Construction Techniques for Highly Accurate Quasi-Interpolation Operators

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Abstract: Under mild additional assumptions this paper constructs quasi -interpolants in the form

$$f_h(x) = \sum_{j=-\infty}^{+\infty} f(hj)\varphi_h\left(\frac{x}{h} - j\right), \quad x \in \mathbb{R}, \ h > 0 \qquad (0.1)$$

with approximation order $\ell - 1$, where $\varphi_h(x)$ is a linear combination of translates $\psi(x - jh)$ of a function ψ in $C^\ell(\mathbb{R})$. Thus the order of convergence of such operators can be pushed up to a limit that only depends on the smoothness of the function ψ. This approach can be generalized to the multivariate setting by using discrete convolutions with tensor products of odd-degree B–splines.

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List of Symbols

ψ	3,5		$C^{\ell+1}$	3,5
$f_h(x)$	3,5		φ_h	3,5
$\hat{\psi}$	5		φ	5
$Qf_h(x)$	5		$C_k(\psi)$	6
$I_{k,h}(x,t)$	6		$c_{k,h,j}(t)$	6
$B_k(f)$	7		ℓ_0	7
ℓ_1	7		ℓ_2	7
$D_{k,m}(\psi)$	7		$T(e^{it})$	10
χ	10		$\chi_{[-1/2,1/2]}$	10
B_{m-1}	10		φ_m	10
$\varphi_{\ell,h}^{(k)}$	12		φ_ℓ	12
$\varphi_\ell^{(k)}$	12		$Q_h^{(k)}f$	12
$\tilde{Q}_h^{(k)}f$	12			

1 Introduction

If the Fourier transform $\hat{\psi}$ of a function $\psi : I\!R \to I\!R$ with certain additional properties has zeros of order ℓ at $2\pi j \neq 0 \quad j \in Z\!\!Z$, then Strang -Fix theory implies that there is a linear combination φ of the translates $\psi(x-j)$ of ψ such that the quasi-interpolation

$$Qf_h(x) = \sum_{j \in Z} f(jh)\varphi(\frac{x}{h} - j) \tag{1.1}$$

is convergent and has approximation order ℓ with respect to $h \to 0$. This paper treats two problems :

1. Is the special quasi- interpolation

   $$\sum_{j \in Z} f(jh)\psi(\frac{x}{h} - j) \tag{1.2}$$

   defined by ψ itself convergent and what is its order of approximation? We will investigate the influence of the behaviour of the Fourier transform $\hat{\psi}$ of ψ near zero to the approximation order of (1.2).

2. The classical Strang-Fix condition [10] is a necessary and sufficient condition for a quasi- interpolant of the above form (1.1) to have a certain approximation order. But can we construct other quasi- interpolants from ψ with higher approximation order? How far can the approximation order of variations of (1.2) be increased for a fixed given ψ?

In the second situation, we shall leave the classical "stationary" setup for quasi–interpolants of the form (1.1). This idea is not new. It was indicated in [9] by Dyn and Ron. See also [6] and [1], where similar estimates to those of our Section 2 were established. Usually, the sampling distance h of the data $f(jh)$ is identical to both the shift distance δ and the scaling parameter σ for the basic function ψ in the sense that the quasi–interpolant is in the span of translates $\psi\left(\frac{\cdot - j\delta}{\sigma}\right)$. In contrast to this, we finally use shift $\delta = h^2$ and scale h with the sampling distance h, and we are interested in explicit constructions of quasi–interpolants

$$Qf_h(x) = \sum_{j \in Z\!\!Z} f(jh)\phi_h\left(\frac{x}{h} - j\right), \tag{1.3}$$

where φ_h is a linear combination of translates $\psi(\cdot - kh)$, such that we altogether work in the span of functions

$$\psi\left(\frac{\cdot - jh - kh^2}{h}\right), \qquad j, k \in Z\!\!Z$$

to recover data $f(jh), \ j \in Z\!\!Z$.

