Comparison of Radial Basis Function Interpolants

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Abstract

This paper compares radial basis function interpolants on different spaces. The spaces are generated by other radial basis functions, and comparison is done via an explicit representation of the norm of the error functional. The results pose some new questions for further research.

§1. Introduction

We consider interpolation of real-valued functions f defined on a set $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$. These functions are evaluated on a set $X := \{x_1, \ldots, x_{N_X}\}$ of $N_X \geq 1$ pairwise distinct points $x_1, \ldots, x_{N_X}$ in Ω. If $N \geq 2$, $d \geq 2$ and $\Omega \subseteq \mathbb{R}^d$ are given with Ω containing at least an interior point, it is well known that there is no N-dimensional space of continuous functions on Ω that contains a unique interpolant for every f and every set $X = \{x_1, \ldots, x_{N_X}\} \subset \Omega \subseteq \mathbb{R}^d$ consisting of $N = N_X$ data points.

Thus the family of interpolants must necessarily depend on X. This can easily be achieved by using translates $\Phi(x - x_j)$ of a single continuous real-valued function Φ defined on $\mathbb{R}^d$, and further simplification is obtained by letting Φ be radially symmetric, i.e.:

$$\Phi(x) := \phi(\|x\|_2) \tag{1}$$

with a continuous real-valued function ϕ on $\mathbb{R}_{\geq 0}$ and the L_2 norm $\|\cdot\|_2$.

Interpolants s_f to f can then be constructed via the representation

$$s_f(x) = \sum_{j=1}^{N_X} \alpha_j \Phi(x - x_j), \tag{2}$$

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where the coefficients $\alpha_1, \ldots, \alpha_{N_X} \in \mathbb{R}$ solve the linear system

$$f(x_k) = \sum_{j=1}^{N_X} \alpha_j \Phi(x_k - x_j), \quad 1 \le k \le N_X,$$

provided that the symmetric $N_X \times N_X$ matrix

$$A_{\Phi,X} := \begin{pmatrix} \Phi(x_1 - x_1) & \ldots & \Phi(x_1 - x_{N_X}) \\ \vdots & \ddots & \vdots \\ \Phi(x_{N_X} - x_1) & \ldots & \Phi(x_{N_X} - x_{N_X}) \end{pmatrix}$$

is nonsingular. This is the simplest form of radial basis function interpolation, but for a variety of choices of Φ it is necessary to add polynomials to the interpolant (2).

So let P_q^d denote the space of d-variate polynomials of order not exceeding q, and let the polynomials $p_1, \ldots, p_Q$ be a basis of P_q^d in $\mathbb{R}^d$. The Q additional degrees of freedom of the extended representation

$$s_f(x) = \sum_{j=1}^{N_X} \alpha_j \Phi(x - x_j) + \sum_{\ell=1}^{Q} \beta_\ell p_\ell(x) \tag{3}$$

are compensated by the Q additional equations

$$\sum_{j=1}^{N_X} \alpha_j p_\ell(x_j) = 0, \quad 1 \le \ell \le Q. \tag{4}$$

With the matrix

$$P_X^T := \begin{pmatrix} p_1(x_1) & \ldots & p_1(x_{N_X}) \\ \vdots & \ddots & \vdots \\ p_Q(x_1) & \ldots & p_Q(x_{N_X}) \end{pmatrix}.$$

we can write the interpolation conditions

$$f(x_k) = \sum_{j=1}^{N_X} \alpha_j \Phi(x_k - x_j) + \sum_{\ell=1}^{Q} \beta_\ell p_\ell(x_k), \quad 1 \le k \le N_X$$

together with (4) as a linear system

$$\begin{pmatrix} A_{\Phi,X} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} f_X \\ 0 \end{pmatrix}, \tag{5}$$

where the data from f form a vector $f_X := (f(x_1), \ldots, f(x_{N_X}))^T$. Solvability of this system depends on two conditions. First, the matrix $A_{\Phi,X}$ should be nonsingular on the vectors α satisfying (4). Second, polynomials in P_q^d should be uniquely determined by their values on X, i.e.:

$$\text{If } p \in P_q^d \text{ satisfies } p(x_i) = 0 \text{ for all } x_i \in X \text{ then } p = 0. \tag{6}$$

