *refinement equations* as well, or occasionally *vector refinement equations*.

Vector refinement equations have come up in several papers. The oldest example is probably the multiwavelet construction by Alpert and Rokhlin (1991) (see also Alpert (1993)), where the ϕ_ν are all supported on $[0, 1]$ and are polynomials of degree $r-1$ on their support. In this example the smoothness of the ϕ_ν is of course known; equation (1.1) is useful as a computational tool in going from one multiresolution level to the next. Matrix generalizations of type (1.1) were also discussed in more generality in Goodman, Lee, and Tang (1993) and Goodman and Lee (1994), including how to define wavelets once the scaling functions were known. However, it was not clear how to construct smooth non-polynomial examples, let alone how to connect smoothness with properties of the $\boldsymbol{P}_n$. This was in marked contrast with the case $r = 1$, where the link between smoothness of $\boldsymbol{\phi}$ and properties of the $\boldsymbol{P}_n$ or of the *refinement mask*

$$\boldsymbol{P}(u) := \frac{1}{2} \sum_n \boldsymbol{P}_n \, e^{-iun} \tag{1.2}$$

is well understood, and where this connection can be exploited to construct $\boldsymbol{\phi}$ with arbitrary pre–assigned smoothness as well as many other properties (see Daubechies (1992)). Donovan, Geronimo, Hardin, and Massopust (1994) (hereafter referred to as DGHM) were the first to construct continuous non-polynomial refinable function vectors. They gave examples of special bases of selfsimilar wavelets, generated by continuous scaling functions that satisfy an equation of type (1.1). In their paper, the iterated function technique used in the construction was the key to derive smoothness, rather than properties of the $\boldsymbol{P}_n$. This first example triggered several other constructions (e.g. Strang and Strela (1994)), as well as work on the filter bank implications of (1.1) (Vetterli and Strang (1994), Heller et al. (1994)) and a systematic study of the approximation order of solutions of (1.1) (Heil, Strang, and Strela (1994), Plonka (1995.a)). This last work contains the key to understanding how solutions of (1.1) can be smooth.

As shown in Plonka (1995.a), the space spanned by the functions $\phi_\nu(x-n)$ $(n \in \mathbb{N})$ can only have approximation order m if $\boldsymbol{P}(u)$ has certain particular factorization properties. (We assume that the $\phi_\nu(x-n)$ are also (algebrically) linearly independent.) This is reminiscent of the case $r = 1$, where similarly linearly independent translates of a refinable function ϕ can only provide approximation order m if the refinement mask, often denoted by $m_0(u)$, can be factored as

$$m_0(u) = \left(\frac{1+e^{-iu}}{2}\right)^m q(u) \, , \tag{1.3}$$

where $q(0) = 1$, q is 2π–periodic and non–singular for $u = \pi$. By iterating the formula

$$\hat{\phi}(u) = m_0\left(\frac{u}{2}\right) \hat{\phi}\left(\frac{u}{2}\right) \, ,$$

which is obtained by Fourier transformation from (1.1), and exploiting the factorization (1.3), one then finds

$$|\hat{\phi}(u)| = |\prod_{j=1}^{\infty} m_0(2^{-j}\,u)| \le C\,(1+|u|)^{-m} \prod_{j=1}^{\infty} |q(2^{-j}\,u)| \, ,$$

where the infinite products converge uniformly on compact sets if m_0 or, equivalently, q is Hölder continuous in $u = 0$. Together with estimates of the type $\sup_u |q(u)| \le B$ or, more generally, $\sup_u |q(2^{k-1}u)\,q(2^{k-2}u)\ldots q(u)| \le B^k$ for some $k \in \mathbb{N} \setminus \{0\}$, this leads to

$$|\hat{\phi}(u)| \le C\,(1+|u|)^{-m+\log_2 B} \tag{1.4}$$

(see e.g. Daubechies (1988, 1992), Cohen and Conze (1992)). The factorization (1.3), together with estimates on the factor $q(u)$, therefore leads to decay for $\hat{\phi}$, and hence to smoothness estimates for ϕ. By using more sophisticated methods involving transfer operators, one can refine the brute force estimates (1.4) and formulate necessary and sufficient conditions on $q(u)$ ensuring that ϕ lies in some Sobolev space W^s (see Conze and Raugi (1990), Villemoes (1994), Eirola (1992), Gripenberg (1993), Hervé (1994), Cohen and Daubechies (1994)). Here again, the factorization (1.3) is a key ingredient.

In this paper, we shall see that the factorization for the matrix $\boldsymbol{P}(u)$ discovered by Plonka (1995.a) for the case $r > 1$, can play a similar role, although the discussion is more intricate.
We shall assume that the $\phi_\nu(x-n)$, $\nu = 0, \ldots, r-1$, $n \in \mathbb{Z}$ form a linearly independent basis for their closed linear span V_0, and that they provide approximation order m, i.e., for $f \in W^m$ one has

$$\|f - \mathrm{Proj}_{V_j} f\|_{L^2} \leq C\, 2^{-jm}\, \|f\|_{W^m},$$

where V_j is the scaled space

$$V_j = \{g \in L^2(\mathbb{R});\ g(2^{-j}\cdot) \in V_0\}. \tag{1.5}$$

Then, it is shown in Plonka (1995.a) that there exist $r \times r$ matrices $\boldsymbol{C}_0(u), \ldots, \boldsymbol{C}_{m-1}(u)$ (constructed explicitly in Plonka (1995.a); see also below) such that $\boldsymbol{P}(u)$, defined in (1.2), factors as

$$\boldsymbol{P}(u) = \frac{1}{2^m}\, \boldsymbol{C}_0(2u) \ldots \boldsymbol{C}_{m-1}(2u)\, \boldsymbol{P}^{(m)}(u)\, \boldsymbol{C}_{m-1}(u)^{-1} \ldots \boldsymbol{C}_0(u)^{-1}, \tag{1.6}$$

where $\boldsymbol{P}^{(m)}(u)$ is well–defined. By Fourier transform of (1.1) we obtain

$$\hat{\boldsymbol{\phi}}(u) = \boldsymbol{P}\left(\frac{u}{2}\right)\, \hat{\boldsymbol{\phi}}\left(\frac{u}{2}\right). \tag{1.7}$$

This can be iterated again, and we find

$$\hat{\boldsymbol{\phi}}(u) = \boldsymbol{P}\left(\frac{u}{2}\right)\, \boldsymbol{P}\left(\frac{u}{4}\right) \ldots \boldsymbol{P}\left(\frac{u}{2^n}\right)\, \hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right)\ . \tag{1.8}$$

Substituting (1.6) in (1.8) leads to

$$\begin{aligned} \hat{\boldsymbol{\phi}}(u) \;&=\; 2^{-mn}\, \boldsymbol{C}_0(u) \ldots \boldsymbol{C}_{m-1}(u) \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \\ &\quad \times \boldsymbol{C}_{m-1}\left(\frac{u}{2^n}\right)^{-1} \ldots \boldsymbol{C}_0\left(\frac{u}{2^n}\right)^{-1}\, \hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right)\ . \end{aligned} \tag{1.9}$$

Even at this stage, the case $r > 1$ is more complicated than $r = 1$. The matrices $\boldsymbol{P}(2^{-j}u)$ or $\boldsymbol{P}^{(m)}(2^{-j}u)$ do not commute, and the discussion of the convergence of

an infinite product definition for $\hat{\boldsymbol{\phi}}(u)$ is therefore more complex.
Hervé (1994) studied the convergence of the matrices $\mathbf{\Pi}_n(u)$

$$\mathbf{\Pi}_n(u) := \boldsymbol{P}\left(\frac{u}{2}\right)\,\boldsymbol{P}\left(\frac{u}{4}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\ , \tag{1.10}$$

as $n \to \infty$, and showed that convergence is assured if $\boldsymbol{P}(0) = \mathrm{diag}(1, \mu_1, \ldots, \mu_{r-1})$, with $|\mu_l| < 1$ for $l = 1, \ldots, r-1$, or if $\boldsymbol{P}(0)$ is similar to such a matrix, i.e., $\boldsymbol{P}(0) = \boldsymbol{M}\,\mathrm{diag}(1, \mu_1, \ldots, \mu_{r-1})\,\boldsymbol{M}^{-1}$ for some non–singular $\boldsymbol{M}$. This already excludes the case where $\boldsymbol{P}(0)$ is not diagonalizable. Moreover, our matrices $\boldsymbol{P}^{(m)}(0)$ may well have a spectral radius larger than 1, so that Hervé's results cannot be used for the products $\boldsymbol{P}^{(m)}(u/2)\ldots\boldsymbol{P}^{(m)}(2^{-n}u)$ in (1.8). Heil and Colella (1994) discuss not only the convergence of $\mathbf{\Pi}_n(u)$ (with results similar to Hervé (1994)) but also the convergence of $\mathbf{\Pi}_n(u)\boldsymbol{v}$, where $\boldsymbol{v}$ is a fixed r–dimensional vector. If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}(0)$ with eigenvalue 1, then $\mathbf{\Pi}_n(u)\,\boldsymbol{v}$ may converge even if the spectral radius ρ_0 of $\boldsymbol{P}(0)$ is strictly larger than 1; Heil and Colella call this *constrained convergence*. They prove constrained convergence if $\rho_0 < 2$ and if the largest eigenvalue of $\boldsymbol{P}(0)$ is nondegenerate. We use a different technique that proves convergence of $\mathbf{\Pi}_n(u)\,\boldsymbol{v}$ if $\rho_0 < 2$, without non–degeneracy condition, and that extends to some cases where $\rho_0 \geq 2$, if $\boldsymbol{P}(u)$ has vanishing derivatives at $u = 0$. Once convergence of (1.8) or (1.9) is established, we can proceed to the main topic of this paper, namely how the factorization (1.6), together with estimates on $\boldsymbol{P}^{(m)}$(u) can lead to decay of $|\hat{\phi}_\nu(u)|$ ($\nu = 0, \ldots, r-1$) as $|u| \to \infty$. As in the case $r = 1$, this can be exploited to prove L^2–convergence and pointwise convergence theorems (in the "x–domain") similar to those in Daubechies (1988). One can also introduce matrix transfer operators to prove more precise estimate like in the case $r = 1$.

This paper is organized as follows. In section 2, we recall the precise results on the factorization of $\boldsymbol{P}(u)$ obtained in Plonka (1995.a). We also show that this factorization is necessary in order to obtain smooth functions $\phi_0, \ldots, \phi_{r-1}$. In section 3, we discuss the pointwise convergence of $\mathbf{\Pi}_n(u)\,\boldsymbol{a}$, as $n \to \infty$, for a fixed vector $\boldsymbol{a}$. In section 4, we exploit the factorization (1.6) to prove, under certain additional conditions, that $\lim_{n\to\infty} \mathbf{\Pi}_n(u)\,\boldsymbol{a}$ decays, as a function of u, for $|u| \to \infty$. We show, in section 5, how transfer operators can also be used to evaluate the regularity of the scaling functions. Section 6 gives a short uniqueness discussion: in the previous sections an infinite product solution for (1.7) has been constructed; if this has sufficient decay, then its inverse Fourier transform gives a solution to (1.1). Theorem 6.1 shows that, under certain conditions on the mask, this solution is unique in a wide class of functions. In section 7 we show how the decay estimates proved earlier can be used to translate the pointwise convergence of $\mathbf{\Pi}_n(u)\boldsymbol{a}$ to convergence of the cascade and subdivision algorithms in the "x–domain". Finally, section 8 studies several examples; we apply our analysis to see how the (known) smoothness of spline functions and of the DGHM scaling functions can be recovered, and we construct some new examples with controlled smoothness.

2. Factorization of the refinement mask

We want to recall some results of Plonka (1995.a). We start by some definitions. Let $r \in \mathbb{N}$ be fixed, and let $\boldsymbol{y} \in \mathbb{R}^r$ be a vector of length r with $\boldsymbol{y} \neq \boldsymbol{0}$. Here and in the following, $\boldsymbol{0}$ denotes the zero vector of length r. We suppose that $\boldsymbol{y}$ is of the form

$$\boldsymbol{y} = (y_0, \ldots, y_{l-1}, 0, \ldots, 0)^T \tag{2.1}$$

with $1 \leq l \leq r$ and $y_\nu \neq 0$ for $\nu = 0, \ldots, l-1$. Introducing the direct sum of square matrices $\boldsymbol{A} \oplus \boldsymbol{B} := \mathrm{diag}(\boldsymbol{A}, \boldsymbol{B})$, we define the matrix $\boldsymbol{C_y}$ by

$$\boldsymbol{C_y}(u) := \tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u) \oplus \boldsymbol{I}_{r-l}, \tag{2.2}$$

where $\boldsymbol{I}_{r-l}$ is the $(r-l) \times (r-l)$ unit matrix. If $l > 1$, then $\tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u)$ is defined by

$$\tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u) := \begin{pmatrix} y_0^{-1} & -y_0^{-1} & 0 & \ldots & 0 \\ 0 & y_1^{-1} & -y_1^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & y_{l-2}^{-1} & -y_{l-2}^{-1} \\ -e^{-iu}/y_{l-1} & 0 & \ldots & 0 & y_{l-1}^{-1} \end{pmatrix} \quad ; \tag{2.3}$$

if $l = 1$, i.e., if $\boldsymbol{y} := (y_0, 0, \ldots, 0)^T$ with $y_0 \neq 0$, then $\tilde{\boldsymbol{C}}_{\boldsymbol{y}}(u)$ is the scalar $(1 - e^{-iu})/y_0$, so that $\boldsymbol{C_y}(u)$ is a diagonal matrix of the form

$$\boldsymbol{C_y}(u) := \mathrm{diag}\left(\frac{1 - e^{-iu}}{y_0}, 1, \ldots, 1\right) .$$

It can easily be observed that $\boldsymbol{C_y}(u)$ is invertible for $u \neq 0$. Further, the matrix $\boldsymbol{C_y}$ is chosen such that

$$\boldsymbol{y}^T \boldsymbol{C_y}(0) = \boldsymbol{0}^T.$$

We introduce

$$\boldsymbol{E_y}(u) := (1 - e^{-iu})\, \boldsymbol{C_y}^{-1}(u) \,. \tag{2.4}$$

Assuming that $\boldsymbol{y}$ is of the form (2.1), we obtain that $\boldsymbol{E_y}(u) = \tilde{\boldsymbol{E}}_{\boldsymbol{y}}(u) \oplus (1 - e^{-iu})\, \boldsymbol{I}_{r-l}$ with

$$\tilde{\boldsymbol{E}}_{\boldsymbol{y}}(u) := \begin{pmatrix} y_0 & y_1 & y_2 & \ldots & y_{l-1} \\ y_0 z & y_1 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & y_{l-1} \\ y_0 z & y_1 z & \ddots & y_{l-2} & y_{l-1} \\ y_0 z & y_1 z & \ldots & y_{l-2} z & y_{l-1} \end{pmatrix} \qquad (z := e^{-iu}) \,. \tag{2.5}$$