In the sense of the literature on principal shift–invariant spaces ([5], [6], [14]) we thus work in the scale of spaces $\{S_{h^2}(\psi(\cdot/h))\}_h$, but our special quasi–interpolation operators do not attain the optimal approximation orders (in terms of distances) that are possible in this scale. Those orders are independent of special operators, but we want to stick to simple operators like (1.3) based on sampling at distance h. Thus we do not study the general approximation

orders in these spaces, since we confine ourselves to the special quasi–interpolants (1.3), where φ_h is based on "other" shifts and dilates of ψ. Nor is it relevant at the outset that we finally end up with shifts $\delta = h^2$: our main ingredient is the data sampling at distance h, and this is why we keep h as an index to the quasi–interpolants (1.3). Our results imply

that the attainable order of convergence for quasi–interpolants (1.3) mainly depends on the smoothness of the function ψ, and we provide an explicit construction that makes full use of this fact. Applications include the construction of quasi–interpolants based on data $f(jh)$ that achieve arbitrarily high approximation orders, if ψ is chosen to be a multiquadric or a Gaussian.

2 Convergence Orders of Quasi–Interpolants

We first consider quasi–interpolants of the special form

$$f_h(x) = \sum_{j=-\infty}^{+\infty} f(hj)\psi\left(\frac{x}{h} - j\right), \quad x \in I\!\!R, \ h > 0 \tag{2.1}$$

for functions $f \in C(I\!\!R)$ that are inverse Fourier transforms

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t)e^{ixt}dt, \quad x \in I\!\!R \tag{2.2}$$

of functions $\hat{f} \in L_1(I\!\!R)$. The quasi–interpolant (2.1) is supposed to use a basic function $\psi \in C^\ell(I\!\!R)$ with

$$C_k(\psi) := \max_{0 \leq \xi \leq 1} \sum_{j=-\infty}^{+\infty} |\psi^{(k)}(\xi - j)| < \infty \tag{2.3}$$

for $0 \leq k \leq \ell$.

In the terminology of shift–invariant spaces this is a specific form of approximation in a scale $\{S_h\}$ of principal shift–invariant subspaces

$$S_h := \{g \in S : g = f_h \quad \text{for } f \in S\}$$

of the shift–invariant space

$$S := \{f \in C(I\!\!R), \ \hat{f} \in L_1(I\!\!R), \ (2.2)\}.$$

Note that we do not attempt to characterize S_h in an intrinsic way, e.g. by a summability condition on coefficients in (2.1). For our purposes it suffices to provide L_∞ bounds for the error of quasi–interpolation of $f \in S$ by $f_h \in S_h$.

We start with an integral representation for $f_h^{(k)}$ that is based on Fourier transform techniques as used in [1] and [6] for instance, and we skip over the proof, which can be reduced to an application of Poisson's summation formula to the function whose Fourier transform is $\psi^{(k)}(x + \cdot)$. If $f \in C(I\!\!R)$ satisfies (2.2) with $\hat{f} \in L_1(I\!\!R)$ and if $\psi \in C^\ell(I\!\!R)$ is a function with (2.3) for $0 \leq k \leq \ell$, then the quasi–interpolant (2.1) exists and has derivatives up to order ℓ that can be expressed by

$$f_h^{(k)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t)e^{ixt}I_{k,h}(x,t)dt \tag{2.4}$$

with a continuous h–periodic function

$$I_{k,h}(x,t) := h^{-k} \sum_{j=-\infty}^{+\infty} e^{-i(x-hj)t} \psi^{(k)} \left(\frac{x}{h} - j \right) \tag{2.5}$$

that has a Fourier series representation

$$I_{k,h}(x,t) = \sum_{j=-\infty}^{+\infty} c_{k,h,j}(t) e^{2\pi i j x / h}$$

with coefficients

$$c_{k,h,j}(t) = \left(\frac{i}{h} \right)^k (2\pi j + ht)^k \hat{\psi}(2\pi j + ht). \tag{2.6}$$

Now we assume f to satisfy

$$B_k(f) := \int_{-\infty}^{+\infty} |\hat{f}(t) t^k| dt < \infty \tag{2.7}$$

for $0 \le k \le L$. Then

$$f^{(k)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t)(it)^k e^{ixt} dt$$

for $0 \le k \le L$ and

$$f_h^{(k)}(x) - f^{(k)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t) e^{ixt} (I_{k,h}(x,t) - (it)^k) dt \tag{2.8}$$

for $0 \le k \le \min(\ell, L)$. A similar error representation was used in [17] and [20] to prove error bounds for radial basis function interpolation. Note that this means that we work in a subspace

$$S^k := \{ f \in C^k(I\!R), \ \hat{f} \in L_1(I\!R) \ \text{and} \ (2.7) \}$$

measuring smoothness of functions in S.

To get a bound for this representation of the error, we require additional assumptions on ψ. Let the Fourier transform $\hat{\psi}$ of ψ have zeros of order ℓ_1 in $2\pi j$ for all $j \ne 0$. Furthermore, let $\hat{\psi} - 1$ have a zero of order ℓ_0 in 0.

More specifically we require $\hat{\psi} \in C^{\ell_2}(I\!R)$, $\ell_2 \ge \max(\ell_0, \ell_1)$ and use Taylor's formula to write

$$\hat{\psi}(2\pi j + t) = \hat{\psi}^{(\ell_1)}(2\pi j + t\tau_j(t)) t^{\ell_1} / \ell_1!, \tag{2.9}$$

$$\hat{\psi}(t) - 1 = \hat{\psi}^{(\ell_0)}(t\tau_0(t)) t^{\ell_0} / \ell_0! \tag{2.10}$$

with $\tau_m(t) \in [0,1]$ for all $m \in Z\!\!Z$ and all $t \in I\!R$. Finally, we define

$$D_{k,m}(\psi) := \max_{\substack{|\xi| \le 1 \\ |\eta| \le 1}} \sum_{j \ne 0} |2\pi j + \xi|^k |\hat{\psi}^{(m)}(2\pi j + \eta)| \tag{2.11}$$

for $0 \le m \le \ell_2$ and assume

$$D_{k,m}(\psi) < \infty, \qquad 0 \le k \le K$$

for some nonnegative integer K.

Theorem 2.1 *Assume the quasi–interpolant (2.1) to be generated by a function $\psi \in C^\ell(I\!R)$ with a Fourier transform $\hat{\psi}$ satisfying (2.9), (2.10), and (2.11). If the quasi–interpolant is evaluated for functions $f \in C(I\!R)$ with $B_k(f) < \infty$, $0 \le k \le L$, then*

$$\|f_h^{(k)} - f^{(k)}\|_\infty \le C h^d$$

for $0 \le k \le \min(\ell, L - \ell_0, K, \ell_1)$,

$$d = \min(\ell_0, \ell_1 - k),$$

and all $h > 0$. The constant C is the maximum of

$$\frac{1}{2\pi} \left(B_L(f)(1 + C_k(\psi)) + \frac{B_{k+\ell_0}(f)}{\ell_0!} \|\psi^{(\ell_0)}\|_{\infty,[-1,+1]} + \frac{B_{\ell_1}(f) D_{k,\ell_1}(\psi)}{\ell_1!} \right)$$

for $0 \le k \le \min(\ell, L - \ell_0, K, \ell_1)$.

Proof: We first consider the integrand (2.8) on $|t| > h^{-1}$. If we use (2.3) to get a uniform bound

$$|I_{k,h}(x,t)| \le C_k(\psi) h^{-k}, \quad 0 \le k \le \ell$$

for all $x, t \in I\!R$ and $h > 0$, then we have

$$\frac{1}{2\pi} \int_{|t| \ge 1/h} |\hat{f}(t) e^{ixt}(I_{k,h}(x,t) - (it)^k)| dt$$

$$\le \frac{1}{2\pi} \int_{|t| \ge 1/h} |\hat{f}(t)|(C_k(\psi) h^{-k} + |t|^k) dt$$

$$\le \frac{1}{2\pi} \int_{|t| \ge 1/h} |\hat{f}(t)|(C_k(\psi) + 1)|t|^k dt$$

$$\le \frac{1}{2\pi} (C_k(\psi) + 1) \int_{|t| \ge 1/h} |\hat{f}(t)| \cdot |t|^k |ht|^{L-k} dt$$

$$\le \frac{1}{2\pi} (C_k(\psi) + 1) h^{L-k} B_L(f)$$

for $0 \le k \le \min(\ell, L)$ and all $h > 0$.

For $|t| < 1/h$ we split the integrand via

$$I_{k,h}(x,t) - (it)^k = c_{k,h,0}(t) - (it)^k + \sum_{j \ne 0} c_{k,h,j}(t) e^{2\pi ijx/h}.$$

The first part is bounded by

$$\frac{1}{2\pi} \int_{|t| < 1/h} |\hat{f}(t)| \, |c_{k,h,0}(t) - (it)^k| dt$$

$$= \frac{1}{2\pi} \int_{|t| < 1/h} |\hat{f}(t)| \, |t|^k |\hat{\psi}(ht) - 1| dt$$

$$\le \frac{h^{\ell_0}}{2\pi \ell_0!} \int_{|t| < 1/h} |\hat{f}(t)| \, |t|^{k+\ell_0} |\hat{\psi}^{(\ell_0)}(ht\tau_0(t))| dt$$

$$\le \frac{h^{\ell_0}}{2\pi \ell_0!} B_{k+\ell_0}(f) \cdot \max_{|\xi| \le 1} |\hat{\psi}^{(\ell_0)}(\xi)|$$

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for $k + \ell_0 \leq L$. The second part is

$$
\begin{aligned}
& \frac{1}{2\pi} \int_{|t|<1/h} |\hat{f}(t)| \, |\sum_{j\neq 0} c_{k,h,j}(t)| dt \\
=\ & \frac{h^{-k}}{2\pi} \int_{|t|<1/h} |\hat{f}(t)| \sum_{j\neq 0} |2\pi j + ht|^k |\hat{\psi}(2\pi j + ht)| dt \\
=\ & \frac{h^{-k+\ell_1}}{2\pi \ell_1!} \int_{|t|<1/h} |\hat{f}(t)| \, |t|^{\ell_1} \sum_{j\neq 0} |2\pi j + ht|^k |\hat{\psi}^{(\ell_1)}(2\pi j + ht\tau_j(ht))| dt \\
\leq\ & \frac{h^{\ell_1-k}}{2\pi \ell_1!} B_{\ell_1}(f) D_{k,\ell_1}(\psi).
\end{aligned}
$$
(2.12)

Now everything combines into the assertion of the theorem. □

Remarks: The principal consequence of Theorem 2.1 is that the *rate* of convergence of derivatives is mainly tied to ℓ_0, while the admissible *orders* of derivatives are mainly controlled by ℓ_1. Thus, a small value of ℓ_0 together with a large value of ℓ_1 will provide slow convergence of high–order derivatives, and vice versa. The next section will provide techniques for modification of given quasi–interpolants in order to yield higher values of ℓ_0 and ℓ_1.

3   Increase of Approximation Order

We first consider simple techniques to improve the approximation order of a quasi–interpolant by suitable modifications that imply an increase of ℓ_0. This has been done already by [3], [4], [8], [16] and possibly others, but we shall include the explicit construction recipe for completeness and in order to be able to refer to it in the next section.

Let a quasi–interpolant (2.1) be generated by a given symmetric function ψ with $\hat{\psi} \in C^{\ell_2}(\mathbb{R})$, and let the assumptions of Theorem 2.1 be satisfied. Then $\ell_0 \leq \ell_2$ holds by definition, but there is a finite linear combination φ of translates of ψ such that φ satisfies Theorem 2.1 with ℓ_0 replaced by ℓ_2 and with an appropriately modified constant C. In other words: By taking fixed linear combinations of ψ one can always push ℓ_0 up to ℓ_2.

Indeed, by Taylor's formula we have

$$
\hat{\psi}(t) = q(t^2) + \mathcal{O}(t^{\ell_2}) \qquad \text{for } t \to 0
$$

with a polynomial q satisfying $q(0) = 1$. Then there is another polynomial p with $p(0) = 1$ and

$$
p(t^2)q(t^2) = 1 + \mathcal{O}(t^{\ell_2}) \qquad \text{for } t \to 0,
$$

which implies

$$
p(t^2)\hat{\psi}(t) = 1 + \mathcal{O}(t^{\ell_2}) \qquad \text{for } t \to 0.
$$

Now let the trigonometric polynomial

$$
T(e^{it}) = \sum_j d_j e^{ijt} = p(t^2) + \mathcal{O}(t^{\ell_2})
$$

be a sufficiently good approximation of p around the origin. Then one defines

$$
\varphi(x) = \sum_j d_j \psi(x + j)
$$

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to get

$$\hat{\varphi}(t) = \hat{\psi}(t) \cdot T(e^{it})$$

$$= 1 + \mathcal{O}(t^{\ell_2}) \qquad \text{for } t \to 0.$$

This makes Theorem 2.1 applicable for ℓ_0 replaced by ℓ_2, and the constants C_k and $D_{k,m}$ of the preceding section will take an additional factor $\sum_j |d_j|$.