The discussion of the first condition is simplified if nonsingularity is replaced by positive definiteness:

Definition 1. *A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ with $\Phi(-x) = \Phi(x)$ is conditionally positive definite of order q on $\mathbb{R}^d$, if for all sets $X = \{x_1, \ldots, x_{N_X}\} \subset \mathbb{R}^d$ with N_X distinct points and all vectors $\alpha := (\alpha_1, \ldots, \alpha_{N_X}) \in \mathbb{R}^{N_X}$ with (4) the quadratic form*

$$\sum_{j,k=1}^{N_X} \alpha_j \alpha_k \Phi(x_j - x_k)$$

attains nonnegative values and vanishes only if $\alpha = 0$.

In a fundamental paper C.A. Micchelli [14] related the conditional positive definiteness of functions Φ of the form (1) to complete monotonicity of derivatives of ϕ, and this technique allows to prove conditional positive definiteness for a variety of radial basis functions. We list a few examples:

Multiquadrics $\phi(r) = (c^2 + r^2)^{\beta/2}$ for $\beta \in \mathbb{R}_{>-d} \setminus 2\mathbb{Z}$ and $2q > \beta$ [8],
Thin-plate splines $\phi(r) = r^\beta$ for $\beta \in \mathbb{R}_{>0} \setminus 2\mathbb{Z}$ and $2q > \beta$ [3, 4, 5],
Thin-plate splines $\phi(r) = (-1)^{\beta/2+1} r^\beta \log r$ for $\beta \in 2\mathbb{N}$, $2q > \beta$ [3, 4, 5],
Gaussians $\phi(r) = e^{-\alpha r^2}$ for $\alpha > 0$ and $q \geq 0$.

Our main purpose here is to study the error $f(x) - s_f(x)$ of different radial basis function interpolants on different spaces. Curiously enough, each conditionally positive definite function Φ does not only define an interpolation method, but also defines an inner-product space F_Φ of functions. We describe the construction of such a space in the next section and introduce the exponentially decaying positive definite radial basis functions that generate the Sobolew spaces $W_2^k(\mathbb{R}^d)$. Then we represent the norm of the error functional of a general linear quasi-interpolation method on such a space by a numerically accessible *power function*. Finally, we evaluate the power functions that arise from interpolation with a radial basis function Φ_I on a space F_{Φ_S} defined by a different radial basis function Φ_S. Inspection of the results leads to a number of open questions for further research.

§2. Spaces Generated By Radial Basis Functions

We assume the radial basis function Φ to be conditionally positive definite of order q on $\mathbb{R}^d$ in the sense of Definition 1, and we now construct an associated function space F_Φ using ideas from Madych and Nelson [11].

Let $\Omega \subseteq \mathbb{R}^d$ be given, and let V be the set of all pairs (α, X) with the following properties:

$$X = \{x_1, \ldots, x_{N_X}\} \subseteq \Omega \subseteq \mathbb{R}^d, \ |X| = N_X$$

$$(\alpha, X) \text{ satisfies } (4), \text{ and}$$

$$X \text{ satisfies } (6).$$

To avoid pathological cases we assume Ω to be large enough to contain at least one set X satisfying (6), to make sure that V is non-empty. Then we define the set

$$F_\Phi := P_q^d + \{f_{\alpha,X} \mid (\alpha, X) \in V\} \tag{7}$$

with functions

$$f_{\alpha,X}(x) := \sum_{j=1}^{N_X} \alpha_j \Phi(x - x_j), \quad (\alpha, X) \in V \tag{8}$$

defined on all of $\mathbb{R}^d$. Note that V depends on Ω, and that this fact induces a subtle dependence of F_Φ on Ω which still is a mystery except for the case $\Omega = \mathbb{R}^d$, treated in detail by Madych and Nelson in [12]. Since F_Φ contains *all* finite linear combinations of translates of $\Phi(x)$ with coefficients satisfying (4) and X satisfying (6), the space F_Φ and its closures under different topologies are very natural candidates to study radial basis function approximation and interpolation. Here, we avoid to take closures, because they turn out to be irrelevant to our purposes. We rather investigate (7) as it is via the following

Lemma 1. *The sum in (7) is direct, and F_Φ is a vector space over $\mathbb{R}$. Furthermore, each function $f \in F_\Phi$ has a unique representation*

$$f = p + f_{\alpha,X} \ \text{ with } \ (\alpha, X) \in V, \ \ p \in P_q^d.$$

Proof: Assume that the function (8) is a polynomial $p \in P_q^d$. Then (4) implies

$$\begin{aligned} 0 &= \sum_{k=1}^{N_X} \alpha_k p(x_k) \\ &= \sum_{j,k=1}^{N_X} \alpha_j \alpha_k \Phi(x_j - x_k) \end{aligned}$$

and α must vanish because Φ is conditionally positive definite of order q.

Let two functions $f_{\alpha,X}$ and $f_{\beta,Y}$ with $(\alpha, X),\ (\beta, Y) \in V$ be given. Without loss of generality we assume

$$\begin{aligned} W := X \cup Y =: \{w_1, \ldots, w_{N_W}\} & \\ \supseteq Z := X \cap Y =: \{w_1, \ldots, w_{N_Z}\} & \\ = \{x_1, \ldots, x_{N_Z}\} = \{y_1, \ldots, y_{N_Z}\}, & \end{aligned}$$

where all of the sets $X,\ Y,\ Z$, and W contain pairwise distinct points. Then we can represent the sum of $f_{\alpha,X}$ and $f_{\beta,Y}$ as

$$\sum_{j=1}^{N_Z} (\alpha_j + \beta_j) \Phi(x - w_j) + \sum_{j=N_Z+1}^{N_X} \alpha_j \Phi(x - x_j) + \sum_{j=N_Z+1}^{N_Y} \beta_j \Phi(x - y_j).$$

On all polynomials $p \in P_q^d$ we get

$$\begin{aligned} \sum_{j=1}^{N_Z} (\alpha_j + \beta_j) p(w_j) + \sum_{j=N_Z+1}^{N_X} \alpha_j p(x_j) + \sum_{j=N_Z+1}^{N_Y} \beta_j p(y_j) = \\ \sum_{j=1}^{N_Z} \alpha_j p(x_j) + \sum_{j=N_Z+1}^{N_X} \alpha_j p(x_j) + \sum_{j=1}^{N_Z} \beta_j p(y_j) + \sum_{j=N_Z+1}^{N_Y} \beta_j p(y_j) = 0, \end{aligned}$$

which proves that the sum of $f_{\alpha,X}$ and $f_{\beta,Y}$ is representable in the form $f_{\gamma,W}$ with a suitable vector $\gamma \in \mathbb{R}^{N_W}$ such that $(\gamma, W) \in V$. The last assertion follows from a similar decomposition argument applied to the difference of two representations of a function $f \in F_\Phi$. ■

On the space F_Φ in (7) we now define the bilinear form

$$(p + f_{\alpha,X}, r + f_{\beta,Y})_\Phi := \sum_{j=1}^{N_X}\sum_{k=1}^{N_Y} \alpha_j \beta_k \Phi(x_j - y_k), \tag{9}$$

where p and r are arbitrary polynomials from P_q^d. Lemma 1 makes sure that this definition is consistent. Of course one can complete the pre-Hilbert space F_Φ / P_q^d into a rather interesting space of (generalized) functions on $\mathbb{R}^d$, but we refer the reader to Madych and Nelson [11,12,13] for details which are not relevant here.

§3. Radial Basis Functions for Sobolew Seminorms

We have seen that any conditionally positive definite function defines an inner product on a space of functions on $\mathbb{R}^d$. Conversely, one can ask for the radial basis functions that possibly generate a given inner product. The most prominent example would be the inner products generating Sobolew spaces $W_2^k(\mathbb{R}^d)$ of functions having generalized derivatives up to order k on $\mathbb{R}^d$.

Theorem 1. *For $k > d/2$ the inner product of Sobolew space $W_2^k(\mathbb{R}^d)$ is given by (9) and (1) with the positive definite function*

$$\phi(r) = \frac{2\pi^k}{\Gamma(k)} \mathcal{K}_{k-d/2}(2\pi r) \cdot r^{k-d/2} \tag{10}$$

defined via the Macdonalds (or spherical Bessel) function $\mathcal{K}_\nu$.

Proof: The inner product of $W_2^k(\mathbb{R}^d)$ can be written in the form

$$(f, g)_{W_2^k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{f}(\omega) \overline{\hat{g}}(\omega)(1 + \|\omega\|_2^2)^k d\omega$$

using Fourier transforms (see Yosida [23], p. 155, eq. (30)). Now if

$$\widehat{\Phi(\cdot)}(\omega) = (1 + \|\omega\|_2^2)^{-k}, \tag{11}$$

and if the inverse Fourier transform can be taken, then

$$\begin{aligned}
(f_{\alpha,X}, f_{\beta,Y})_\Phi &= \sum_{\ell=1}^{N_X}\sum_{j=1}^{N_Y} \alpha_\ell \beta_j \Phi(x_\ell - y_j), \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\Phi(\cdot)}(\omega) \sum_{\ell=1}^{N_X} \alpha_\ell e^{i x_\ell \cdot \omega} \sum_{j=1}^{N_Y} \beta_j e^{-i y_j \cdot \omega} d\omega \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}_{\alpha,X}(\omega) \overline{\hat{f}}_{\beta,Y}(\omega)(1 + \|\omega\|_2^2)^k d\omega
\end{aligned}$$

coincides with the inner product in Sobolew space. For $k > d/2$ the equation (11) has the solution (10). To prove this, combine p. 21 of Oberhettinger [16] with p. 158 of Stein and Weiss [19]. The radial basis function in (10) decays exponentially at infinity, behaves like a polynomial of degree $2k - d$ near zero, and is conditionally positive definite of order zero. See e.g.: Abramowitz and Stegun [1] for properties of Bessel functions. For spaces of odd dimension d, the radial basis function (10) can be calculated recursively in terms of monomials and the exponential function. In even dimensions the recursion starts at Bessel functions of index 0 and 1, which are readily available in any software package. ■

At this point we do not comment on the possible closures of F_Φ for various choices of Ω. An additional density argument will be needed for the proof of $\overline{F_\Phi} = W_2^k(\Omega)$, which may be difficult for peculiar cases, *e.g.*, Ω being finite, compact, or of dimension less that d.

§4. Error of Quasi-Interpolants on Radial Function Spaces

Now for any $(\lambda, Y) \in V$ we can define a linear functional

$$\varphi_{\lambda,Y}(f) := \sum_{j=1}^{N_Y} \lambda_j f(y_j)$$

that vanishes on P_q^d by (4) and satisfies

$$\begin{aligned} \varphi_{\lambda,Y}(f_{\alpha,X}) &= \sum_{j=1}^{N_Y}\sum_{k=1}^{N_X} \lambda_j \alpha_k \Phi(x_k - y_j) \\ &= (f_{\lambda,Y}, f_{\alpha,X})_\Phi \\ &= \varphi_{\alpha,X}(f_{\lambda,Y}) \end{aligned} \tag{12}$$

for all $(\alpha, X) \in V$. Thus $\varphi_{\lambda,Y}$ is a continuous linear functional on F_Φ with respect to the seminorm $|\cdot|_\Phi$ induced by the bilinear form $(\cdot,\cdot)_\Phi$. Its corresponding norm is

$$\|\varphi_{\lambda,Y}\|_\Phi = |f_{\lambda,Y}|_\Phi,$$

because its representer via the bilinear form is $f_{\lambda,Y}$. The square of this norm can be explicitly evaluated as

$$\begin{aligned} \|\varphi_{\lambda,Y}\|_\Phi^2 &= |f_{\lambda,Y}|_\Phi^2 \\ &= \sum_{j=1}^{N_Y}\sum_{k=1}^{N_Y} \lambda_j \lambda_k \Phi(y_k - y_j). \end{aligned} \tag{13}$$

Now we specialize $\varphi_{\lambda,Y}$ to be the error functional of some quasi-interpolant

$$s_{f,u,X}(x) := \sum_{i=1}^{N_X} u_i(x) f(x_i) \tag{14}$$

that depends only on the data of a function f on a set $X = \{x_1, \ldots, x_{N_X}\}$ and which is exact on P_q^d. Here, the point x is fixed in $\Omega \setminus X$ and the values $u_1(x), \ldots, u_{N_X}(x)$ are just real numbers. We set $Y := X \cup \{x\}$ and $\lambda := (1, -u_1(x), \ldots, -u_{N_X}(x)) \in \mathbb{R}^{N_X+1}$ to get $(\lambda, Y) \in V$ from the exactness of the quasi-interpolant (14) on P_q^d. Then we use (13) to derive the error bound

$$(f(x) - s_{f,u,X}(x))^2 \leq |f|_\Phi^2 \left(\Phi(0) - 2\sum_{i=1}^{N_X} u_i(x)\Phi(x - x_i) + \sum_{i=1}^{N_X}\sum_{j=1}^{N_X} u_i(x)u_j(x)\Phi(x_i - x_j) \right)$$

for all $f \in F_\Phi$. We call

$$P_{X,u,\Phi}(x) := \left(\Phi(0) - 2\sum_{i=1}^{N_X} u_i(x)\Phi(x - x_i) + \sum_{i,j=1}^{N_X} u_i(x)u_j(x)\Phi(x_i - x_j) \right)^{1/2} \tag{15}$$

the *power function* of the quasi-interpolant (14), because it precisely describes the quality of the quasi-interpolant at x. This is reminiscent of the notion of a power function of a statistical decision function.

Theorem 2. *Let the radial basis function Φ be conditionally positive definite of order q on $\mathbb{R}^d$ in the sense of Definition 1. Then any quasi-interpolant (14) that is exact on P_q^d satisfies*

$$\sup_{f \in F_\Phi \setminus P_q^d} \frac{|f(x) - s_{f,u,X}(x)|}{|f|_\Phi} = P_{X,u,\Phi}(x). \tag{16}$$

Discussion:

1. The sup in (16) can be extended to the Hilbert space completion of F_Φ without change of $P_{X,u,\Phi}(x)$. This is why we do not care about completions in this paper.
2. Though F_Φ will depend on the domain Ω, the right-hand side of (16) is *independent* of Ω, which is a rather startling fact at first sight. It can be explained by the observation that the sup is attained for the special function
$$f_x(y) := \Phi(x - y) - \sum_{j=1}^{N_X} u_i(x)\Phi(x_i - y)$$
which is in F_Φ and its completion whenever $Y := X \cup \{x\}$ is contained in Ω.
3. The expression (15) for the power function $P_{X,u,\Phi}(x)$ can be numerically evaluated at any x where the quasi-interpolant is defined. This allows a convenient numerical comparison between different quasi-interpolants on the same space F_Φ. We shall do this in section 6. ■

§5. Optimal Interpolants on Radial Function Spaces

Following Wu and Schaback [22] we now optimize the function (15) with respect to the N_X real variables $u_1(x), \ldots, u_{N_X}(x)$ under the constraints of exactness on P_q^d, which read as

$$p_\ell(x) = \sum_{j=1}^{N_X} u_j(x) p_\ell(x_j), \quad 1 \leq \ell \leq Q.$$

This yields a finite-dimensional convex optimization problem with linear constraints, and the solution vector $u_1^*(x), \ldots, u_{N_X}^*(x)$ with Lagrange multipliers $v_1^*(x), \ldots, v_Q^*(x)$ is characterized by the necessary and sufficient optimality conditions

$$\begin{pmatrix} A_{\Phi,X} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} u^*(x) \\ v^*(x) \end{pmatrix} = \begin{pmatrix} R(x) \\ S(x) \end{pmatrix} \tag{17}$$

with vectors

$$R(x) := (\Phi(x - x_1), \ldots, \Phi(x - x_{N_X}))^T, \quad S(x) := (p_1(x), \ldots, p_Q(x))^T$$

and matrices as in (5). Thus (17) is uniquely solvable for all $x \in \mathbb{R}^d$, and since for $x = x_j$ the right-hand side coincides with the j-th column of the coefficient matrix, we get $u_i^*(x_j) = \delta_{ij}$. This proves

Theorem 3. *Among all linear quasi-interpolants of the form (14) that are exact on P_q^d and have data points in X, the radial basis function interpolation with centers in X is pointwise optimal with respect to minimization of the function (16).*

Classical spline theory would minimize the seminorm induced by (9) under all interpolants in the space F_Φ. This variational approach was used in the radial basis function context by Duchon [3,4,5] and Madych and Nelson [11,12,13]. It is

a) a minimization of a "smoothness functional"
b) on an infinite-dimensional space
c) under all interpolants in the space,

while the approach of Wu and Schaback [22] is

a) a minimization of the norm of the pointwise error functional
b) on a finite-dimensional space
c) under all quasi-interpolants that need not necessarily lie in the space.

Under general conditions these two approaches are not equivalent. To give a short account of the relation between the two problems, we include a simple proof of

Theorem 4. *The optimal radial basis function interpolant in the sense of Theorem 3 is also optimal with respect to the minimization of the seminorm induced by (9) under all interpolants in the space F_Φ.*

Proof: The optimal interpolant $s_{f,X}$ to a function f with data on X which minimizes the seminorm induced by (9) is characterized by the usual interpolation conditions and the property

$$(s_{f,X}, g)_\Phi = (s_{f,X}, f_{\lambda,Y})_\Phi = 0$$

for all $g = p + f_{\lambda,Y} \in F_\Phi$ with $(\lambda, Y) \in V$ that vanish on X. It suffices to show that this orthogonality condition is satisfied by the interpolant $s^*_{f,X} = f_{\alpha^*,X} + p^*$ of the form (3) with $(\alpha^*, X) \in V$. But with (12) we easily get

$$\begin{aligned}(s^*_{f,X}, g)_\Phi &= (s^*_{f,X}, f_{\lambda,Y})_\Phi \\ &= (f_{\alpha^*,X}, f_{\lambda,Y})_\Phi \\ &= \varphi_{\alpha^*,X}(f_{\lambda,Y}) \\ &= \varphi_{\alpha^*,X}(g) \\ &= 0.\end{aligned}$$

■

We note that Laurent [10] has a beautiful result that guarantees equivalence of the two variational approaches considered here, provided that they both have solutions and that they are based on inner-product spaces. Going further back, the minimization of the norm of the representer of a linear functional on a Hilbert space dates seems to have been started by Golomb and Weinberger [7] and was carried forth by a series of others, including de Boor and Lynch [2], Sard [17], Larkin [9], and Dyn [6]. Somewhat related to this approach is the theory of optimal recovery, as given by Micchelli and Rivlin in [15].

§6. Numerical Results

We now perform numerical comparisons between different radial basis function interpolations on different spaces. Each space will be defined via a radial basis function Φ_S, and each interpolation will be carried out with a radial basis function Φ_I, where the corresponding orders of conditional positive definiteness are q_S and q_I, respectively.

We do not compare the interpolants themselves, but evaluate the functions (15). This will require $q_I \geq q_S$, because the exactness order of the interpolant must at least equal the order of positive definiteness of the radial basis function defining the space (see the hypothesis of Theorem 2). Otherwise the interpolation error at x simply is not a continuous linear functional on the space in question. Tables 1 and 2 thus will have no entries in case $q_I < q_S$.

If $\Phi_S = \Phi_I$, there will be no better interpolant on F_{Φ_S} because of Theorem 3. We underlined these cases in Tables 1 and 2.

The numerical results were obtained in space dimension $d = 1$ for simplicity. The sets X contained $2n$ points in $[-1, +1]$ with spacing $h := 2/(2n-1)$, and we set $x = 0$ to evaluate (15). Table 2 contains the actual values of (15) for $n = 50$ being fixed, while Table 1 gives approximate error orders p_{IS} for interpolation with Φ_I in F_{Φ_S} in the sense that (15) behaves like $\mathcal{O}(h^{p_{IS}})$ for $h \to 0$.

Spaces $\Rightarrow$ / Interpolations $\Downarrow$	Mq $q_S = 1$	i.Mq $q_S = 0$	r^3 $q_S = 2$	W_2^2 $q_S = 0$	r^1 $q_S = 1$	W_2^1 $q_S = 0$
Mq, $q_I = 1$	$\underline{\infty}$	∞	-	1.5	0.5	0.5
Mq, $q_I = 2$	∞	∞	1.5	1.5	0.5	0.5
i.Mq, $q_I = 0$	-	$\underline{\infty}$	-	1.5	-	0.5
i.Mq, $q_I = 1$	∞	∞	-	1.5	0.5	0.5
i.Mq, $q_I = 2$	∞	∞	1.5	1.5	0.5	0.5
r^3, $q_I = 2$	4.0	4.0	$\underline{1.5}$	1.5	0.5	0.5
W_2^2, $q_I = 2$	4.0	4.0	1.5	1.5	0.5	0.5
W_2^2, $q_I = 0$	-	4.0	-	$\underline{1.5}$	-	0.5
r^1, $q_I = 1$	2	2	-	1.5	$\underline{0.5}$	0.5
W_2^1, $q_I = 1$	2	2	-	1.5	0.5	0.5
W_2^1, $q_I = 0$	-	2	-	1.5	-	$\underline{0.5}$

Table 1. Estimated convergence orders.

Mq = Multiquadrics with $\beta = 1/2$
i.Mq = inverse Multiquadrics, $\beta = -1/2$
W_2^k = Sobolew function

Spaces $\Rightarrow$ / Interpolations $\Downarrow$	Mq $q_S = 1$	i.Mq $q_S = 0$	r^3 $q_S = 2$	W_2^2 $q_S = 0$	r^1 $q_S = 1$	W_2^1 $q_S = 0$
Mq, $q_I = 1$	$\underline{0.002178}$	0.061979	-	0.001675	0.155065	0.206437
Mq, $q_I = 2$	0.002178	0.061979	0.003082	0.001675	0.155065	0.206437
i.Mq, $q_I = 0$	-	$\underline{0.061883}$	-	0.001665	-	0.205494
i.Mq, $q_I = 1$	0.002182	0.061883	-	0.001665	0.154357	0.205494
i.Mq, $q_I = 2$	0.002182	0.061833	0.003064	0.001665	0.154357	0.205494
r^3, $q_I = 2$	0.002672	0.070031	$\underline{0.002986}$	0.001623	0.148943	0.198287
W_2^2, $q_I = 2$	0.002672	0.070038	0.002986	0.001623	0.148941	0.198284
W_2^2, $q_I = 0$	-	0.070038	-	$\underline{0.001623}$	-	0.198284
r^1, $q_I = 1$	0.010857	0.182091	-	0.002215	$\underline{0.142857}$	0.190178
W_2^1, $q_I = 1$	0.010893	0.182285	-	0.002230	0.148941	0.190178
W_2^1, $q_I = 0$	-	0.182285	-	0.002241	-	$\underline{0.190178}$

Table 2. Power function value at zero for 50 data points.

There are some remarkable observations to be made from the tables:

1. *Quasi-optimal convergence:* Nonoptimal interpolation processes with radial basis functions Φ_I on spaces defined by radial basis functions $\Phi_S \neq \Phi_I$ seem to achieve the order of the optimal interpolation on F_{Φ_S} if the optimal orders satisfy $p_{II} \geq p_{SS}$. That is, radial basis function interpolants that are optimal on small and very smooth spaces apparently are quasi-optimal on larger and less smooth spaces.

2. *Superconvergence:* On even smoother spaces than their native space, a radial basis function interpolant may show an even better behavior. That is, nonoptimal interpolation processes with radial basis functions Φ_I on spaces defined by radial basis functions $\Phi_S \neq \Phi_I$ achieve a higher order than their optimal order on their basic space, if $p_{II} < p_{SS}$.
3. There is hardly any difference induced by variations of the polynomial orders q_I and q_S. The only visible deviation is in the last two entries of the W_2^2 column of Table 2.
4. The values of (15) are astonishingly close to each other and to the optimal value.

The first two observations can be commented within the context of classical one-dimensional natural splines, which also solve a variational problem and which are a special case of the theory of this paper. On the Sobolew space $H_2^k[-1,+1]$ with seminorm $|f|_k^2 := \|f^{(k)}\|_2^2$, the natural polynomial splines of order $2k$ are optimal and have the optimal error order $k-1/2$, which is a saturation order (see Schumaker [18]). Natural splines of higher order $2n > 2k$ attain this order on $H_2^k[-1,+1]$, too, and are quasi-optimal in the sense of the first observation. On the space $C^{2k}[a,b]$, however, natural splines of order $2k$ will have an error order $2k$ in the interior of the domain, and this phenomenon was called superconvergence in the second observation. Since $2k$ is a saturation order on $C^{2k}[a,b]$, no improvement is possible in this special case. These facts were proven by Swartz and Varga in [20] for the univariate spline case, but there is no proof so far for multivariate splines or general radial basis functions. This raises the question of saturation orders and saturation spaces for general cases of radial basis function interpolation. The underlined optimal orders p_{II} for $\Phi_S = \Phi_I$ were theoretically proven by Madych and Nelson [11,12,13] and Wu and Schaback [22].

Of course, the above experiments were only done in the interior of the domain and for uniform meshes. Performance may be quite different if these assumptions are not satisfied. The classical spline case teaches us that the behavior near the boundary will not show superconvergence, and there is some theoretical investigation under way that suggests quasi-optimality to be dependent on asymptotically quasi-uniform meshes.

§7. References

1. Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York 1970
2. de Boor, C., and R. E. Lynch, On splines and their minimum properties, *J. Math. Mech***15** (1966), 953–969
3. Duchon, J., Interpolation des fonctions de deux variables suivant le principe de la flexion des plaques minces, *RAIRO Anal. Num.* **10** (1976), 5–12
4. Duchon, J., Splines minimizing rotation-invariant seminorms in Sobolev spaces, in *Constructive Theory of Functions in Several Variables*, W.

Schempp and K. