Convergence of Planar Curve Interpolation Schemes

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Abstract

This note provides convergence orders for a number of parametric interpolation schemes for planar curves. The methods use piecewise quadratic or cubic polynomials and are globally GC^1 or GC^2. The data are either of Lagrange or of Hermite type; convergence orders range between 4 and 6.

§1. Introduction

Let $f : [0, L] \to \mathbb{R}^2$ be a smooth planar curve, parametrized by arclength. We consider interpolation processes using data

$$f_i := f(t_i), \quad 0 \le i \le N, \tag{1}$$

$$f_i' := f'(t_i), \quad 0 \le i \le N, \tag{2}$$

$$\kappa_i := \kappa_f(t_i), \quad 0 \le i \le N \tag{3}$$

at unknown parameter values

$$0 = t_0 < t_1 < \ldots < t_N = L, \tag{4}$$

where κ_f denotes the curvature of f. We employ the notation Hj for values $j = 0, 1, 2$ to describe the situation of Hermite interpolation of order j; i.e., when equation $(i+1)$ is required to hold for $0 \le i \le j \le 2$.

The interpolants should be piecewise polynomials p of degree $k = 2$ or $k = 3$ having breakpoints at the data. The polynomial pieces are written in Bernstein–Bezier (BB) representation as $p_i(t)$, $1 \le i \le N$, $t \in [0, 1]$, between $f(t_{i-1})$ and $f(t_i)$. Continuity should be of class GC^l with $l = 1$ or $l = 2$. So the interpolation processes considered here are roughly described by the three numbers j, k, and l.

The error is measured either as in [1] or (equivalently) using the maximum deviation between f and p taken on lines perpendicular to the lines joining adjacent interpolation points $f(t_{i-1})$ and $f(t_i)$.

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We consider the limit $h \to 0, h := \max_{1 \le i \le N}(t_i - t_{i-1})$, and derive the order m of the error with respect to $h \to 0$. Table 1 summarizes a number of results for quick reference.

Hj j	GC^l l	∂p k	$\mathcal{O}(h^m)$ m	Remarks, references, and additional assumptions
2	2	3	6	if $\kappa_f \neq 0$, see [1]
2	2	3	4	if $\kappa_f \geq 0$, see [1]
1	1	2	4	if $\kappa_f \neq 0$, see 2.1
1	1	3	4	see 2.2
1	2	3	4	see 2.3
1	2	3	6/4 ^{2})	under $H2$–conditions^1), see 3^3)
0	2	2	4	if $\kappa_f \neq 0$, global scheme, see $[2]^3$)
0	1	2	4	if $\kappa_f \neq 0$, local scheme, see 2.4^3)
0	1	3	4	see 2.5^3)
0	2	3	4	see 2.6^3)

^{1}) under assumptions of the two cases of cubic $H2$–interpolation

^{2}) the two possible convergence orders of cubic $H2$–interpolation

^{3}) under assumption of a uniformly bounded mesh ratio, $0 < c \le h_{i-1}/h_i \le C < \infty,\ h_i := t_i - t_{i-1}$.

Table 1. Approximation orders.

§2. Fourth–order interpolation methods

The rest of this short note consists of comments to some of the table entries. The proof of the stated convergence order m always follows the technique of deBoor, Höllig, and Sabin in [1]. Therefore only some hints concerning the definition of the methods and certain variations in the standard convergence proof are necessary.

2.1 Quadratic GC^1 interpolation to $H1$ data

Just take the polynomial piece $p_i(t)$ as the quadratic polynomial in BB form with control points f_{i-1}, Z_i, f_i, where Z_i is the intersection of tangents T_{i-1} and T_i at f_{i-1} and f_i.

2.2 Cubic GC^1 interpolation to $H1$ data

Let $p_i(t)$ be the cubic polynomial in BB form defined by control points $f_{i-1}, Z^+_{i-1}, Z^-_i, f_i$, where the points Z^-_i and Z^+_i lie on the "left" and "right" of $f(t_i)$ on the tangent T_i at $f(t_i)$ at a distance of $\|f_{i-1} - f_i\|/3$ and $\|f_i - f_{i+1}\|/3$ to $f(t_i)$, respectively.

2.3 Cubic GC^2 interpolation to $H1$ data

First use the method of 2.2 and evaluate the curvature (or the mean over the jump of the curvature) of the GC^1 interpolant at the interpolation points. This gives $\mathcal{O}(h^2)$ estimates for κ_i that can be used to calculate a GC^2 solution to the corresponding $H2$ problem, and the standard technique of [1] gives an error of $\mathcal{O}(h^4)$.

2.4 Quadratic GC^1 interpolation to $H0$ data

The methods of [2] allow an expansion

$$\frac{\gamma(h_1,h_2)}{h_1-h_2} = \frac{1}{2}\theta_1 + \frac{1}{3}\theta_2(h_1+h_2) + \mathcal{O}(h_1^2) + O(h_2^2) \tag{5}$$

for the angle $\gamma(h_1,h_2)$ between the chords $f(t+h_1)-f(t)$ and $f(t+h_2)-f(t)$, where $\theta_1 = \kappa_f(t)$ and $\theta_2 = \kappa'_f(t)$. Using four successive values of f near t and substituting chord length $d_n := \operatorname{sgn} h_n \|f(t+h_n) - f(t)\|$ for the unknown arclength h_n between $f(t+h_n)$ and $f(t)$ yields two equations of type (5). Solving these will provide an $\mathcal{O}(h^2)$ estimate for $\theta_1 = \kappa_f(t)$ and an $\mathcal{O}(h)$ estimate for $\theta_2 = \kappa'_f(t)$, if the mesh ratio is uniformly bounded. Putting these into the expansion (see [2])

$$\alpha(h) = \frac{1}{2}\theta_1 h + \frac{1}{3}\theta_2 h^2 + \mathcal{O}(h^3) \tag{6}$$

for the angle $\alpha(h)$ between the tangent at $f(t)$ and the chord $f(t+h)-f(t)$ gives an $\mathcal{O}(h^3)$ approximation to $\alpha(h)$. The estimated $H1$ data values can be used to calculate a solution along the lines of 2.1. Substitution of arclength h by chordlength d is feasible as long as the asymptotic behavior $d(h) = h + \mathcal{O}(h^3)$ is sufficient for the required approximation order. Note that we regard curvature, arclength, chordlength, and angles as *signed* quantities.

2.5 Cubic GC^1 interpolation to $H0$ data

Proceed as in 2.4 to get third order accurate data for the $H1$ problem, and then apply 2.2.

2.6 Cubic GC^2 interpolation to $H0$ data

Use 2.5, 2.2, and 2.3 in a suitable way.

§3. A "bootstrapping" sixth–order method for $H1$ data

The methods of 2.2 and 2.3 gave fourth order accuracy by use of a second–order approximation of curvature values κ_i. To raise the overall error order to 6 in a parametrization–independent way, one has to provide fourth–order accurate estimates of κ_i. This can be done using an expansion

$$\theta(h) = \sum_{i=1}^{4} \theta_i h^i + \mathcal{O}(h^5) \tag{7}$$

as in [1], where $\theta(h)$ is the angle between tangents at $f(t)$ and $f(t+h)$. Furthermore, the curvature $\kappa_f(t+h)$ is $\theta'(t+h)$. Now, for five consecutive $H1$ data points around t, four angles of the form $\theta(h_i)$ are given and one is tempted to solve a simple polynomial interpolation problem for data $(h_i, \theta(h_i))$ to get estimates of order $\mathcal{O}(h^{5-i})$ for $\theta_i,\ i = 1, \ldots, 4$.

Unfortunately, the values h_i are unknown arclengths which can no more be substituted by chordlengths d_i, because the required accuracy is too high. Thus, we employ a "bootstrapping" technique to gradually raise the accuracy of estimates of h_i:

Step 1: Set $h_n^{(1)} := d_n,\ \theta_3 = \theta_4 = 0$ and use (7) to calculate approximations $\theta_i^{(1)}$ of order $\mathcal{O}(h^{3-i})$ for $i = 1, 2$ with the three "nearest " values of f to $f(t)$.

Step 2: Put the chordlengths $d = d_n,\ n = 1, 2, 3, 4$ and the resulting approximations $\theta_i^{(1)},\ i = 1, 2$ into the expansion

$$h(d) = d + d^3 \frac{\theta_1^2}{24} + d^4 \frac{\theta_1 \theta_2}{12} + \mathcal{O}(d^5) \tag{8}$$

of the arclength h between $f(t+h)$ and $f(t)$ as a function of the chordlength $d = \|f(t+h) - f(t)\|$. Such an expansion can easily be calculated by REDUCE, taking all nonlinearities into account. This gives new values $h_n^{(2)} = h_n + \mathcal{O}(h^5)$, and now the arclength values are accurate enough to apply polynomial interpolation of $(h_i, \theta(h_i))$ to get estimates of order $\mathcal{O}(h^{5-i})$ for $\theta_i,\ i = 1, \ldots, 4$, if the mesh ratio is uniformly bounded.

The standard error analysis in [1] then yields an overall interpolation error of $\mathcal{O}(h^6)$ for f with $\kappa_f \neq 0$, and of $\mathcal{O}(h^4)$ for f with $\kappa_f \geq 0$.

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