4 Increase of the Order of Derivatives

We now want to modify a function ψ in such a way that ℓ_1 is increased. This will boost up the bound on the order of convergent derivatives, and it will be done via smoothing by convolution. We shall first consider convolution with B–splines, and later we shall employ discrete convolution to get a numerically accessible quasi–interpolant based on the introduction of additional shifts of ψ with spacing h^2. Our assumptions on ψ will be much weaker than those required by Theorem 2.1, but the quasi -interpolation constructed by function φ, resulting from our construction, will have the same error bounds as given by Theorem 2.1 with $\ell_1 = \ell$.

Theorem 4.1 *Let $\psi \in C^\ell$ with $\hat{\psi} \in L_1(\mathbb{R}) \cap C^\ell(\mathbb{R})$ be a basis function for quasi–interpolation, and assume*

$$C_0(\psi) = \max_{0 \le \xi \le 1} \sum_{j=-\infty}^{+\infty} |\psi(\xi - j)| < \infty, \ .$$

If B_ℓ is the ℓ–th order symmetric uniform B–spline, then the basis function $\varphi_\ell = \psi * B_{\ell-1}$ satisfies Theorem 2.1 with $\ell_1(\varphi_\ell) = \ell$. Thus φ_ℓ satisfies the Strang -Fix condition of order ℓ and

$$C_k(\varphi_\ell) < \infty, \quad 0 \le k \le \ell. \tag{4.1}$$

Proof: Let $\chi = \chi_{[-1/2,1/2]}$ be the characteristic function on $[-1/2, 1/2]$, and define the B–spline

$$B_{m-1} = \chi * \cdots * \chi$$

as the m–fold convolution of χ with itself. Now we define

$$\varphi_m := B_{m-1} * \psi = \chi * \varphi_{m-1}$$

for $1 \le m \le \ell$, $\varphi_0 := \psi$. Then

$$\hat{\varphi}_\ell(t) = \hat{B}_{\ell-1}(t) \cdot \hat{\psi}(t) = \hat{\psi}(t) \cdot \text{sinc}^\ell \left(\frac{t}{2} \right)$$

has the required behaviour at all points $2\pi j$, $j \in \mathbb{Z} \setminus \{0\}$. It remains to show (4.1), and this follows from

$$C_k(\varphi_\ell) = C_0(D^k \varphi_\ell) = C_0 \left(\psi * B_{\ell-1}^{(k)} \right)$$

$$\le C_0(\psi) \| B_{\ell-1}^{(k)} \|_1,$$

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where we interpret the norm in the last line as the total mass of the measure $B_{k-1}^{(k)}$ for $k = \ell$.
■

Now we have constructed a function φ_ℓ from the function ψ by convolution such that the function φ_ℓ satisfies the Strang - Fix condition of order ℓ and Theorem 2.1 with $\ell_1(\varphi) = \ell$. Unfortunately, the function φ_ℓ is defined by convolution and is not expressible via translates of the original function ψ. Thus we now take the discrete convolution instead of the usual convolution. If we replace the integration by a high-order quadrature formula for equidistant data, we can replace φ_ℓ by a function $\varphi_{\ell,h}$, depending on ℓ and h, which is a linear combination of translates $\psi(x - jh)$, such that

$$||\varphi_{\ell,h}^{(k)} - \varphi_\ell^{(k)}||_\infty = \mathcal{O}(h^{\ell-k}) \tag{4.2}$$

holds for $h \to 0$. If we define $Q_h f$ as the quasi- interpolation of (2.1) with $\psi := \varphi_{\ell,h}$ and $\tilde{Q}_h f$ with $\psi := \varphi_\ell$, then for $f \in L_1(I\!R)$ we find

$$
\begin{aligned}
&||Q_h^{(k)} f - \tilde{Q}_h^{(k)} f||_\infty \\[2mm]
&= \max_x \left| \sum_{j=-\infty}^\infty f(jh)(\varphi_{\ell,h}^{(k)}(\frac{x}{h} - j) - \varphi_\ell^{(k)}(\frac{x}{h} - j)) \right| \\[2mm]
&\leq \left(\sum_{j=-\infty}^\infty |f(jh)| \right) \max_y (|\varphi_{\ell,h}^{(k)}(y) - \varphi_\ell^{(k)}(y)|) \\[2mm]
&= \mathcal{O}(h^{\ell-k-1}).
\end{aligned}
\tag{4.3}
$$

We summarize the above discussion to our main theorem

Theorem 4.2 *Let ψ have the properties*

$$\psi \in C^\ell, \quad ||\psi^{(\ell)}||_\infty < \infty, \quad C_0(\psi) < \infty,$$

and let its Fourier transform $\hat{\psi}$ satisfy

$$\hat{\psi} \in L_1(I\!R), \quad \hat{\psi} \in C^\ell(I\!R)$$

with (2.11) for $0 \leq k, m \leq \ell$. Then there is a function $\varphi_{\ell,h}$ consisting of linear combinations of the translates $\psi(x - jh)$, such that the quasi interpolation (2.1) with $\psi := \varphi_{\ell,h}$ has the error estimate

$$||f_h^{(k)}(x) - f^{(k)}(x)||_\infty = \mathcal{O}(h^{\ell-k-1}), \ 0 \leq k \leq \ell - 1$$

for any function $f \in L_1$ with $B_{2\ell}(f) < \infty$.

Proof : By the construction of Section 3 there is a trigonometric polynomial

$$T(e^{it}) = \sum d_j e^{ijt},$$

and by Theorem 4.1 there is a B-spline of degree ℓ such that the function

$$\varphi_\ell = \sum d_j B^\ell * \psi(\cdot - j)$$

satisfies the complete Strang - Fix condition of order ℓ, in other words it satisfies the conditions of Theorem 2.1 with $\ell_0 = \ell_1 = \ell$. Using discrete convolution with a quadrature

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formula we get a function $\varphi_{\ell,h}$ such that the order of the approximation is estimated as in (4.3). Then Theorem 2.1 yields our assertion. ∎

Remark : Our construction can be summarized as follows: If $\psi \in C^\ell$ satisfies the mild additional conditions of Theorem 4.2, we can construct quasi -interpolants in the form

$$
\begin{aligned}
Qf_h(x) &= \sum_{j=-\infty}^{+\infty} f(hj)\varphi_{\ell,h}\left(\frac{x}{h} - j\right), \quad x \in I\!\!R, \ h > 0 \\
&= \sum_{j=-\infty}^{+\infty} f(hj) \cdot \sum_k c_k^{(\ell)}(h)\psi\left(\frac{x}{h} - j - kh\right)
\end{aligned}
\tag{4.4}
$$

with approximation order $\ell - 1$, where $\varphi_{\ell,h}(x)$ is a linear combination of translates $\psi(x - jh)$ of the function ψ but depends on h. Note that the data are still sampled at points with distance h, while the set of scaled translates of ψ now uses shifts h and h^2. In the terminology of approximation from shift–invariant spaces these functions live in a scale $\{S_{h^2}\}$ instead of $\{S_h\}$, and therefore a general optimal approximation in these spaces can and will have a better approximation order. However, our special linear quasi–interpolant uses function values of spacing h only. The increase of approximation order, as described in this section, is achieved by using more basis functions, not by using more data. The order of convergence of such operators can be pushed up to a limit that only depends on the smoothness of the function ψ. This approach can be generalized to the multivariate setting by using discrete convolutions with tensor products of odd-degree B -splines.

Example 1: For multiquadrics $\phi(x) = \sqrt{c^2 + x^2}$, we choose at first $\psi(x) = (\phi(x+1) - 2\phi(x) + \phi(x-1))$. Then $\ell_1(\psi) = 2$, and with the Strang-Fix condition we get the approximation order 2, which is similar to results in [2] [7] [19]. But references [11] [20] imply that the order of approximation by interpolation exceeds any ℓ, if the function f is very smooth and if c is fixed. Note here that the usual Strang- Fix theory scales c like $\mathcal{O}(h)$ for $h \to 0$. Our result bridges the gap between these two approaches. Since $\psi^{(k)}(x) = \mathcal{O}(x^{-(k+3)})$, we can construct for any ℓ a quasi-interpolant with approximation order $\ell - 1$ for functions $f \in L_1$ with $B_{2\ell}(f) < \infty$.

Example 2: (see also [1]). The Gaussian distribution $\psi(x) = e^{-a^2 x^2}$ is often criticized in the literature on radial basis functions, because it fails to satisfy the Strang-Fix conditions. But by [17] for some very smooth functions f the order of approximation can exceed any ℓ. Using Theorem 4.2 for any ℓ we can always construct a quasi-interpolant with approximation order $\ell - 1$ for $f \in L_1$ and $B_{2\ell}(f) < \infty$.

Example 3: The function $\psi(x) = (1 - x^2)_+^{\ell+1}$ satisfies no Strang-Fix condition, but from our approach the approximation order of the quasi- interpolation (4.4) is at least $\ell - 1$. We can get a similar result for B -splines, too.

References

[1] Beatson, R.K., & W.A. Light, *Quasi–interpolation in the obsence of polynomial repro-duction.* In: Numerical Methods of Approximation Theory, Vol. 9, Birkhäuser, D. Braess and L.L. Schumaker (eds.), 1992, 21–39.

[2] Beatson, R. K. & M. J. D. Powell, *Univariate multiquadric approximation: quasi inter-polation to scattered data*, DAMTP 1990/NA, University of Cambridge.

[3] Boor, C. de, *The polynomials in the linear space of integer translates of a compactly supported function.* Constructive Approximation 3 (1987), 199–208.

[4] Boor, C. de, *Quasi–interpolants and approximation power of multivariate splines.* In: Computation of Curves and Surfaces, Kluwer, Dordrecht, W. Dahmen, M. Gasca and C.A. Micchelli (eds.), 1990, 313–345.

[5] Boor, C. de, R.A. DeVore, & A. Ron, *Approximation from shift–invariant subspaces of $L_2(\mathbb{R}^d)$.* Trans. Amer. Math. Soc. 341 (1994), 787–806.

[6] Boor, C. de & A. Ron, *Fourier analysis of approximation orders from principal shift–invariant spaces.* Constructive Approximation 8 (1992), 427–462.

[7] Buhmann, M. D., *Multivariable interpolation using Radial Basis Functions.* Ph.D. dis-sertation, University of Cambridge 1989.

[8] Chui, C.K. & H. Diamond, *A natural formulation of quasi–Interpolation by multivariate splines*, Proc. Am. Math. Soc. 99 (1987), 643–646.

[9] Dyn, N. & A. Ron, *Local approximation by certain spaces of multivariate exponential polynomials, approximation order of exponential box splines and related interpolation problems*, Trans. Amer. Math. Soc. 319 (1990), 381–404.

[10] Light, W. A., *Some aspects of radial basis function approximation.* In: Approxima-tion Theory, Spline Functions and Applications, S.P.Singh (ed.), NATO ASI Series 356 (1992), 163-190.

[11] Madych, W. A. & S. A. Nelson, *Error Bounds for multiquadric Interpolation.* In: Ap-proximation Theory VI: vol 2, C. K. Chui, L. L. Schumaker and J.D. Ward (eds.) 1989, 413-416.

[12] Micchelli, C. A., *Interpolation of scattered data: distance matrix and conditionally pos-itive definite functions.* Constructive Approximation 2 (1986), 11-22.

[13] Powell, M. J. D., *Radial basis functions for multivariable interpolation: a review.* In: Numerical Analysis, D. F. Griffiths and G. A. Watson (eds.), Longman Scientific & Technical (Harlow), 1987, 223-241.

[14] Ron, A, *Approximation orders from principal shift–invariant spaces generated by a radial basis function.* In: Numerical Methods of Approximation Theory, Vol. 9, Birkhäuser, D. Braess and L.L. Schumaker (eds.), 1992, 245–268.

[15] Schaback, R., *Error estimates and condition numbers for radial basis function interpo-lation.* Advances in Computational Mathematics 3 (1995), 251–264.

[16] Strang, G., & G. Fix,, *A Fourier analysis of the finite element variational method*, C.I.M.E. II Ciclo 1971. In: Constructive Aspects of Functional Analysis, G. Geymonat (ed.), 1973, 793–840.

[17] Wu, Z., *Die Kriging Methode zur Lösung mehrdimensionaler Interpolationsprobleme*. Ph. D. dissertation, Universität Göttingen 1986.

[18] Wu, Z., *Hermite-Birkhoff interpolation of scattered data by radial basis function*. Approx. Theory & its Appl. 8:2 (1992), 1–10.

[19] Wu, Z., & R. Schaback, *Shape preserving properties and convergence of univariate multi-quadric quasi–interpolation*. Acta Mathematicae Applicatae Sinica 10:4 (1994), 441–446.

[20] Wu, Z. & R. Schaback, *Local error estimates for radial basis function interpolation of scattered data*, IMA Journal of Numerical Analysis 13 (1993), 13-27.