Note that $\boldsymbol{E_y}$ can be written in the form

$$\boldsymbol{E_y}(u) = \boldsymbol{E_y}(0) - i(1 - e^{-iu})\, (\mathrm{D}\boldsymbol{E_y})(0) \,, \tag{2.6}$$

where D denotes the differential operator with respect to ω, $\mathrm{D} := \mathrm{d}/\mathrm{d}\omega$. The vector $\boldsymbol{y}$ need not to be such that its zero entries always occur at the tail. If the non–zero entries of the vector $\boldsymbol{y}$ are given in a different order than in (2.1), then the matrices $\boldsymbol{C}_{\boldsymbol{y}}$ and $\boldsymbol{E}_{\boldsymbol{y}}$ are defined just by reshuffling the rows and columns accordingly.
We can now formulate the factorization results for $\boldsymbol{P}(u)$. The following theorem is a special case of a theorem proved in Plonka (1995.a):

Theorem 2.1 *Let $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of compactly supported functions, and let $\{\phi_\nu(\cdot - n) : n \in \mathbb{Z},\ \nu = 0, \ldots, r-1\}$ form a linearly independent basis of their closed linear span V_0. Then V_0 provides approximation order m if and only if the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ satisfies the following conditions:*
The elements of $\boldsymbol{P}$ are trigonometric polynomials, and there are vectors $\boldsymbol{y}_k \in \mathbb{R}^r$; $\boldsymbol{y}_0 \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ such that for $n = 0, \ldots, m-1$ we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k}\boldsymbol{P})(0) &= 2^{-n}\, (\boldsymbol{y}_n)^{\mathrm{T}}, \\ \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k}\boldsymbol{P})(\pi) &= \boldsymbol{0}^{\mathrm{T}}\ . \end{aligned} \tag{2.7}$$

Furthermore, the equalities (2.7) imply that there are vectors $\boldsymbol{x}_k \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ such that $\boldsymbol{P}$ factorizes

$$\boldsymbol{P}(u) = \frac{1}{2^m} \boldsymbol{C}_{\boldsymbol{x}_0}(2u) \ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\, \boldsymbol{P}^{(m)}(u)\, \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1} \ldots \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}\ , \tag{2.8}$$

where the $(r \times r)$–matrices $\boldsymbol{C}_{\boldsymbol{x}_k}$ are defined by $\boldsymbol{x}_k$ $(k = 0, \ldots, m-1)$ via (2.2) and $\boldsymbol{P}^{(m)}(u)$ is an $(r \times r)$–matrix with trigonometric polynomials as entries.

The vectors $\boldsymbol{x}_l$ $(l = 0, \ldots, m-1)$ in Theorem 2.1 are completely defined in terms of the vectors $\boldsymbol{y}_k$ $(k = 0, \ldots, m-1)$. In particular, we have $(\boldsymbol{x}_0)^T = (\boldsymbol{y}_0)^T$, $(\boldsymbol{x}_1)^T = (-i)(\boldsymbol{y}_0)^T\, (\mathrm{D}\boldsymbol{C}_{\boldsymbol{y}_0})(0) + (\boldsymbol{y}_1)^T\, \boldsymbol{C}_{\boldsymbol{y}_0}(0)$ (cf. Plonka (1995.a)). With the assumptions in Theorem 2.1, approximation order m is equivalent with exact reproduction of algebraic polynomials of degree $m-1$ in V_0. Vice versa, if algebraic polynomials of degree $m-1$ can be exactly reproduced in V_0, i.e., if there are vectors $\boldsymbol{y}_l^n \in \mathbb{R}^r$ $(l \in \mathbb{Z}, n = 0, \ldots, m-1)$ such that

$$\sum_{l \in \mathbb{Z}} (\boldsymbol{y}_l^n)^T\, \boldsymbol{\phi}(x - l) = x^n \qquad (x \in \mathbb{R};\ n = 0, \ldots, m-1)\ ,$$

then $\boldsymbol{y}_l^n$ can be written in the form

$$\boldsymbol{y}_l^n = \sum_{k=0}^{n} \binom{n}{k}\, l^{n-k}\, \boldsymbol{y}_0^k\ ,$$

and the vectors $\boldsymbol{y}_0^k$ $(k = 0, \ldots, m-1)$ satisfy the equalities (2.7) with respect to the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$.

Now, assume that $\boldsymbol{\phi}$ is a refinable function vector with a refinement mask $\boldsymbol{P}$ satisfying the conditions (2.7) for the vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{m-1}$ ($\boldsymbol{y}_0 \neq 0$). Further, let $\boldsymbol{M} \in \mathbb{R}^{r \times r}$ be an invertible matrix and

$$\boldsymbol{\phi}^\sharp(x) := \boldsymbol{M}\, \boldsymbol{\phi}(x).$$

Then $\boldsymbol{\phi}^\sharp$ is also a refinable function vector with the refinement mask $\boldsymbol{P}^\sharp(u) := \boldsymbol{M}\, \boldsymbol{P}(u)\, \boldsymbol{M}^{-1}$, since

$$\begin{aligned} \hat{\boldsymbol{\phi}}^\sharp(u) &= \boldsymbol{M}\, \hat{\boldsymbol{\phi}}(u) = \boldsymbol{M}\, \boldsymbol{P}(u/2)\, \hat{\boldsymbol{\phi}}(u/2) \\ &= \boldsymbol{M}\, \boldsymbol{P}(u/2)\, \boldsymbol{M}^{-1}\, \hat{\boldsymbol{\phi}}^\sharp(u/2)\ . \end{aligned}$$

Observe that $\boldsymbol{P}^\sharp$ is obtained by a similarity transformation from $\boldsymbol{P}$, i.e., $\boldsymbol{P}$ and $\boldsymbol{P}^\sharp$ possess the same spectrum. Furthermore, $\boldsymbol{P}^\sharp(u)$ satisfies the conditions (2.7) for $n = 0, \ldots, m-1$ with vectors $\boldsymbol{y}_0^\sharp, \ldots, \boldsymbol{y}_{m-1}^\sharp$, given by

$$(\boldsymbol{y}_\nu^\sharp)^T = (\boldsymbol{y}_\nu)^T\, \boldsymbol{M}^{-1} \quad (\nu = 0, \ldots, m-1)\ .$$

Hence, $\boldsymbol{P}^\sharp$ can also be factored as in (2.8) with $\boldsymbol{C}$–matrices defined by certain vectors $\boldsymbol{x}_0^\sharp, \ldots, \boldsymbol{x}_{m-1}^\sharp$. In particular, we have $(\boldsymbol{x}_0^\sharp)^T := (\boldsymbol{y}_0^\sharp)^T = (\boldsymbol{y}_0)^T\, \boldsymbol{M}^{-1}$. Note that this implies that the factorization (2.8) is not invariant under basis transformations. For instance, in the case where we consider a single factorization,

$$\boldsymbol{P}(u) = \frac{1}{2} \boldsymbol{C}_{\boldsymbol{y}_0}(2u) \boldsymbol{P}^{(1)}(u) \boldsymbol{C}_{\boldsymbol{y}_0}(u)^{-1} \tag{2.9}$$

with $\boldsymbol{y}_0 = (y_\nu)_{\nu=0}^{r-1}$ ($y_0 \neq 0$), we could choose instead to carry out first the basis transformation

$$\boldsymbol{M} = \begin{pmatrix} y_0 & y_1 & y_2 & \ldots & y_{r-1} \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{2.10}$$

For $\boldsymbol{P}^\sharp(u) = \boldsymbol{M}\boldsymbol{P}(u)\boldsymbol{M}^{-1}$ the equations (2.7) now hold with $(\boldsymbol{y}_0^\sharp)^T = (1, 0, \ldots, 0)$, and we can factor $\boldsymbol{P}^\sharp(u)$ accordingly. Multiplying the factored expression by $\boldsymbol{M}^{-1}$ on the left and $\boldsymbol{M}$ on the right, we obtain

$$\boldsymbol{P}(u) = \frac{1}{2} \boldsymbol{D}_{\boldsymbol{y}_0}(2u) \boldsymbol{Q}^{(1)}(u) \boldsymbol{D}_{\boldsymbol{y}_0}(u)^{-1}\ , \tag{2.11}$$

where $\boldsymbol{D}_{\boldsymbol{y}_0}(u)$ is now defined by

$$\boldsymbol{D}_{\boldsymbol{y}_0}(u) := \boldsymbol{M}^{-1}\, \mathrm{diag}\, \Big(1 - z,\, 1, \ldots, 1\Big) = \begin{pmatrix} 1 - z & -\frac{y_1}{y_0} z & -\frac{y_2}{y_0} z & \ldots & -\frac{y_{r-1}}{y_0} z \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Other choices of $\boldsymbol{M}$ would lead to yet other factorizations. In most applications, the original factorization (2.8) turns out to be the most useful. We shall use the existence of this different factorization (2.11) as a tool to study the spectrum of $\boldsymbol{P}^{(1)}(0)$.

In the second part of this section, we show that the factorization of the refinement mask is necessary in order to obain smooth functions.

Lemma 2.2 *Let* $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ *be a refinable vector of compactly supported functions, i.e., we have*

$$\boldsymbol{\phi}(x) = \sum_{n=0}^{N} \boldsymbol{P}_n\, \boldsymbol{\phi}(2x-n). \tag{2.12}$$

Further, let $\{\phi_\nu(\cdot - n) : n \in \mathbb{Z},\ \nu = 0,\ldots,r-1\}$ *form a Riesz basis of their closed linear span* V_0. *If* $\phi_\nu \in C^{m-1}(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, *then* V_0 *provides approximation order* m. *In particular, there are vectors* $\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{m-1}$ $(\boldsymbol{x}_\nu \neq \boldsymbol{0})$ *such that the refinement mask* $\boldsymbol{P}$ *of* $\boldsymbol{\phi}$ *factorizes in the form* (2.8).

Proof: From the Riesz basis property, there exist dual scaling functions $\tilde{\phi}_\nu \in V_0$, $\nu = 0,\ldots,r-1$ such that

$$\langle \phi_\mu(\cdot - k), \tilde{\phi}_\nu(\cdot - l)\rangle = \delta_{\mu,\nu}\delta_{k,l},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2(\mathbb{R})$. These functions are defined by

$$(\hat{\tilde{\phi}}_0(u),\ldots,\hat{\tilde{\phi}}_{r-1}(u))^T = \boldsymbol{G}(u)^{-1}\,\hat{\boldsymbol{\phi}}(u),$$

where the matrix elements of $\boldsymbol{G}(u)$ are defined by

$$\begin{aligned} g_{\mu,\nu}(u) &= \sum_{n\in\mathbb{Z}} \hat{\phi}_\mu(u+2n\pi)\overline{\hat{\phi}_\nu(u+2n\pi)} \\ &= \sum_{k\in\mathbb{Z}} \langle \phi_\mu, \phi_\nu(\cdot - k)\rangle e^{iku} \qquad (\mu,\nu = 0,\ldots,r-1). \end{aligned}$$

The Riesz basis property is equivalent with the fact that $\boldsymbol{G}(u)$ is uniformly nonsingular. Since its entries are trigonometric polynomials, it follows that the functions $\tilde{\phi}_\nu$ have exponential decay. We thus can define the polynomials

$$p_{n,\nu}(x) = \int_{-\infty}^{\infty} y^n\, \tilde{\phi}_\nu(y-x)\ \mathrm{d}y.$$

We shall prove that for $n < m$, we have

$$x^n = \sum_{k\in\mathbb{Z}} \sum_{\nu=0}^{r-1} p_{n,\nu}(k)\phi_\nu(x-k), \tag{2.13}$$

for all $x \in \mathbb{R}$.(Note that for a fixed x, the above sum has a finite number of non zero terms.)

We proceed by induction on n. For $n = 0$, we remark that $\boldsymbol{\phi}$ cannot vanish at every integer: by repeated application of the refinement equation (2.12), we would obtain that $\boldsymbol{\phi}$ vanishes at all dyadic rationals $2^{-j}k$ ($j \in \mathbb{N}$, $k \in \mathbb{Z}$) and thus is identically zero. Let l and ν_0 be such that $\phi_{\nu_0}(l) = C \neq 0$ and define $f_j = \phi_{\nu_0}(2^{-j} \cdot + l)$. For $j \geq 0$, we have $f \in V_{-j} \subset V_0$ (with V_{-j} as in (1.5)), and thus

$$f_j(x) = \sum_{k \in \mathbb{Z}} \sum_{\nu=0}^{r-1} \langle f_j, \tilde{\phi}_\nu(\cdot - k) \rangle \phi_\nu(x-k). \tag{2.14}$$

As j goes to $+\infty$, $f_j(x)$ tends to C uniformly on every compact set, and for a fixed $k \in \mathbb{Z}$, $\langle f_j, \tilde{\phi}_\nu(\cdot - k)\rangle$ tends to $C\, p_{0,\nu}(k)$. We thus obtain (2.13) from (2.14), by letting j go to infinity.

Now suppose that (2.13) is proved up to order $n-1$. For the same reason as above, we can find l and ν_0 such that $\mathrm{D}^n \phi_{\nu_0}(l) = C \neq 0$. We then define

$$f_j(x) = 2^{nj} n!\, [\phi_{\nu_0}(2^{-j}x + l) - \sum_{s=0}^{n-1} \mathrm{D}^s \phi_{\nu_0}(l)\, (2^{-j}x)^s/s!].$$

From the recursion hypothesis, we have

$$f_j(x) = \sum_{k \in \mathbb{Z}} \sum_{\nu=0}^{r-1} \langle f_j, \tilde{\phi}_\nu(\cdot - k) \rangle \phi_\nu(x-k). \tag{2.15}$$

As j goes to $+\infty$, $f_j(x)$ tends to Cx^n uniformly on every compact set, and for a fixed $k \in \mathbb{Z}$, $\langle f_j, \tilde{\phi}_\nu(\cdot - k)\rangle$ tends to $C\, p_{n,\nu}(k)$. We thus obtain (2.13) from (2.15), by letting j go to infinity. Hence, we have proved that all polynomials of degree $m-1$ are linear combinations of the functions ϕ_ν, ($\nu = 0, \ldots, r-1$).

By Theorem 2.2 in Plonka (1995.a), it follows that V_0 provides approximation order m. Hence, by Theorem 2.1, $\boldsymbol{P}(u)$ can be factorized as in Theorem 2.1. ∎

3. Convergence of infinite matrix products

Ultimately, we are interested in L^1-solutions $\boldsymbol{\phi}(x)$ of (1.1), and their smoothness, if they have any. We also want the space spanned by the $\phi_\nu(x-n)$ ($\nu = 0, \ldots, r-1$, $n \in \mathbb{Z}$) to have a certain approximation order. For $\boldsymbol{\phi} \in L^1$, the Fourier transform $\hat{\boldsymbol{\phi}}$ is a well-defined and continuous vector-valued function that must satisfy (1.7) for all u. In particular, we must have

$$\hat{\boldsymbol{\phi}}(0) = \boldsymbol{P}(0)\hat{\boldsymbol{\phi}}(0)\ .$$

On the other hand, if we want any non-zero approximation order, then we must have $\hat{\boldsymbol{\phi}}(0) \neq 0$, since $\hat{\boldsymbol{\phi}}(0) = 0$ would imply $\int \phi_\nu(x-n)\ \mathrm{d}x = 0$ for all ν, n, making it impossible to construct the function 1 as a combination of the $\phi_\nu(x-n)$. Together, these two observations imply that we should take $\hat{\boldsymbol{\phi}}(0) = \boldsymbol{a}$, where $\boldsymbol{a}$ is a left eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Note that we know that 1 has to be an eigenvalue of $\boldsymbol{P}(0)$ because of (2.7). In all the examples we shall consider in practice, $\boldsymbol{\phi}$ will be compactly supported; more generally, $\boldsymbol{\phi}$ should have good (exponential) decay, so that $\hat{\boldsymbol{\phi}}$ will be smooth. This means that we expect that in

$$\begin{aligned} \hat{\boldsymbol{\phi}}(u) &= \boldsymbol{P}\left(\frac{u}{2}\right)\cdots \boldsymbol{P}\left(\frac{u}{2^n}\right)\hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) \\ &= \boldsymbol{P}\left(\frac{u}{2}\right)\cdots \boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a} + \boldsymbol{P}\left(\frac{u}{2}\right)\cdots \boldsymbol{P}\left(\frac{u}{2^n}\right)\left[\hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) - \hat{\boldsymbol{\phi}}(0)\right] , \end{aligned}$$

the second term should become negligibly small in the limit for $n \to \infty$. This suggests that we define

$$\hat{\boldsymbol{\Upsilon}}_n(u) := \boldsymbol{P}\left(\frac{u}{2}\right)\cdots \boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a} = \boldsymbol{\Pi}_n(u)\boldsymbol{a}$$

and study its limit for $n \to \infty$. In this section, we shall discuss the existence of this limit, pointwise in u. In what follows, $\|\boldsymbol{v}\|$ will denote the Euclidean norm of $\boldsymbol{v} \in \mathbb{R}^d$, i.e., $\|\boldsymbol{v}\| = [v_0^2 + \cdots + v_{r-1}^2]^{1/2}$, and $\|\boldsymbol{V}\| := \max \|\boldsymbol{V}\boldsymbol{v}\|/\|\boldsymbol{v}\|$ will be the corresponding matrix norm (spectral norm) for $\boldsymbol{V} \in \mathbb{R}^{r\times r}$. Recall that the spectral norm of a matrix $\boldsymbol{V}$ can be defined by the spectral radius of $\overline{\boldsymbol{V}}^T \boldsymbol{V}$, i.e., $\|\boldsymbol{V}\| = \|\boldsymbol{V}\|_2 := (\rho(\overline{\boldsymbol{V}}^T \boldsymbol{V}))^{1/2}$.

Lemma 3.1 *Suppose that $\boldsymbol{a}$ is an eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Further, suppose that $\boldsymbol{P}$ satisfies*

$$\|\boldsymbol{P}(u) - \boldsymbol{P}(0)\| \le C\,|u|^\alpha , \tag{3.1}$$

for some $\alpha > 0$, and that

$$\|\boldsymbol{P}(0)\| \le 2^\alpha .$$

Then the infinite product

$$\hat{\boldsymbol{\Upsilon}}(u) := \lim_{n\to\infty} \boldsymbol{\Pi}_n(u)\,\boldsymbol{a} \tag{3.2}$$

converges pointwise for any $u \in \mathbb{R}$. The convergence is uniform on compact sets.

Proof: The estimate (3.1) implies that

$$\|\boldsymbol{P}(u)\| \le \|\boldsymbol{P}(0)\| + C\,|u|^\alpha \le \|\boldsymbol{P}(0)\|\, e^{C'|u|^\alpha}.$$

Hence, we have

$$\begin{aligned} \|\boldsymbol{P}\left(\frac{u}{2}\right)\ldots \boldsymbol{P}\left(\frac{u}{2^l}\right)\| &\le e^{C'|u|^\alpha[2^{-\alpha}+\ldots+2^{-l\alpha}]}\,\|\boldsymbol{P}(0)\|^l \\ &\le e^{C_\alpha|u|^\alpha}\,\|\boldsymbol{P}(0)\|^l \end{aligned}$$

since $[2^{-\alpha} + \ldots + 2^{-l\alpha}] \le \frac{2^{-\alpha}}{1-2^{-\alpha}} < \infty$ for $\alpha > 0$. Using this estimate and observing that

$$\begin{aligned} \boldsymbol{\Pi}_k(u)\,\boldsymbol{a} - \boldsymbol{a} &= \left[\boldsymbol{P}\left(\frac{u}{2}\right)\ldots \boldsymbol{P}\left(\frac{u}{2^k}\right) - \boldsymbol{P}(0)^k\right]\boldsymbol{a} \\ &= \sum_{l=1}^{k} \boldsymbol{P}\left(\frac{u}{2}\right)\ldots \boldsymbol{P}\left(\frac{u}{2^{l-1}}\right)\left[\boldsymbol{P}\left(\frac{u}{2^l}\right) - \boldsymbol{P}(0)\right]\boldsymbol{P}(0)^{k-l}\,\boldsymbol{a}\ , \end{aligned} \tag{3.3}$$

it follows that for any $k \in \mathbb{N}$

$$\begin{aligned} \|\mathbf{\Pi}_k(u)\,\boldsymbol{a} - \boldsymbol{a}\| &\leq C\, e^{C_\alpha |u|^\alpha}\, |u|^\alpha \sum_{l=1}^{\infty} \left[\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right]^l \\ &\leq C'\, e^{C_\alpha |u|^\alpha}\, |u|^\alpha , \end{aligned}$$

where we assume that $\boldsymbol{a}$ is normalized, $\|\boldsymbol{a}\| = 1$, for the sake of convenience. Now, remarking that $\mathbf{\Pi}_{N+k}(u)\boldsymbol{a} - \mathbf{\Pi}_N(u)\boldsymbol{a} = \mathbf{\Pi}_N(u)[\mathbf{\Pi}_k(2^{-N}u)\boldsymbol{a} - \boldsymbol{a}]$, we obtain

$$\|\mathbf{\Pi}_{N+k}(u)\boldsymbol{a} - \mathbf{\Pi}_N(u)\boldsymbol{a}\| \leq C'\, e^{C_\alpha |u|^\alpha (1+2^{-N\alpha})}\, |u|^\alpha \left(\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right)^N ,$$

and hence

$$\lim_{m,n\to\infty} \|\mathbf{\Pi}_m(u)\,\boldsymbol{a} - \mathbf{\Pi}_n(u)\,\boldsymbol{a}\| = 0.$$

Thus, (3.2) converges pointwise for all $u \in \mathbb{R}$. The convergence is uniform on compact sets. ∎

This result is often sufficient. Note that, when the entries of $\boldsymbol{P}(u)$ are trigonometric polynomials, (3.1) is always satisfied with $\alpha = 1$ and can be satisfied for integer values $\alpha > 1$ if and only if $\boldsymbol{P}(u)$ has vanishing derivative at the origin. The argument can be pushed a little further, allowing for the replacement of $\|\boldsymbol{P}(0)\|$ by the spectral radius of $\boldsymbol{P}(0)$,

$$\rho_0 = \rho(\boldsymbol{P}(0)) := \max\{|\lambda| :\ \boldsymbol{P}(0)\,\boldsymbol{x} = \lambda\,\boldsymbol{x},\ \boldsymbol{x} \neq \mathbf{0}\}\ .$$

Theorem 3.2 *Let $\boldsymbol{a}$ be an eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Suppose that $\boldsymbol{P}(u)$ satisfies (3.1), and that*

$$\rho_0 < 2^\alpha\ . \tag{3.4}$$

Then $\hat{\mathbf{\Upsilon}}(u)$, defined by (3.2), converges pointwise for all $u \in \mathbb{R}$, and the convergence is uniform on compact sets. Moreover, $\hat{\mathbf{\Upsilon}}(u)$ is Hölder continuous in $u = 0$.

Proof: 1. Again, we assume $\|\boldsymbol{a}\| = 1$ for the sake of convenience. Let $\boldsymbol{Q}(u) := \boldsymbol{P}(u) - \boldsymbol{P}(0)$. Then it follows from (3.1) that $\|\boldsymbol{Q}(u)\| \leq C\,|u|^\alpha$ with $\alpha > 0$. Further, observe that $\|\boldsymbol{P}(0)^k\| \leq C_\epsilon\,(\rho_0 + \epsilon)^k$. Then we have

$$\begin{aligned} \|\mathbf{\Pi}_N(u)\| &= \|\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^N}\right)\| \\ &= \|[\boldsymbol{P}(0) + \boldsymbol{Q}\left(\frac{u}{2}\right)]\ldots[\boldsymbol{P}(0) + \boldsymbol{Q}\left(\frac{u}{2^N}\right)]\| \\ &= \|\sum_{l=0}^{N} \sum_{m_1+\ldots+m_{l+1}=N-l} \boldsymbol{P}(0)^{m_1}\,\boldsymbol{Q}\left(\frac{u}{2^{m_1+1}}\right)\,\boldsymbol{P}(0)^{m_2}\,\boldsymbol{Q}\left(\frac{u}{2^{m_1+m_2+2}}\right)\ldots \\ &\quad \times\boldsymbol{Q}\left(\frac{u}{2^{m_1+m_2+\ldots+m_l+l}}\right)\,\boldsymbol{P}(0)^{m_{l+1}}\|. \end{aligned}$$

The second summation above is taken over all positive integer $m_1, \ldots, m_{l+1}$ such that $m_1 + \ldots + m_{l+1} = N - l$. Introducing $b = 2^{-\alpha} < 1$, this leads to

$$\|\mathbf{\Pi}_N(u)\| \leq \sum_{l=0}^{N} \sum_{m_1+\ldots+m_{l+1}=N-l} C_\epsilon^{l+1} (\rho_0 + \epsilon)^{N-l} C^l |u|^{\alpha l} 2^{-\alpha \sum_{k=0}^{l}(m_k+1)(l+1-k)}$$

$$\leq C_\epsilon b^{-N+l} b^{l(l+1)/2} \sum_{l=0}^{N} (\rho_0 + \epsilon)^{N-l} (|u|^\alpha C_\epsilon C)^l \sum_{m_1+\ldots+m_{l+1}=N-l} b^{[m_1(l+1)+\ldots+m_{l+1}]}.$$

2. Next we find an upper bound for the sum over $m_1, \ldots, m_{l+1}$. Consider the sum

$$A_{M,L} := \sum_{m_1+\ldots+m_L=M} b^{m_1+2m_2+\ldots Lm_L}. \tag{3.5}$$

For $A_{M,L}$ we find the recursion (putting $m = m_1$)

$$A_{M,L} = \sum_{m=0}^{M} b^M A_{M-m,L-1} = b^M \sum_{m=0}^{M} A_{M-m,L-1}$$

with $A_{M,1} = b^M$ and $A_{0,L} = 1$. We show by induction that

$$A_{M,L} \leq \frac{b^M}{(1-b)^{L-1}}. \tag{3.6}$$

For $L = 1$ and $M \in \mathbb{N}$, (3.6) is satisfied. Now, assume that (3.6) holds for $L \geq 1$ and $M \in \mathbb{N}$. Then we obtain by the recursion formula

$$A_{M,L+1} = b^M \sum_{m=0}^{M} A_{M-m,L} \leq \frac{b^M}{(1-b)^{L-1}} \sum_{m=0}^{M} b^m \leq \frac{b^M}{(1-b)^L}.$$

3. Substituting (3.6) into the expression for $\|\mathbf{\Pi}_N(u)\|$ obtained above, we find

$$\begin{aligned}
\|\mathbf{\Pi}_N(u)\| &\leq C_\epsilon \sum_{l=0}^{N} (\rho_0+\epsilon)^{N-l} (|u|^\alpha C_\epsilon C)^l b^{-N+l} b^{l(l+1)/2} A_{N-l,l+1} \\
&\leq C_\epsilon \sum_{l=0}^{N} (\rho_0+\epsilon)^{N-l} (|u|^\alpha C_\epsilon C)^l \frac{b^{l(l+1)/2}}{(1-b)^l} \\
&\leq C_\epsilon (\rho_0+\epsilon)^N \sum_{l=0}^{N} \left[\frac{|u|^\alpha C_\epsilon C}{(\rho_0+\epsilon)(1-b)}\right]^l b^{l(l+1)/2}.
\end{aligned} \tag{3.7}$$

The sum in (3.7) converges uniformly for $|u| \leq \Omega$, since $b < 1$. Hence, we can estimate

$$\|\mathbf{\Pi}_N(u)\| \leq C_{\epsilon,\Omega} (\rho_0 + \epsilon)^N. \tag{3.8}$$

4. Now, with the same argument as in the proof of Lemma 3.1, we have by (3.3)

$$\begin{aligned} \|\mathbf{\Pi}_k(u)\,\boldsymbol{a}-\boldsymbol{a}\| &\le C\sum_{l=1}^{k} C_{\epsilon,\Omega}\,(\rho_0+\epsilon)^{l-1}\left(\frac{|u|}{2^l}\right)^{\alpha} \\ &\le C'_{\epsilon,\Omega}\,|u|^{\alpha}\sum_{l=1}^{k}\left(\frac{\rho_0+\epsilon}{2^{\alpha}}\right)^{l}. \end{aligned} \tag{3.9}$$

Hence, uniform boundedness of $\|\mathbf{\Pi}_k(u)\boldsymbol{a}-\boldsymbol{a}\|$ is ensured, if $\rho_0<2^{\alpha}$, by choosing ϵ sufficiently small. Again, it follows that

$$\begin{aligned} \|\mathbf{\Pi}_{N+k}(u)\boldsymbol{a}-\mathbf{\Pi}_N(u)\boldsymbol{a}\| &\le C_{\epsilon,\Omega}\,C'_{\epsilon,\Omega}(\rho_0+\epsilon)^N\frac{|u|^{\alpha}}{2^{N\alpha}}\sum_{l=1}^{k}\left(\frac{\rho_0+\epsilon}{2^{\alpha}}\right)^{l} \\ &\le C''_{\epsilon,\Omega}\,|u|^{\alpha}\left(\frac{\rho_0+\epsilon}{2^{\alpha}}\right)^{N}, \end{aligned}$$

where the last term is uniformly small in k if N is sufficiently large. Thus, for fixed u, $\mathbf{\Pi}_k(u)\boldsymbol{a}$ is a Cauchy sequence for $\rho_0<2^{\alpha}$, implying that we have pointwise convergence of $\mathbf{\Upsilon}(u)$. Moreover, the convergence is uniform on compact sets. The Hölder continuity of $\mathbf{\Upsilon}(u)$ in $u=0$ directly follows from (3.9). ∎

4. Decay of infinite matrix products

Having shown that $\hat{\mathbf{\Upsilon}}(u)$ is well-defined (under some conditions on $\boldsymbol{P}(u)$), we now proceed to study how the factorization (2.8) of the refinement mask $\boldsymbol{P}(u)$ can lead to decay in u of $\hat{\mathbf{\Upsilon}}(u)$ for $|u|\to\infty$. Let us suppose that $\boldsymbol{P}(u)$ can be factored in the form

$$\boldsymbol{P}(u)=\frac{1}{2^m(1-e^{-iu})^m}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}^{(m)}(u)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(u),$$

where the $\boldsymbol{C}$– and $\boldsymbol{E}$–matrices are defined as in (2.2) and (2.4) and where the vectors $\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{m-1}$ are all different from the zero vector. We can now rewrite $\hat{\mathbf{\Upsilon}}(u)$ as

$$\begin{aligned} \hat{\mathbf{\Upsilon}}(u) &:= \lim_{n\to\infty}\mathbf{\Pi}_n(u)\,\boldsymbol{a}=\lim_{n\to\infty}\left\{\left[\frac{1}{2^n(1-e^{-iu/2^n})}\right]^m\boldsymbol{C}_{\boldsymbol{x}_0}(u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)\right. \\ &\quad\left.\times\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}\right\}. \end{aligned}$$

We note again that, since $\boldsymbol{x}_0^T\,\boldsymbol{P}(0)=\boldsymbol{x}_0^T$ with $\boldsymbol{x}_0\neq 0$, 1 is an eigenvalue of $\boldsymbol{P}(0)$, and we take $\boldsymbol{a}$ to be a right eigenvector of $\boldsymbol{P}(0)$ for that eigenvalue. We also assume that $\boldsymbol{x}_0^T\,\boldsymbol{a}\neq 0$; if the eigenvalue 1 of $\boldsymbol{P}(0)$ is nondegenerate, then this is automatically satisfied. Note that for $u=0$ we have $\hat{\mathbf{\Upsilon}}(0)=\lim_{n\to\infty}\boldsymbol{P}(0)^n\,\boldsymbol{a}=\boldsymbol{a}$. We will establish conditions under which

$$\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}\|$$

tends to a finite limit for $n \to \infty$; since $\lim_{n\to\infty} |2^{-n}\,(1-e^{-iu/2^n})^{-n}| = |u|^{-1}$ for $u \neq 0$, this then implies

$$\begin{aligned} \|\hat{\boldsymbol{\Upsilon}}(u)\| \;&\le\; (1+|u|)^{-m}\, \|\boldsymbol{C}_{\boldsymbol{x}_0}(u)\ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)\| \\ &\quad\times \lim_{n\to\infty} \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots \boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}\|. \end{aligned} \tag{4.1}$$

Let us define the vectors $\boldsymbol{e}_k := (e_{k,\nu})_{\nu=0}^{r-1}$ by

$$e_{k,\nu} := \begin{cases} 1 & \text{if} \quad x_{k,\nu} \neq 0, \\ 0 & \text{if} \quad x_{k,\nu} = 0, \end{cases} \tag{4.2}$$

where $x_{k,\nu}$ are the components of the vectors $\boldsymbol{x}_k$ $(k = 0,\ldots,m-1)$ introduced above.

Theorem 4.1 *Let $\boldsymbol{P}$ be an $r \times r$–matrix of the form*

$$\boldsymbol{P}(u) = \frac{1}{2^m}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\ldots \boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}^{(m)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1}\ldots \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1},$$

where the matrices $\boldsymbol{C}_{\boldsymbol{x}_k}$ are defined by the vectors $\boldsymbol{x}_k \neq \boldsymbol{0}$ $(k = 0,\ldots,m-1)$ via (2.2) *and where $\boldsymbol{P}^{(m)}(u)$ is an $(r \times r)$ matrix with trigonometric polynomials as entries. Suppose that $\boldsymbol{P}^{(m)}(0)\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$ where $\boldsymbol{e}_{m-1}$ is defined by* (4.2)*. Further, suppose that*

$$\rho_m := \rho(\boldsymbol{P}^{(m)}(0)) < 2 \tag{4.3}$$

and let for $k \ge 1$

$$\gamma_k := \frac{1}{k}\log_2 \sup_u \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^k}\right)\|. \tag{4.4}$$

Then there exists a constant $C > 0$ such that for all $u \in \mathbb{R}$

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \le C\,(1+|u|)^{-m+\gamma_k}. \tag{4.5}$$

Note that the requirement $\boldsymbol{P}^{(m)}\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$ is automatically satisfied in the case of interest to us, i.e., if $\boldsymbol{P}(u)$ is the refinement mask for the vector of functions $\phi_0(x),\ldots,\phi_{r-1}(x)$ whose integer translates provide approximation order m; see Plonka (1995.a).

Proof: 1. From (2.6) it follows that

$$\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u)\ldots \boldsymbol{E}_{\boldsymbol{x}_0}(u) = \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots \boldsymbol{E}_{\boldsymbol{x}_0}(0) + \sum_{k=1}^{m}(1-e^{-iu})^k\,\boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-1},\ldots,\boldsymbol{x}_0}$$

with some matrices $\boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-1},\ldots,\boldsymbol{x}_0}$ depending on $\boldsymbol{E}_{\boldsymbol{x}_\nu}(0)$ and $(\mathrm{D}\boldsymbol{E}_{\boldsymbol{x}_\nu})(0)$ $(\nu = 0,\ldots,m-1)$. Hence, we can write

$$\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots \boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a} = \boldsymbol{T}_{1,n}(u) + \boldsymbol{T}_{2,n}(u)$$

with
$$\boldsymbol{T}_{1,n}(u) := \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\ \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\dots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\,\boldsymbol{a}$$
and
$$\boldsymbol{T}_{2,n}(u) := \sum_{k=1}^{m}(1-e^{-iu/2^n})^k\,\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\ \boldsymbol{v}_k$$
where $\boldsymbol{v}_k := \boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-1},\dots,\boldsymbol{x}_0}\,\boldsymbol{a}$ $(k = 1,\dots,m)$.
2. We can estimate the second term $\boldsymbol{T}_{2,n}(u)$ with the same argument as in (3.8),
$$\begin{aligned}\|\boldsymbol{T}_{2,n}(u)\| &\le C\sum_{k=1}^{m}|2^{-n}u|^k\,\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\|\,\|\boldsymbol{v}_k\|\\ &\le C_{|u|}\,2^{-n}\,\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\|\\ &\le C_{\epsilon,|u|}\,2^{-n}\,(\rho_m+\epsilon)^n.\end{aligned}$$
Since the spectral radius ρ_m of $\boldsymbol{P}^{(m)}(0)$ is supposed to be < 2, it follows that for all $u \in \mathbb{R}$
$$\lim_{n\to\infty}\|\boldsymbol{T}_{2,n}(u)\| = 0.$$
3. We now concentrate on $\boldsymbol{T}_{1,n}(u)$. From the structure of $\boldsymbol{E}_{\ \boldsymbol{x}_k}(0)$ and the definition (4.2) of $\boldsymbol{e}_k$ it follows that, for any vector $\boldsymbol{b}$
$$\boldsymbol{E}_{\boldsymbol{x}_k}(0)\,\boldsymbol{b} = (\boldsymbol{x}_k)^T\boldsymbol{b}\,\boldsymbol{e}_k \quad (k = 0,\dots,m-1).$$
Repeating this argument, we obtain
$$\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\dots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\,\boldsymbol{a} = [(\boldsymbol{x}_0)^T\boldsymbol{a}]\,[(\boldsymbol{x}_1)^T\boldsymbol{e}_0]\dots[(\boldsymbol{x}_{m-1})^T\boldsymbol{e}_{m-2}]\,\boldsymbol{e}_{m-1}.$$
This leads to
$$\boldsymbol{T}_{1,n}(u) = [(\boldsymbol{x}_0)^T\boldsymbol{a}]\,[(\boldsymbol{x}_1)^T\boldsymbol{e}_0]\dots[(\boldsymbol{x}_{m-1})^T\boldsymbol{e}_{m-2}]\,\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\ \boldsymbol{e}_{m-1}.$$
Since $\boldsymbol{P}^{(m)}(0)\,\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$, and $\rho_m < 2$, we find by Theorem 3.2 that $\lim_{n\to\infty}\boldsymbol{T}_{1,n}(u)$ is well–defined for all u, and uniformly bounded on compact sets.
4. Take now any $u \in \mathbb{R}$. If $|u| \le 1$ then by the Hölder continuity of $\boldsymbol{P}^{(m)}(u)$ with Hölder exponent $\alpha \ge 1$ there is a C such that $\|\boldsymbol{T}_{1,n}(u)\| \le C$. If $|u| > 1$, define L such that $2^{L-1} < |u| \le 2^L$. Thus,
$$\begin{aligned}\|\lim_{n\to\infty}\boldsymbol{T}_{1,n}(u)\| &\le \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^L}\right)\|\,\|\lim_{n\to\infty}\boldsymbol{T}_{1,n}\left(\frac{u}{2^L}\right)\|\\ &\le C\,\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(m)}\left(\frac{u}{2^L}\right)\|.\end{aligned}$$
By the definition of γ_k it follows that
$$\|\lim_{n\to\infty}\boldsymbol{T}_{1,n}(u)\| \le C'\,2^{L\,\gamma_k} \le C''\,(1+|u|)^{\gamma_k},$$

i.e., by (4.1) and the observations above we find a constant C such that

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1+|u|)^{-m+\gamma_k}. \quad \blacksquare$$

Remarks.

1. It follows from (4.5) that the components of $\boldsymbol{\Upsilon}(x)$ are continuous if $\boldsymbol{P}$ satisfies the above conditions and if $\gamma_k < m-1$.

2. For the proof of Theorem 4.1 we have assumed that $\rho(\boldsymbol{P}^{(m)}(0)) < 2$. As we will see in Lemma 4.3 below, this can be ensured if the largest eigenvalue of $\boldsymbol{P}(0)$ apart from the eigenvalue 1 is smaller than 2^{-m+1}.

3. In order to avoid that $\boldsymbol{T}_{1,n}(u)$ collapses to 0 as $n \to \infty$, i.e., $\lim_{n\to\infty} \|\boldsymbol{T}_{1,n}(u)\| = 0$, which would imply $\hat{\boldsymbol{\Upsilon}}(u) = 0$, we have to make sure that

$$[(\boldsymbol{x}_0)^T \boldsymbol{a}]\,[(\boldsymbol{x}_1)^T \boldsymbol{e}_0] \ldots [(\boldsymbol{x}_{m-1})^T \boldsymbol{e}_{m-2}] \neq 0. \tag{4.6}$$

Note that this is already satisfied if there is an index ν $(0 \leq \nu \leq r-1)$ such that the νth component of $\boldsymbol{x}_k$ does not vanish for all $k = 0,\ldots,m-1$. On the other hand, since $\boldsymbol{x}_l$ is a left eigenvector, and $\boldsymbol{e}_{l-1}$ a right eigenvector of $\boldsymbol{P}^{(l)}(0)$, both for the eigenvalue 1, (4.6) is also satisfied if the eigenvalue 1 of $\boldsymbol{P}^{(l)}(0)$ is nondegerate, for all l.

More detailed estimates show that decay of $\hat{\boldsymbol{\Upsilon}}(u)$ is also possible in some cases where $\rho(\boldsymbol{P}^{(m)}) \geq 2$.

Corollary 4.2 *Let $\boldsymbol{P}$ be again an $r \times r$–matrix of the form*

$$\boldsymbol{P}(u) := \frac{1}{2^m}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}^{(m)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1}\ldots\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1},$$

where the matrices $\boldsymbol{C}_{\boldsymbol{x}_k}$ are defined by the vectors $\boldsymbol{x}_k \neq \boldsymbol{0}$ $(k = 0,\ldots,m-1)$ via (2.2) and where $\boldsymbol{P}^{(m)}(u)$ is an $(r \times r)$ matrix with trigonometric polynomials as entries. Suppose that $\boldsymbol{P}^m(0)\boldsymbol{e}_{m-1} = \boldsymbol{e}_{m-1}$. Further, suppose that

$$\|\boldsymbol{P}^{(m)}(u) - \boldsymbol{P}^{(m)}(0)\| \leq C\,|u|^{\alpha} \tag{4.7}$$

and that the $\boldsymbol{E}$-matrices defined in (2.4) satisfy

$$\|\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(u) - \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\| \leq C\,|u|^{\beta}. \tag{4.8}$$

Now, if $\rho_m < 2^{\min\{\alpha,\beta\}}$, then there exists a constant $C > 0$ such that for all $u \in \mathbb{R}$

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1+|u|)^{-m+\gamma_k},$$

where γ_k is defined in (4.4).

Proof: Observe that

$$\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}\| \le \|\boldsymbol{S}_n(u)\| + \|\boldsymbol{T}_{1,n}\|,$$

with

$$\begin{aligned}\boldsymbol{S}_n(u) \;&:=\; \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\\ &\quad\times\left[\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)-\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\right]\boldsymbol{a}\end{aligned}$$

and where $\boldsymbol{T}_{1,n}$ is defined as in the proof of Theorem 4.1. With the same argument as in Theorem 3.2 (cf. (3.8)) we obtain by (4.8) that

$$\begin{aligned}\|\boldsymbol{S}_n(u)\| \;&\le\; \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\|\\ &\quad\times\|[\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)-\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)]\,\boldsymbol{a}\|\\ &\le\; C_{\epsilon,\Omega}\,(\rho_m+\epsilon)^n\,C\,\frac{|u|^\beta}{2^{n\beta}}.\end{aligned}$$

Thus $\boldsymbol{S}_n(u)$ tends to zero for $n\to\infty$ if $\rho_m < 2^\beta$. Further, since $\boldsymbol{e}_{m-1}$ is an eigenvector of $\boldsymbol{P}^{(m)}(0)$, we can apply Lemma 3.1 in order to show that $\boldsymbol{T}_{1,n}$ is convergent for $(\rho_m+\epsilon) < 2^\alpha$. Hence,

$$\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\,\boldsymbol{a}$$

is well–defined if $\rho_m < 2^{\min\{\alpha,\beta\}}$. Following point 4 of the proof of Theorem 4.1 we can find a constant C such that

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \le C\,(1+|u|)^{-m+\gamma_k}. \qquad \blacksquare$$

Since $\boldsymbol{P}^{(m)}(u)$ is completely determined by $\boldsymbol{P}(u)$, the conditions $\rho_m < 2$ or $\rho_m < 2^\gamma$ are restrictions on $\boldsymbol{P}(u)$. The following lemma shows that there is a simple connection between the spectra of $\boldsymbol{P}(0)$ and $\boldsymbol{P}^{(m)}(0)$, which makes it possible to recast bounds on ρ_m as spectral bounds on $\boldsymbol{P}(0)$ as well.

Lemma 4.3 *Let $\boldsymbol{P}(u)$ be an $r\times r$–matrix of the form*

$$\boldsymbol{P}(u) = \frac{1}{2}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\,\boldsymbol{P}^{(1)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}, \tag{4.9}$$

where $\boldsymbol{C}_{\boldsymbol{x}_0}$ is defined by $\boldsymbol{x}_0 \ne \boldsymbol{0}$ via (2.2), and assume that $\boldsymbol{P}^{(1)}(0)\,\boldsymbol{e}_0 = \boldsymbol{e}_0$ (with $\boldsymbol{e}_0$ defined by $\boldsymbol{x}_0$ via (4.2)). Then, $\boldsymbol{P}(0)$ possesses a spectrum of the form $\{1,\mu_1,\ldots,$ $\mu_{r-1}\}$ if and only if $\boldsymbol{P}^{(1)}(0)$ possesses a spectrum of the form $\{1,2\mu_1,\ldots,2\mu_{r-1}\}$.

Proof: 1. First, observe that the factorization (4.9) implies that $\boldsymbol{P}(0)$ has the eigenvalue 1 with left eigenvector $\boldsymbol{x}_0$. At the same time, $\boldsymbol{x}_0$ is a left eigenvector of $\boldsymbol{P}(\pi)$ for the eigenvalue 0, i.e., we have

$$(\boldsymbol{x}_0)^T \, \boldsymbol{P}(0) = (\boldsymbol{x}_0)^T, \qquad (\boldsymbol{x}_0)^T \, \boldsymbol{P}(\pi) = \boldsymbol{0}^T$$

(cf. Plonka (1995.a), Theorem 4.1).
2. Without loss of generality, we assume that $\boldsymbol{x}_0$ is of the form $\boldsymbol{x}_0 = (x_{0,0}, \ldots, x_{0,l-1}, 0, \ldots, 0)^T$ with $1 \leq l \leq r$ and $x_{0,\nu} \neq 0$ for $\nu = 0, \ldots, l-1$. Now, consider $\boldsymbol{P}^\sharp(u) := \boldsymbol{M} \, \boldsymbol{P}(u) \, \boldsymbol{M}^{-1}$ with

$$\boldsymbol{M} := \begin{pmatrix} x_{0,0} & x_{0,1} & \ldots & x_{0,l-1} \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \oplus \boldsymbol{I}_{r-l},$$

where $\boldsymbol{I}_{r-l}$ is the $(r-l) \times (r-l)$ unit matrix (cf. section 2). Since $\boldsymbol{M}$ is invertible, $\boldsymbol{P}^\sharp(0)$ possesses the same spectrum as $\boldsymbol{P}(0)$. The left eigenvector of $\boldsymbol{P}^\sharp(0)$ for the eigenvalue 1 is $\boldsymbol{x}^\sharp := (\boldsymbol{x}_0)^T \, \boldsymbol{M}^{-1} = (1, 0, \ldots, 0)^T$. Analogously, $\boldsymbol{x}^\sharp$ is the left eigenvector of $\boldsymbol{P}^\sharp(\pi)$ for the eigenvalue 0. Hence, we find the factorization

$$\boldsymbol{P}^\sharp(u) = \frac{1}{2} \, \boldsymbol{C}^\sharp(2u) \, \boldsymbol{P}^{\sharp\,(1)}(u) \, \boldsymbol{C}^\sharp(u)^{-1} \tag{4.10}$$

with

$$\boldsymbol{C}^\sharp(u) := \text{diag}\,(1 - e^{-iu}, \, 1, \ldots, 1).$$

Observe that $\boldsymbol{P}^\sharp(0)$ has the structure

$$\boldsymbol{P}^\sharp(0) = \begin{pmatrix} 1 & 0 \ldots 0 \\ \boxed{\boldsymbol{r}(0)} & \boxed{\boldsymbol{R}(0)} \end{pmatrix},$$

where $\boldsymbol{r}(0)$ is a vector of length $r-1$ and $\boldsymbol{R}(0)$ an $(r-1) \times (r-1)$–matrix, it follows by the factorization (4.10) that $\boldsymbol{P}^{\sharp\,(1)}(0)$ is of the form

$$\boldsymbol{P}^{\sharp\,(1)}(0) = \begin{pmatrix} 1 & 0 \ldots 0 \\ \boxed{\boldsymbol{0}} & \boxed{2\,\boldsymbol{R}(0)} \end{pmatrix}.$$

Consequently, $\boldsymbol{P}^{\sharp\,(1)}(0)$ has the spectrum $\{1, 2\mu_1, \ldots, 2\mu_{r-1}\}$ if and only if $\boldsymbol{P}^\sharp(0)$ has the spectrum $\{1, \mu_1, \ldots, \mu_{r-1}\}$.
3. We show next that $\boldsymbol{P}^{\sharp\,(1)}(0)$ and $\boldsymbol{P}^{(1)}(0)$ have the same spectrum. The factorizations (4.9) and (4.10) imply the following connection between $\boldsymbol{P}^{\sharp\,(1)}(0)$ and $\boldsymbol{P}^{(1)}(0)$:

$$\boldsymbol{P}^{(1)}(u) = \boldsymbol{A}(2u) \, \boldsymbol{P}^{\sharp\,(1)}(u) \, \boldsymbol{A}(u)^{-1},$$

where

$$\begin{aligned}\boldsymbol{A}(u) &:= \boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}\,\boldsymbol{M}^{-1}\,\boldsymbol{C}^{\sharp}(u) \\ &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z & x_{0,1} & x_{0,2} & \dots & x_{0,r-1} \\ z & 0 & x_{0,2} & \dots & x_{0,r-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z & 0 & \dots & 0 & x_{0,r-1} \end{pmatrix} \oplus \boldsymbol{I}_{r-l} \qquad (z := e^{-iu}).\end{aligned}$$

Since $\boldsymbol{A}(0)$ is invertible, it follows that $\boldsymbol{P}^{\sharp(1)}(0)$ and $\boldsymbol{P}^{(1)}(0)$ are similar, and thus the spectra of $\boldsymbol{P}(0)$ and $\boldsymbol{P}^{(1)}(0)$ are connected as given in Lemma 4.3. ■

It follows that the spectrum of $\boldsymbol{P}^{(m)}(0)$ is likewise given by $\{1, 2^m\mu_1, \dots, 2^m\mu_{r-1}\}$. The requirement that $\rho_m < 2^\lambda$ (as in Corollary 4.2 thus translates into

$$\max\{|\mu_1|, \dots, |\mu_{r-1}|\} < 2^{\lambda - m}\ . \tag{4.11}$$

Remark.
If $m > \lambda$, which need not be true in general, but which we expect to be true in most cases ($\lambda = 1$ except if both $\boldsymbol{P}(u)$ and $\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u) \dots \boldsymbol{E}_{\boldsymbol{x}_0}(u)$ have vanishing derivatives at $u = 0$), then (4.11) automatically implies that ρ_0, the spectral radius of $\boldsymbol{P}(0)$, equals 1. It also implies that the eigenvalue 1 of $\boldsymbol{P}(0)$ is nondegenerate. Since $\gamma_k \geq 0$ for all k, we need to have $m \geq 2$ in order to ensure decay faster than $(1 + |u|)^{-1-\epsilon}$ for $|\hat{\boldsymbol{\Upsilon}}(u)|$.

5. Transfer operators

In this section, we want to investigate regularity estimates for $\boldsymbol{\Upsilon}$ in terms of Sobolev estimates, using transfer operators. The Sobolev exponent s of $\boldsymbol{\Upsilon}$ is defined by

$$s = \sup\{\delta;\ \int_{-\infty}^{\infty} \|\hat{\boldsymbol{\Upsilon}}(u)\|^2\,(1 + |u|^2)^\delta\ \,\mathrm{d}u < +\infty\}.$$

We assume that the factorization (2.8) and the hypothesis (4.3) of Theorem 4.1 are satisfied. As we saw in the proof of Theorem 4.1, we have

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C(1 + |u|)^{-m}\|T_1(u)\|,$$

where

$$\boldsymbol{T}_1(u) = \lim_{n \to +\infty} \boldsymbol{T}_{1,n}(u)$$

and

$$\boldsymbol{T}_{1,n}(u) := \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \dots \boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right) \boldsymbol{E}_{\boldsymbol{x}_{m-1}}(0) \dots \boldsymbol{E}_{\boldsymbol{x}_0}(0)\boldsymbol{a}.$$

It follows that if we can prove an estimate of the type

$$\int_{-2^n\pi}^{2^n\pi} \|\boldsymbol{T}_1(u)\|^2 \ \mathrm{d}u \le C\, 2^{2n\gamma}, \tag{5.1}$$

then $\boldsymbol{\Upsilon}$ is in the Sobolev space H^s for all $s < m-\gamma$.
The estimate (5.1) is related to the spectral property of the transition operator $\mathcal{T}$ that acts on 2π-periodic $r \times r$ matrices $\boldsymbol{M}(u)$ according to

$$(\mathcal{T}\boldsymbol{M})(2u) := \boldsymbol{P}^{(m)}(u)\boldsymbol{M}(u)(\boldsymbol{P}^{(m)})^*(u)+\boldsymbol{P}^{(m)}(u+\pi)\boldsymbol{M}(u+\pi)(\boldsymbol{P}^{(m)})^*(u+\pi), \tag{5.2}$$

where $(\boldsymbol{P}^{(m)})^* := \overline{(\boldsymbol{P}^{(m)})}^T$. As in the scalar case, this operator leaves a finite dimensional space E containing the identity invariant, if $\boldsymbol{P}^{(m)}$ has trigonometric polynomial entries (cf. Cohen and Daubechies (1994)). Let σ be the spectral radius of $\mathcal{T}$ restricted to E.

Theorem 5.1 *The estimate* (5.1) *holds for all* $\gamma > \frac{\log(\sigma)}{2\log 2}$. *Consequently,* $\boldsymbol{\Upsilon}$ *is in* H^s *for all* $s < m - \frac{\log(\sigma)}{2\log 2}$.

Proof: For all $n > 0$, we have

$$\begin{aligned} &\int_{-\pi}^{\pi} \mathcal{T}^n\boldsymbol{M}(u)\ \mathrm{d}u = \int_{-\pi}^{\pi} \mathcal{T}\,\mathcal{T}^{n-1}\boldsymbol{M}(u)\ \mathrm{d}u \\ =\ &\int_{-2\pi}^{2\pi} \boldsymbol{P}^{(m)}(u/2)\,\mathcal{T}^{n-1}\,\boldsymbol{M}(u/2)(\boldsymbol{P}^{(m)})^*(u/2)\ \mathrm{d}u \\ =\ &\cdots\cdots \\ =\ &\int_{-2^n\pi}^{2^n\pi} \boldsymbol{P}^{(m)}(u/2)\ldots\boldsymbol{P}^{(m)}(2^{-n}u)\boldsymbol{M}(2^{-n}u)(\boldsymbol{P}^{(m)})^*(2^{-n}u)\ldots(\boldsymbol{P}^{(m)})^*(u/2)\ \mathrm{d}u. \end{aligned}$$

If we take $\boldsymbol{M} = \boldsymbol{I}$ and apply the trace operation, we thus obtain the estimate

$$\int_{-2^n\pi}^{2^n\pi} \mathrm{Tr}[\boldsymbol{P}^{(m)}(u/2)\ldots\boldsymbol{P}^{(m)}(2^{-n}u)(\boldsymbol{P}^{(m)})^*(2^{-n}u)\ldots(\boldsymbol{P}^{(m)})^*(u/2)]\ \mathrm{d}u \le C_\epsilon(\sigma+\epsilon)^n,$$

for all $n > 0$. Since $\|A\|_2 = \sqrt{\mathrm{Tr}(AA^*)}$ is an equivalent norm for finite matrices, it follows that

$$\int_{-2^n\pi}^{2^n\pi} \|\boldsymbol{P}^{(m)}(u/2)\ldots\boldsymbol{P}^{(m)}(2^{-n}u)\|^2\ \mathrm{d}u \le C_\epsilon(\sigma+\epsilon)^n.$$

This last estimate clearly implies (5.1) for all $\gamma > \frac{\log(\sigma)}{2\log 2}$, if we observe that $\boldsymbol{T}_1(u) = \boldsymbol{P}^{(m)}(u/2)\ldots\boldsymbol{P}^{(m)}(2^{-n}u)\boldsymbol{T}_1(2^{-n}u)$ and that $\boldsymbol{T}_1(u)$ is uniformly bounded on compact sets. ■

If the conditions of Theorem 4.1 are satisfied, with $\gamma_k < m-1$ for some $k \leq 1$, then $\hat{\boldsymbol{\Upsilon}}$ is well-defined and integrable, so that $\boldsymbol{\Upsilon}(x)$, its inverse Fourier transform, is well-defined as well. Since $\hat{\boldsymbol{\Upsilon}}(u)$ is obviously a solution to (1.7), $\boldsymbol{\Upsilon}(x)$ is a solution to (1.1). Is it the only one? The following theorem lists some conditions that ensure uniqueness.

Theorem 6.1 *Suppose that the conditions of Theorem* 4.1 *are satisfied, with* $\inf_{k\geq 1}\gamma_k < m-1$, *and that the eigenvalue* 1 *of* $\boldsymbol{P}(0)$ *is nondegenerate. Then* $\boldsymbol{\Upsilon}(x)$ *is a compactly supported continuous solution to* (1.1). *Moreover, if* $\boldsymbol{\phi}(x)$ *is any other* L^1*-solution to* (1.1) *such that* $\int \boldsymbol{\phi}(x)\,\mathrm{d}x \neq \mathbf{0}$ *and* $\int (1+|x|)\|\boldsymbol{\phi}(x)\|\,\mathrm{d}x < \infty$, *then* $\boldsymbol{\phi}(x)$ *is a multiple of* $\boldsymbol{\Upsilon}(x)$.

Proof: 1. We assume, as in Theorem 4.1, that all the entries of $\boldsymbol{P}(u)$ are trigonometric polynomials. Let us, for this point only, consider u to be complex rather than real. The argument that $\|\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\|$ is bounded uniformly in $n \geq 1$ and in $u \in \{z; |z| < 1\}$ holds for complex u as well. Since $\|\boldsymbol{P}(u)\| \leq Ce^{R|\mathrm{Im}u|}$, it then follows that for any $|u| > 1$, $2^k \leq |u| < 2^{k+1}$,

$$\|\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}\| \leq C^k e^{R|\mathrm{Im}u|(1/2+1/4+\ldots+1/2^k)}C_1 \leq C'(1+|u|)^{k\log_2 C}\, e^{R|\mathrm{Im}u|}.$$

It follows that $\hat{\boldsymbol{\Upsilon}}(u) = \lim_{n\to\infty}\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}$ satisfies the same bound, implying that $\boldsymbol{\Upsilon}$ is a compactly supported distribution. On the other hand, $\boldsymbol{\Upsilon}(x)$ is bounded and continuous because, by Theorem 4.1, $|\hat{\boldsymbol{\Upsilon}}(u)| \leq C(1+|u|)^{-1-\epsilon}$ for real u. Note that $\hat{\boldsymbol{\Upsilon}}$ is a C^∞–function since its Fourier transform has compact support.

2. If $\boldsymbol{\phi}(x)$ is another L^1-solution, then $\hat{\boldsymbol{\phi}}(0) \neq 0$ must be an eigenvector for $\boldsymbol{P}(0)$ with eigenvalue 1, so that $\hat{\boldsymbol{\phi}}(0) = c\boldsymbol{a}$ for some $c \neq 0$. Since $\int |x|\|\boldsymbol{\phi}(x)\|dx < \infty$, we also have $\|\hat{\boldsymbol{\phi}}(u) - \hat{\boldsymbol{\phi}}(0)\| \leq C|u|$. Hence, for any fixed u,

$$\begin{aligned}
&\|\hat{\boldsymbol{\phi}}(u) - c\hat{\boldsymbol{\Upsilon}}(u)\| \\
&= \lim_{n\to\infty}\|\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\left[\hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) - \hat{\boldsymbol{\phi}}(0)\right]\| \\
&\leq C'\lim_{n\to\infty}\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\boldsymbol{E}\boldsymbol{x}_{m-1}\left(\frac{u}{2^n}\right)\cdots\boldsymbol{E}\boldsymbol{x}_0\left(\frac{u}{2^n}\right) \\
&\quad\times\left[\hat{\boldsymbol{\phi}}\left(\frac{u}{2^n}\right) - \hat{\boldsymbol{\phi}}(0)\right]\| \\
&\leq C'\lim_{n\to\infty}\left[C_{\epsilon,|u|}(\rho_m+\epsilon)^n C_1 C\,\frac{|u|}{2^{-n}}\right] = 0\ ,
\end{aligned}$$

since $\rho_m < 2$. Thus, $\boldsymbol{\phi} \equiv c\boldsymbol{\Upsilon}$. ∎

All the examples studied in the literature so far correspond to $\boldsymbol{P}(u)$ in which all the entries are trigonometric polynomials, and that is why we have mostly restricted ourselves to this case. Nevertheless, most of our analysis carries over to the non-polynomial case. In Plonka (1995.a), the original version of Theorem 2.1 does not require the ϕ_ν to be compactly supported, nor the entries of $\boldsymbol{P}$ to be trigonometric polynomials; only sufficient decay in x for $\boldsymbol{\phi}(x)$ and a sufficiently high regularity of $\boldsymbol{P}(u)$ are required. As shown in section 3 (where the $\boldsymbol{P}(u)$ were not restricted to trigonometric polynomials), this then implies $|\hat{\boldsymbol{\Upsilon}}(u) - \hat{\boldsymbol{\Upsilon}}(0)| \leq C|u|$ (since $\boldsymbol{P}$ is Hölder continuous in $u = 0$ with Hölder exponent at least 1). This, in turn, is the only ingredient necessary in point 2 of the proof of Theorem 6.1, which establishes uniqueness of the solution within a certain class of functions with mild decay. Compact support of $\boldsymbol{\Upsilon}(x)$ is, of course, no longer assured.

7. Convergence of the cascade algorithm

If $\boldsymbol{P}(u)$ is m ($m \geq 1$) times factorizable (in the sense of Plonka (1995.a)), i.e.,

$$\boldsymbol{P}(u) = \frac{1}{2^m(1-e^{-iu})^m}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}^{(m)}(u)\,\boldsymbol{E}_{\boldsymbol{x}_{m-1}}(u)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(u),$$

if the spectral radius of $\boldsymbol{P}^{(m)}(0)$ is less than 2, and if, for some fixed k,

$$\gamma_k = \frac{1}{k}\log_2 \sup_u \|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^k}\right)\| < m,$$

then our analysis in the previous sections has shown that

$$\hat{\boldsymbol{\Upsilon}}(u) = \lim_{n\to\infty}\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}$$

(with $(\boldsymbol{x}_0)^T\boldsymbol{a} = 1$) is well-defined, and that

$$\|\hat{\boldsymbol{\Upsilon}}(u)\| \leq C\,(1+|u|)^{-m+\gamma_k}.$$

Moreover, we have $\|\hat{\boldsymbol{\Upsilon}}(u) - \hat{\boldsymbol{\Upsilon}}(0)\| \leq C\,|u|$ for $|u| \leq 1$. So far, this convergence is only pointwise, in the Fourier domain. For practical applications one is often interested in convergence of iterative schemes that generate the function $\boldsymbol{\Upsilon}$ in the "x–domain". One has to distinguish two types of schemes:

- The cascade algorithm, introduced in Daubechies (1988), consists in iterating the mapping $f \mapsto \sum_{n=0}^N \boldsymbol{P}_n f(2x-n)$ on a well chosen initial function vector $f(x)$.
- The subdivision or refinement algorithm consists in iterative refinements of a "vector sequence" $\boldsymbol{s}_0(k)$ by rules of the type

  $$\boldsymbol{s}_n(2^{-n}k) = \sum_m \boldsymbol{P}_{k-2m}^T\,\boldsymbol{s}_{n-1}(2^{-n+1}m)$$

  (see Cavaretta, Dahmen and Micchelli (1991) or Dyn (1992) for an overview of subdivision schemes).

In the scalar case, it can easily be checked that n iterations of the cascade algorithm, initiated on the "hat-function" $\Delta(x) := \max\{0, 1-|x|\}$, are equivalent to the linear interpolation of the points generated by n iterations of the subdivision algorithm, initiated on a Dirac sequence $\delta(k)$. However, the subdivision process is often preferred, because of its local nature.
In the vector case, these relations are more complex: if one iterates n times the cascade algorithm on an initial vector function of the type $\Delta(x)\,\boldsymbol{b}$ where $\boldsymbol{b}$ is a fixed vector, then the result $\boldsymbol{\Phi}_n$ is expressed in the Fourier domain by

$$\hat{\boldsymbol{\Phi}}_n(u) = \hat{\Delta}(2^{-n}u)\boldsymbol{P}(u/2)\ldots\boldsymbol{P}(2^{-n}u)\boldsymbol{b}. \tag{7.1}$$

In contrast, if one iterates n times the subdivision algorithm on an inital vector sequence $\boldsymbol{s}_0(k)$, the resulting sequence $\boldsymbol{s}_n(k)$ is related to $\boldsymbol{s}_0$ by

$$(\hat{\boldsymbol{s}}_n(u))^T = (\hat{\boldsymbol{s}}_0(u))^T\boldsymbol{P}(u/2)\ldots\boldsymbol{P}(2^{-n}u).$$

Here $(\hat{\boldsymbol{s}}_n(u))^T$ is the row vector composed by the Fourier series of each component of $\boldsymbol{s}_n$. After linear interpolation, we obtain a row vector function $(\tilde{\boldsymbol{\Phi}}_n(x))^T$ given by

$$(\hat{\tilde{\boldsymbol{\Phi}}}_n(u))^T = \hat{\Delta}(2^{-n}u)(\hat{\boldsymbol{s}}_0(u))^T\boldsymbol{P}(u/2)\ldots\boldsymbol{P}(2^{-n}u).$$

This shows that the j-th component of $\boldsymbol{\Phi}_n$ can be obtained by applying the subdivision algorithm on the initial vector sequence $(s_{0,l}(k) = \delta_{l,j}\delta_{k,0}$ (i.e. the sequence $\cdots 0001000\cdots$ in the j-th component, and the zero sequence in all other components), then taking the scalar product with the vector $\boldsymbol{b}$ and interpolating linearly the resulting scalar sequence.
Let us now investigate the convergence of the cascade algorithm, keeping in mind these more sophisticated relations with subdivision schemes. In order to simplify the study of convergence, we shall use the function $\frac{\sin(\pi x)}{\pi x}$ as a starting point, rather than the hat function $\Delta(x)$, which is equivalent to considering band-limited interpolation of the sequences generated from the subdivision scheme. We thus define

$$\hat{\boldsymbol{\Phi}}_0^{b.l.}(u) := \chi_{[-\pi,\pi]}(u)\hat{\boldsymbol{\Upsilon}}(0) = \chi_{[-\pi,\pi]}(u)\boldsymbol{a}\,, \qquad \hat{\boldsymbol{\Phi}}_n^{b.l.}(u) := \boldsymbol{P}\left(\frac{u}{2}\right)\hat{\boldsymbol{\Phi}}_{n-1}^{b.l.}\left(\frac{u}{2}\right). \tag{7.2}$$

Note that $\hat{\boldsymbol{\Phi}}_0(0) = \hat{\boldsymbol{\Upsilon}}(0)$. The following result deals with the convergence of the cascade algorithm in the uniform norm.

Theorem 7.1 *Let $\boldsymbol{P}$ be an $r\times r$ matrix with the assumptions of Theorem 4.1. If γ_k defined in (4.4) satisfies $\gamma_k < m-1$, then we have*

$$\lim_{n\to\infty} \|\hat{\boldsymbol{\Phi}}_n^{b.l.} - \hat{\boldsymbol{\Upsilon}}\|_{L^1} = 0.$$

As a consequence $\boldsymbol{\Phi}_n^{b.l.}(x)$ converges uniformly to $\boldsymbol{\Upsilon}(x)$.

Proof: We have

$$\hat{\mathbf{\Phi}}_n^{b.l.}(u) = \chi_{[-\pi,\pi]}(2^{-n}u)\boldsymbol{P}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}\left(\frac{u}{2^n}\right)\boldsymbol{a}.$$

With the assumptions of Theorem 4.1, we already know that $\hat{\mathbf{\Phi}}_n^{b.l.}$ converges pointwise (and uniformly on compact sets) to $\hat{\mathbf{\Upsilon}}$.
Using the factorization of $\boldsymbol{P}(u)$, we obtain

$$\begin{aligned}\hat{\mathbf{\Phi}}_n^{b.l.}(u) &= \chi_{[-\pi,\pi]}(2^{-n}u)\left[\frac{1}{2^n(1-e^{-iu/2^n})}\right]^m \boldsymbol{C}_{\boldsymbol{x}_0}(u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)\\ &\quad\times\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\boldsymbol{E}_{\boldsymbol{x}_{m-1}}\left(\frac{u}{2^n}\right)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}\left(\frac{u}{2^n}\right)\boldsymbol{a},\end{aligned}$$

and thus

$$\|\hat{\mathbf{\Phi}}_n^{b.l.}(u)\| \leq C_1(1+|u|)^{-m}\,\chi_{[-\pi,\pi]}(2^{-n}u)\,\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\|.$$

Now, from the assumptions of Theorem 4.1, we have

$$\chi_{[-\pi,\pi]}(2^{-n}u)\,\|\boldsymbol{P}^{(m)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(m)}\left(\frac{u}{2^n}\right)\| \leq C_2(1+|u|)^{\gamma_k},$$

where C_2 does not depend on n. We thus have the uniform estimate

$$\|\hat{\mathbf{\Phi}}_n^{b.l.}(u)\| \leq C(1+|u|)^{-m+\gamma_k}.$$

Since $m-\gamma_k > 1$, we can apply dominated convergence and the result follows. ■

Remarks.
1. Because the hat function $\Delta(x)$ and the sinc function $\frac{\sin(\pi x)}{\pi x}$ agree on integers, one easily checks that the vector functions $\mathbf{\Phi}_n$ defined by (7.1) (with $\boldsymbol{b}$ replaced by $\boldsymbol{a}$) and the band-limited $\mathbf{\Phi}_n^{b.l.}$ agree in the dyadic rationals $2^{-n}\mathbb{Z}$,

$$\mathbf{\Phi}_n(2^{-n}k) = \mathbf{\Phi}_n^{b.l.}(2^{-n}k)\ \ ,\ \ k\in\mathbb{Z}.$$

Now $\mathbf{\Phi}_n$ is just the linear interpolation of the $\mathbf{\Phi}_n(2^{-n}k$; because $\mathbf{\Upsilon}$ is Hölder continuous, and $\sup_k\|\mathbf{\Phi}_n(2^{-n}k)-\mathbf{\Upsilon}(2^{-n}k)\|\to 0$ as $n\to\infty$, it follows that $\mathbf{\Phi}_n$ converges uniformly to $\mathbf{\Upsilon}$ as well.
2. The same arguments will also give L^2–convergence, assuming only $\gamma_k < m-\frac{1}{2}$.
3. If $m-\gamma_k > m'+1$, convergence results in $C^{m'}$ can also be obtained starting from the same cardinal sine function.
4. The graphs for the examples in section 7 are, in fact, graphs of close approximations $\mathbf{\Phi}_n$ to the true solutions $\boldsymbol{\phi}$, obtained by the subdivision iteration described just before in Theorem 7.1.

8. Examples

In this section we want to apply the analysis of the previous sections to various examples. We will see that the known smoothness of B–splines with multiple knots and of DGHM scaling functions can be recovered. Further, we construct a new example with controlled smoothness.

8.1. B–splines with multiple knots

Let $r \in \mathbb{N}$ and $m \in \mathbb{N}_0$ be given fixed integers. We consider equidistant knots with multiplicity r, $x_l := \lfloor l/r \rfloor$ $(l \in \mathbb{Z})$, where $\lfloor x \rfloor$ means the integer part of $x \in \boldsymbol{R}$. Let $N_\nu^{m,r}$ $(\nu \in \mathbb{Z})$ denote the cardinal B–spline of order m and defect r with respect to the knots $x_\nu, \ldots, x_{\nu+m}$ given by the following formulas:
For $m = 0$ and $\nu = 0, \ldots, r-2$ let $N_\nu^{0,r} := \mathrm{D}^{r-1-\nu}\delta/(r-1-\nu)$ and let $N_{r-1}^{0,r} := \delta$, where δ denotes the Dirac distribution. For $m \geq 1$ and $x_\nu = x_{\nu+m} = 0$, we define $N_\nu^{m,r}$ according to the distribution theory by

$$N_\nu^{m,r} := \frac{\mathrm{D}^{r-m-1-\nu}\delta}{r-1-\nu}.$$

Further, let $N_{r-1}^{1,r} := (\chi_{[0,1)} + \chi_{(0,1]})/2$. Assume that for $l \in \mathbb{Z}$ and $\nu = 0, \ldots, r-1$ we have $N_{\nu+lr}^{1,r} := N_\nu^{1,r}(\cdot - l)$. Now, for $m \geq 2$ and $x_{\nu+m} > x_\nu$, let $N_\nu^{m,r}$ $(\nu \in \mathbb{Z})$ be defined by the recursion formula

$$(x_{\nu+m} - x_\nu)\, N_\nu^{m,r}(x) := (x - x_\nu)\, N_\nu^{m-1,r}(x) + (x_{\nu+m} - x)\, N_{\nu+1}^{m-1,r}(x).$$

Note that for $\nu \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$N_{\nu+lr}^{m,r} = N_\nu^{m,r}(\cdot - l) \qquad (l \in \mathbb{Z})$$

and for $m \geq r$,

$$\hat{N}_\nu^{m,r}(0) = \int_{-\infty}^{\infty} N_\nu^{m,r}(x)\ \mathrm{d}x = \frac{1}{m}.$$

It is well–known that for $m > r$, we have $N_\nu^{m,r} \in C^{m-r-1}(\mathbb{R})$. We put $\boldsymbol{N}_m := (N_\nu^{m,r})_{\nu=0}^{r-1}$ and $\hat{\boldsymbol{N}}_m := (\hat{N}_\nu^{m,r})_{\nu=0}^{r-1}$. In particular, we obtain

$$\hat{\boldsymbol{N}}_0(u) = \left(\frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^1}{1}, 1 \right)^T.$$

As shown in de Boor (1976), the spline functions $N_\nu^{m,r}(\cdot - l)\,(m \geq r;\, l \in \mathbb{Z};\, \nu = 0, \ldots r-1)$ form a Riesz basis of their closed linear span V_0. Furthermore, V_0 provides approximation order m.
The vector $\boldsymbol{N}_m$ satisfies a vector refinement equation

$$\hat{\boldsymbol{N}}_m(2u) = \boldsymbol{P}_m(u)\, \hat{\boldsymbol{N}}_m(u),$$

where the refinement mask $\boldsymbol{P}_m$ is of the form

$$\boldsymbol{P}_m(u) = \frac{1}{2^m}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}_0(u)\,\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1}\ldots\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1} \tag{8.1}$$

with matrices $\boldsymbol{C}_{\boldsymbol{x}_k}$ defined by the vectors of spline knots $\boldsymbol{x}_k := (x_{m-k},\ldots, x_{m-k+r-1})^T$ $(k = 0,\ldots,m-1)$ via (2.2) and with the refinement mask of $\boldsymbol{N}_0$

$$\boldsymbol{P}_0(u) := \boldsymbol{P}_0(0) = \text{diag}\,(2^{r-1},\ldots,2^0)$$

(cf. Plonka (1995.b)). In particular, we have the recursion

$$\boldsymbol{P}_m(u) = \frac{1}{2}\,\boldsymbol{C}(2u)\,\boldsymbol{P}_{m-1}(u)\,\boldsymbol{C}(u)^{-1},$$

with $\boldsymbol{C}$ defined by $(x_m,\ldots,x_{m+r-1})^T$, where $\boldsymbol{P}_{m-1}$ is the refinement mask of the B–spline vector $\boldsymbol{N}_{m-1}$ of order $m-1$.
Now, let us apply the theory of the previous sections to the refinement mask $\boldsymbol{P}_m$. Repeated application of Lemma 4.3 yields that $\boldsymbol{P}_m(0)$ possesses the spectrum $\{1, 2^{r-1-m},\ldots,2^{-m+1}\}$. Since $\|\boldsymbol{P}_0(u) - \boldsymbol{P}_0(0)\| \le C|u|^\alpha$ holds for all $\alpha > 0$, we have by Lemma 3.1 pointwise convergence of

$$\lim_{n\to\infty}\prod_{l=1}^{n}\boldsymbol{P}_0\left(\frac{u}{2^l}\right)\,\boldsymbol{a}$$

with $\boldsymbol{a} := (0,\ldots,0,1)^T$ for all $u \in \mathbb{R}$. Hence, it follows for all $m \in \mathbb{N}_0$ that

$$\hat{\boldsymbol{\Upsilon}}_m(u) := \lim_{n\to\infty}\prod_{l=1}^{n}\boldsymbol{P}_m\left(\frac{u}{2^l}\right)\,\boldsymbol{a}, \tag{8.2}$$

where $\boldsymbol{a} := (\underbrace{0,\ldots,0}_{r-m-1},\underbrace{1,\ldots,1}_{m+1})^T$ for $r > m+1$ and $\boldsymbol{a} := (1,\ldots,1)^T$ for $r \le m+1$, is at least pointwise convergent. As we will see later, Theorem 6.1 implies that for $m > r$ the solution $\boldsymbol{\Upsilon}_m(x)$ coincides with $\boldsymbol{N}_m(x)$.
Observe that for $r = 1$, we have the well–known refinement equation for cardinal B–splines

$$P_m(u) = \left(\frac{1+e^{-iu}}{2}\right)^m.$$

Now, assume that $r \ge 2$. Then $\rho(\boldsymbol{P}_0(0)) = 2^{r-1} \ge 2$, such that we can not apply our analysis in Theorem 4.1 to the factorization (8.1). But using Lemma 4.3, we find that $\boldsymbol{P}_{r-1}(0)$ with

$$\boldsymbol{P}_{r-1}(u) := \frac{1}{2^{r-1}}\,\boldsymbol{C}_{\boldsymbol{x}_{m-r+1}}(2u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(2u)\,\boldsymbol{P}_0(u)\,\boldsymbol{C}_{\boldsymbol{x}_{m-1}}(u)^{-1}\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-r+1}}(u)^{-1}$$

possesses the spectrum $\{1,\,1,\,2^{-1},\ldots,2^{-r+2}\}$, i.e.,

$$\rho(\boldsymbol{P}_{r-1}(0)) = 1.$$

Note, that $\boldsymbol{P}_{r-1}$ is the refinement mask of $\boldsymbol{N}_{r-1}$. Thus, we can apply Theorem 4.1 to the factorization

$$\boldsymbol{P}_m(u) = \boldsymbol{C}_{\boldsymbol{x}_0}(2u)\ldots\boldsymbol{C}_{\boldsymbol{x}_{m-r}}(2u)\,\boldsymbol{P}_{r-1}(u)\,\boldsymbol{C}_{\boldsymbol{x}_{m-r}}(u)^{-1}\ldots\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}$$

and find that

$$\|\hat{\boldsymbol{\Upsilon}}_m(u)\| \le C\,(1+|u|)^{-m+r-1+\gamma_1}$$

with

$$\gamma_1 = \log_2 \sup_u \|\boldsymbol{P}_{r-1}\left(\frac{u}{2}\right)\|. \tag{8.3}$$

Lemma 8.1 *For γ_1 given in* (8.3) *we have $\gamma_1 = 0$.*

Proof: Observe that $N_0^{r-1,r}$ is defined by the knots $\underbrace{0,\ldots,0}_{r}$, that means $N_0^{r-1,r} = \frac{\delta}{r-1}$. Further, $N_\nu^{r-1,r}$ $(\nu = 1,\ldots,r-1)$, defined by $\underbrace{0,\ldots,0}_{r-\nu},\underbrace{1,\ldots,1}_{\nu}$, coincide with the Bernstein polynomials of degree $r-2$, i.e.

$$N_\nu^{r-1,r}(x) = \binom{r-2}{\nu-1}\,x^{\nu-1}\,(1-x)^{r-1-\nu}.$$

Hence the refinement mask of $\boldsymbol{N}_{r-1}$ can explicitely be given by

$$\boldsymbol{P}_{r-1}(u) = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A}_{r-2}(u) \end{pmatrix}$$

with

$$\boldsymbol{A}_{r-2}(u) = \frac{1}{2}\Big(\boldsymbol{A}_{r-2}^0 + \boldsymbol{A}_{r-2}^1\,e^{-iu}\Big),$$

where $\boldsymbol{A}_{r-2}^0$ and $\boldsymbol{A}_{r-2}^1$ are triangular matrices of the form

$$\boldsymbol{A}_{r-2}^0 := \left(\frac{1}{2^j}\binom{j}{i}\right)_{i,j=0}^{r-2}, \qquad \boldsymbol{A}_{r-2}^1 := \left(\frac{1}{2^{r-2-j}}\binom{r-2-j}{i-j}\right)_{i,j=0}^{r-2}$$

(see e.g. Micchelli and Pinkus (1991)). Recall that the spectral norm $\|\boldsymbol{V}\|_2$ of a matrix $\boldsymbol{V} := (v_{ij})_{i,j=1}^n$ can be estimated by the product of the matrix 1–norm and the matrix ∞–norm

$$\|\boldsymbol{V}\|_1 := \max_{1\le j\le n}\sum_{i=1}^n |v_{ij}|, \qquad \|\boldsymbol{V}\|_\infty := \max_{1\le i\le n}\sum_{j=1}^n |v_{ij}|, \tag{8.4}$$

i.e.,

$$\|\boldsymbol{V}\|_2^2 \le \|\boldsymbol{V}\|_1\,\|\boldsymbol{V}\|_\infty$$

(see e.g. Lancaster and Tismenetsky (1985)). Since all entries of $\boldsymbol{A}^0_{r-2}$ and $\boldsymbol{A}^1_{r-2}$ are nonnegative, it follows that

$$\begin{aligned}\sup_u \|\boldsymbol{A}_{r-2}(u)\|_1 &= \|\boldsymbol{A}_{r-2}(0)\|_1 = \max_{0\le j\le r-2} \frac{1}{2}\sum_{i=0}^{r-2}\left(\frac{1}{2^j}\binom{j}{i} + \frac{1}{2^{r-2-j}}\binom{r-2-j}{i-j}\right) \\ &= \frac{1}{2^{j+1}}\sum_{i=0}^{j}\binom{j}{i} + \frac{1}{2^{r-1-j}}\sum_{i=j}^{r-2}\binom{r-2-j}{i-j} = \frac{1}{2}+\frac{1}{2} = 1.\end{aligned}$$

Analogously , we find that

$$\sup_u \|\boldsymbol{A}_{r-2}(u)\|_\infty = \|\boldsymbol{A}_{r-2}(0)\|_\infty = 1.$$

Hence, we have $\sup_u \|\boldsymbol{P}_{r-1}(u)\|_2 = 1$, and thus $\gamma_1 = 0$. ∎

By Theorem 4.1 it follows for $\nu = 0, \ldots, r-1$ that

$$\|\hat{\boldsymbol{\Upsilon}}_m\| \le C\,(1+|u|)^{-m+r-1},$$

i.e., the elements of $\boldsymbol{\Upsilon}_m$ are $(m-r-1)$–times continuously differentiable. Since $\gamma_1 < m-r$, Theorem 6.1, ensuring the uniqueness of the limit, yields for $m > r$ that $\boldsymbol{\Upsilon}_m(x) = \boldsymbol{N}_m(x)$. Further, we also have uniform and L^2–convergence of the associated cascade algorithm.

We want to check, whether the smoothness result can be improved by Corollary 4.2. By

$$\|\boldsymbol{E}_{\boldsymbol{x}_{m-r}}(u)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(u) - \boldsymbol{E}_{\boldsymbol{x}_{m-r}}(0)\ldots\boldsymbol{E}_{\boldsymbol{x}_0}(0)\| = \|\sum_{k=1}^{m-r-1}(1-e^{-iu})^k\,\boldsymbol{G}^{(k)}_{\boldsymbol{x}_{m-r},\ldots,\boldsymbol{x}_0}\|$$

and $\|\boldsymbol{G}^{(1)}_{\boldsymbol{x}_{m-r},\ldots,\boldsymbol{x}_0}\| > 0$, we have (4.8) only with $\beta = 1$. Hence, our result can not be improved.

8.2. DGHM–scaling functions

Now, we consider the example of two scaling functions treated in Donovan, Geronimo, Hardin, Massopust (1994). In the special case $s = s_1 = s_2$ of their construction, let $\hat{\boldsymbol{\phi}}$ be a solution of (1.7) with the refinement mask

$$\boldsymbol{P}(u) := \frac{1}{2}\,(\boldsymbol{P}_0 + \boldsymbol{P}_1\,e^{-iu} + \boldsymbol{P}_2\,e^{-2iu} + \boldsymbol{P}_3\,e^{-3iu}), \tag{8.5}$$

where

$$\boldsymbol{P}_0 := \begin{pmatrix} -\frac{s^2-4s-3}{2\,(s+2)} & 1 \\ -\frac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & \frac{3s^2+s-1}{2\,(s+2)}\end{pmatrix}, \quad \boldsymbol{P}_1 := \begin{pmatrix} -\frac{s^2-4s-3}{2\,(s+2)} & 0 \\ -\frac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & 1\end{pmatrix},$$

$$\boldsymbol{P}_2 := \begin{pmatrix} 0 & 0 \\ -\frac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & \frac{3s^2+s-1}{2\,(s+2)}\end{pmatrix}, \quad \boldsymbol{P}_3 := \begin{pmatrix} 0 & 0 \\ -\frac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & 0\end{pmatrix}.$$

The refinement mask $\boldsymbol{P}(u)$ can be factorized

$$\boldsymbol{P}(u) = \frac{1}{4}\,\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\,\boldsymbol{C}_{\boldsymbol{x}_1}(2u)\,\boldsymbol{P}^{(2)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_1}(u)^{-1}\,\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1}, \tag{8.6}$$

where $\boldsymbol{C}_{\boldsymbol{x}_0}$, $\boldsymbol{C}_{\boldsymbol{x}_1}$ are defined by $\boldsymbol{x}_0 := (-\frac{3(s^2-1)}{s+2},\, 1)^T$ and $\boldsymbol{x}_1 := (1,\, 1)^T$ via (2.2) and

$$\boldsymbol{P}^{(2)}(u) := \frac{1}{2}\begin{pmatrix} 2 & 0 \\ \frac{(s^2-3s-1)z^2+(-10s^2-8s+6)z+(s^2-3s-1)}{(s+2)} & 4s(1+z) \end{pmatrix} \quad (z := e^{-iu}).$$

Since $\boldsymbol{P}^{(2)}(0)$ possesses the spectrum $\{1,\, 4s\}$, by Lemma 4.3 the refinement mask $\boldsymbol{P}(0)$ has the spectrum $\{1,\, s\}$. Observe that $(1,\, \frac{(s-1)^2}{s+2})^T$ is a right eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1 so that Theorem 3.2 yields that for $|s| < 2$, the infinite product

$$\hat{\boldsymbol{\phi}}(u) := \lim_{n\to\infty} \prod_{l=1}^{n} \boldsymbol{P}\left(\frac{u}{2^l}\right) \begin{pmatrix} 1 \\ \frac{(s-1)^2}{s+2} \end{pmatrix} \tag{8.7}$$

converges pointwise for $u \in \mathbb{R}$. By simple computations, we find that $\boldsymbol{P}$ does not satisfy (3.1) with $\alpha > 1$, such that convergence of the infinite product can not be shown for $|s| > 2$. In order to apply Theorem 4.1, we even need that $|4s| < 2$ and hence $|s| < 1/2$. As before, Corollary 4.2 will not provide an improvement of the results, since $\boldsymbol{P}^{(2)}$ does not satisfy (4.7) with $\alpha > 1$.
We apply Theorem 4.1 and obtain for $k \in \mathbb{N}$ the estimate

$$\|\hat{\boldsymbol{\phi}}(u)\| \leq C\,(1 + |u|)^{-2+\gamma_k}$$

with

$$\gamma_k := \frac{1}{k}\log_2 \sup_u \|\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right) \ldots \boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|.$$

We show

Lemma 8.2 *For a fixed s with $|s| < 1/2$, there is a number $k \in \mathbb{N}$ such that $\gamma_k < 1$.*

Proof: 1. Since $\boldsymbol{P}^{(2)}(u)$ is of the form

$$\boldsymbol{P}^{(2)}(u) = \begin{pmatrix} 1 & 0 \\ a(u) & b(u) \end{pmatrix}$$

with

$$\begin{aligned} a(u) &:= \frac{(s^2-3s-1)e^{-2iu} + (-10s^2-8s+6)e^{-iu} + (s^2-3s-1)}{2\,(s+2)}, \\ b(u) &:= 2s\,(1+e^{-iu}), \end{aligned}$$

we obtain

$$\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\,\boldsymbol{P}^{(2)}\left(\frac{u}{4}\right) = \begin{pmatrix} 1 & 0 \\ a(u/2) + a(u/4)b(u/2) & b(u/2)b(u/4) \end{pmatrix}.$$

By induction it follows that

$$\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)=\begin{pmatrix} 1 & 0 \\ \sum_{l=1}^{k} a\left(\frac{u}{2^l}\right)\prod_{n=1}^{l-1} b\left(\frac{u}{2^n}\right) & \prod_{l=1}^{k} b\left(\frac{u}{2^l}\right)\end{pmatrix}.$$

2. Note that the spectral norm of a matrix $\boldsymbol{V}$ can be estimated by the Frobenius norm of $\boldsymbol{V} := (v_{ij})_{i,j=1}^n$, i.e.,

$$\|\boldsymbol{V}\|_2 \le \left(\sum_{i,j=1}^{n} |v_{ij}|^2\right)^{1/2} := \|\boldsymbol{V}\|_F.$$

We will show that, for any fixed s with $|s|<1/2$ there is a $k\in\mathbb{N}$ such that

$$\frac{1}{k}\log_2 \sup_u \|\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|_F < 1.$$

First, observe that $b_k := \sup_u |\prod_{l=1}^{k} b(2^{-l}u)| \le |4s|^k$. Further, for

$$a_k := \sup_u |\sum_{l=0}^{k} a\left(\frac{u}{2^l}\right)\prod_{n=1}^{l-1} b\left(\frac{u}{2^n}\right)|$$

we obtain the estimate

$$\begin{aligned} |a_k| &\le \sum_{l=1}^{k}\left|\left((s^2-3s-1)(1+e^{-2iu/2^l})+(-10s^2-8s+6)e^{-iu/2^l}\right)\right|\frac{|4s|^{l-1}}{2(s+2)} \\ &\le \frac{1}{2(s+2)}\sum_{l=1}^{k}\left(2|s^2-3s-1|+|-10s^2-8s+6|\right)|4s|^{l-1}. \end{aligned}$$

Since $|s|<1/2$, we have $2|s^2-3s-1|<9/2$ and $|-10s^2-8s+6|<38/5$. Thus,

$$|a_k| \le \frac{1}{2(s+2)}\,\frac{121}{10}\sum_{l=1}^{k}|4s|^{l-1} \le \frac{121}{30}\sum_{l=1}^{k}|4s|^{l-1}.$$

For the Frobenius norm of $\boldsymbol{P}^{(2)}(u/2)\ldots\boldsymbol{P}^{(2)}(u/2^k)$ we find

$$\begin{aligned} \sup_u \|\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|_F &\le (1+|a_k|^2+|b_k|^2)^{1/2} \\ &\le \left(1+\left(\frac{121}{30}\sum_{l=1}^{k}|4s|^{l-1}\right)^2+|4s|^{2k}\right)^{1/2}. \end{aligned}$$

3. First, we consider the case that s is fixed with $0\le|s|<1/4$. Then,

$$\sup_u \|\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\ldots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|_F \le \left(2+\left(\frac{121}{30}\,k\right)^2\right)^{1/2}.$$

Now, choosing k such that $k > \frac{1}{2}\log_2\left(2 + \left(\frac{121k}{30}\right)^2\right)$ (this is satisfied for $k \geq 5$) we obtain that

$$\frac{1}{k}\log_2 \sup_u |\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|_F \leq \frac{1}{k}\log_2\left(2 + \left(\frac{121k}{30}\right)^2\right)^{1/2} < 1.$$

4. Now, we deal with the case that s is fixed with $1/4 \leq |s| < 1/2$. Then $|a_k| \leq \frac{121}{30}\, k\, |4s|^{k-1}$, such that

$$\sup_u \|\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|_F \leq \left(1 + \left(\frac{121}{30}k\right)^2 |4s|^{2k-2} + |4s|^{2k}\right)^{1/2}.$$

Choosing k such that $\log_2|4s| < 1 + \frac{1-\log_2((121k/30)^2+4)}{2k-2}$ (this is e.g. satisfied if $\log_2|4s| < 1 - \frac{\log_2 k}{4k-4}$), then it follows

$$(2k-2)\log_2|4s| + \log_2\left(\left(\frac{121}{30}k\right)^2 + |4s|^2\right) < 2k-1$$

and hence

$$1 + \left(\frac{121}{30}k\right)^2 |4s|^{2k-2} + |4s|^{2k} < 2^{2k}.$$

For the Frobenius norm it follows that

$$\begin{aligned}&\frac{1}{k}\log_2 \sup_u \|\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\dots\boldsymbol{P}^{(2)}\left(\frac{u}{2^k}\right)\|_F \\ \leq\;& \frac{1}{2k}\log_2\left(1 + \left(\frac{121}{30}k\right)^2 |4s|^{2k-2} + |4s|^{2k}\right) < 1. \quad \blacksquare\end{aligned}$$

Since for $|s| < 1/2$ there is a $k \in \mathbb{N}$ such that $\gamma_k < 1$, it follows that the elements of the solution $\boldsymbol{\phi}$ of the the vector refinement equation (1.1) with the refinement mask $\boldsymbol{P}$ defined in (8.5) are continuous. The uniqueness of the solution is ensured by Theorem 6.1. Further, by the analysis in section 7, we have uniform and L^2–convergence of the associated cascade algorithm.

Remarks.

1. The continuity of ϕ_0 and ϕ_1 for $|s| < 1/2$ is also proved in DGHM (1994) by means of fractal interpolation. In their paper it is already shown that ϕ_0, ϕ_1 are Lipschitz continuous for $|s| < 1/2$, i.e., there exits an $M < \infty$ such that for all $x, y \in [0,1]$ we have $|\phi_\nu(x) - \phi_\nu(y)| \leq M\,|x-y|$ $(\nu = 0,1)$. Further, if $1/2 < |s| < 1$, then ϕ_0, ϕ_1 have the Hölder exponent $\alpha = -\log|s|/\log 2$.
2. The solutions ϕ_0 and ϕ_1 are symmetric, and they have a very short support. In particular, supp $\phi_0 = [0,1]$, supp $\phi_1 = [0,2]$, and we have

$$\phi_0 = \phi_0(\cdot - 1/2), \qquad \phi_1 = \phi_1(\cdot - 1).$$

The closed linear span V_0 of the integer translates of ϕ_0 and ϕ_1 provides the approximation order 2. Using the results in section 2, this fact is a simple consequence of the factorization (8.6) of the refinement mask $\boldsymbol{P}$. Note that in DGHM (1994) it is proved that the hat function $\Delta(x) := \max\{0,\, 1-|x|\}$ is contained in V_0, which already implies that V_0 has approximation order 2.
3. In the case $s=-0.2$, it is shown in DGHM (1994) that the integer translates of ϕ_0 and ϕ_1 form an orthogonal basis of V_0.

8.3. Scaling functions with controlled smoothness

We consider solutions of the vector refinement equation (1.1) with the refinement mask

$$\begin{aligned}\boldsymbol{P}(u) &:= \frac{1}{4}\begin{pmatrix} e^{-iu}+2e^{-2iu}+e^{-3iu} & (e^{-iu}-2e^{-3iu}+e^{-5iu})/2 \\ 1/32 & e^{-iu}\end{pmatrix} \\ &= \frac{1}{4}\boldsymbol{C}_{\boldsymbol{x}_0}(2u)\,\boldsymbol{C}_{\boldsymbol{x}_1}(2u)\,\boldsymbol{P}^{(2)}(u)\,\boldsymbol{C}_{\boldsymbol{x}_1}(u)^{-1}\,\boldsymbol{C}_{\boldsymbol{x}_0}(u)^{-1},\end{aligned}$$

where

$$\boldsymbol{C}_{\boldsymbol{x}_0}(u)=\boldsymbol{C}_{\boldsymbol{x}_1}(u) := \begin{pmatrix} 1-e^{-iu} & 0 \\ 0 & 1\end{pmatrix} \qquad \boldsymbol{P}^{(2)}(u) := e^{-iu}\begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{\sin(u/2)^2}{8} & 1\end{pmatrix}.$$

Observe that $\boldsymbol{P}^{(2)}(0)$ and $\boldsymbol{P}(0)$ possess the spectra $\{1,\,1\}$ and $\{1, 1/4\}$, respectively. Hence, by Theorem 3.2 the infinite product

$$\boldsymbol{\phi}(u) := \lim_{n\to\infty}\prod_{l=1}^{n}\boldsymbol{P}\left(\frac{u}{2^l}\right)\begin{pmatrix}1\\ 1/96\end{pmatrix}$$

converges pointwise, since $(1, 1/96)$ is a right eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. Further, considering the product

$$\boldsymbol{P}^{(2)}(u)\,\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right) = \begin{pmatrix} e^{-3iu/2}+\frac{e^{-iu}(1-e^{-iu/2})^2}{64} & e^{-3iu/2} \\ \frac{e^{-iu/2}(1-e^{-iu})^2}{32}+\frac{e^{-iu}\,(1-e^{-iu})^2}{32} & \frac{e^{-iu/2}(1-e^{-iu})^2}{64}+e^{-3iu/2}\end{pmatrix},$$

we find for the matrix 1–norm and the matrix ∞–norm (defined in (8.4))

$$\sup_u \|\boldsymbol{P}^{(2)}(u)\,\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\|_1 \le \frac{33}{16}, \qquad \sup_u \|\boldsymbol{P}^{(2)}(u)\,\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\|_\infty \le \frac{33}{16}.$$

Hence, for the spectral norm it follows that

$$\begin{aligned}&\sup_u \|\boldsymbol{P}^{(2)}(u)\,\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\|_2 \\ \le\; &\left(\sup_u \|\boldsymbol{P}^{(2)}(u)\,\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\|_1 \sup_u \|\boldsymbol{P}^{(2)}(u)\,\boldsymbol{P}^{(2)}\left(\frac{u}{2}\right)\|_\infty\right)^{1/2} \le \frac{33}{16}.\end{aligned}$$

Now, applying Theorem 4.1 with

$$\gamma_2 = \frac{1}{2} \log_2 \sup_u \| \boldsymbol{P}^{(2)}(u)\, \boldsymbol{P}^{(2)}\left(\frac{u}{2}\right) \|_2 \leq \frac{1}{2} \log_2 \left(\frac{33}{16}\right) \approx 0.52219706$$

we have that

$$\|\hat{\boldsymbol{\phi}}(u)\| \leq C\,(1 + |u|)^{-2+\gamma_2},$$

i.e., the elements ϕ_0, ϕ_1 of the solution $\boldsymbol{\phi}$ are continuous functions. For the support of ϕ_0 and ϕ_1 we obtain

$$\text{supp } \phi_0 = [\frac{2}{3}, \frac{10}{3}], \qquad \text{supp } \phi_1 = [\frac{1}{3}, \frac{5}{3}].$$

Furthermore, we have the symmetry relations

$$\phi_0(2 + x) = \phi_0(2 - x), \qquad \phi_1(1 + x) = \phi_1(1 - x).$$

It can be shown that the integer translates of ϕ_0 and ϕ_1 form a Riesz basis of their closed linear span V_0. Then factorization of the refinement mask already implies that V_0 provides approximation order 2 (cf. section 2). In particular, the equalities (2.7) are satisfied with $\boldsymbol{y}_0 := (1,\, 0)^T$ and $\boldsymbol{y}_1 := (2,\, 0)^T$. Actually, the approximation order 2 is already provided by the closed linear span of the integer translates of ϕ_0. We have indeed

$$\begin{aligned} \boldsymbol{P}(0) &= \begin{pmatrix} 1 & 0 \\ \frac{1}{128} & \frac{1}{4} \end{pmatrix}, \qquad \boldsymbol{P}(\pi) = \begin{pmatrix} 0 & 0 \\ \frac{1}{128} & -\frac{1}{4} \end{pmatrix}, \\ (\mathrm{D}\boldsymbol{P})(0) &= \begin{pmatrix} -2i & 0 \\ 0 & -\frac{i}{4} \end{pmatrix}, \qquad (\mathrm{D}\boldsymbol{P})(\pi) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{4} \end{pmatrix}, \end{aligned}$$

so that for odd l:

$$\hat{\boldsymbol{\phi}}(2\pi l) = \boldsymbol{P}(\pi)\, \hat{\boldsymbol{\phi}}(\pi l)$$

and hence $\hat{\phi}_0(2\pi l) = 0$. For even l we have

$$\hat{\boldsymbol{\phi}}(2\pi l) = \boldsymbol{P}(0)\, \hat{\boldsymbol{\phi}}(\pi l),$$

i.e., $\hat{\phi}_0(2\pi l) = \hat{\phi}_0(\pi l)$. Thus, $\hat{\phi}_0(2\pi l) = 0$ for $l \in \mathbb{Z} \setminus \{0\}$. Analogously, for the derivative of $\hat{\boldsymbol{\phi}}$ it follows, for odd l,

$$(\mathrm{D}\hat{\boldsymbol{\phi}})(2\pi l) = \frac{1}{2} \left((\mathrm{D}\boldsymbol{P})(\pi)\, \hat{\boldsymbol{\phi}}(\pi l) + \boldsymbol{P}(\pi)\, (\mathrm{D}\hat{\boldsymbol{\phi}})(\pi l) \right)$$

so that $(\mathrm{D}\hat{\phi}_0)(2\pi l) = 0$. For even l we have

$$(\mathrm{D}\hat{\boldsymbol{\phi}})(2\pi l) = \frac{1}{2} \left((\mathrm{D}\boldsymbol{P})(0)\, \hat{\boldsymbol{\phi}}(\pi l) + \boldsymbol{P}(0)\, (\mathrm{D}\hat{\boldsymbol{\phi}})(\pi l) \right),$$

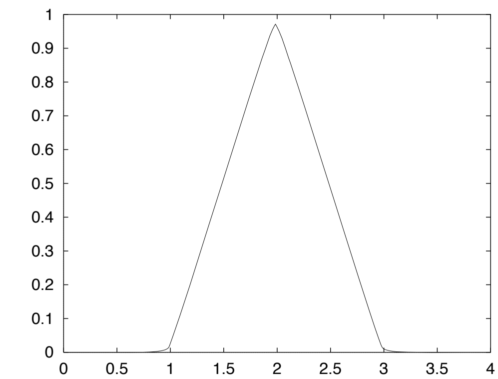

Figure 1: Graph of ϕ_0

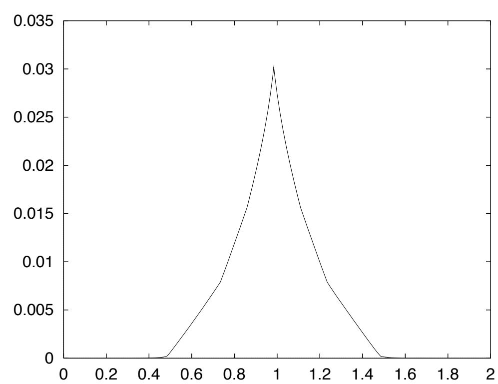

Figure 2: Graph of ϕ_1

i.e., $(\mathrm{D}\hat{\phi}_0)(2\pi l) = \frac{1}{2}(\mathrm{D}\hat{\phi}_0)(\pi l)$. Thus, $\hat{\phi}_0$ satisfies the Strang–Fix conditions of order 2,

$$\begin{aligned} \hat{\phi}_0(2\pi l) &= 0 \quad (l \in \mathbb{Z} \setminus \{0\}), \quad \hat{\phi}_0 \neq 0, \\ (\mathrm{D}\hat{\phi})(2\pi l) &= 0 \quad (l \in \mathbb{Z}). \end{aligned}$$

Finally, we note that, in this example, there is no function f in the space V_0 which is already refinable by itself. In the most other examples considered in the literature, even if the elements of $\boldsymbol{\phi}$ are not refinable by themselves, there exist refinable functions in the span of their integer translates; in the spline example (see section 8.1) the space V_0, spanned by the B–splines of order m with r–fold knots, contains the cardinal B–splines N_{m-k} $(k = 0, \ldots, r-1)$, in the case of the DGHM–scaling functions, V_0 contains the hat function.

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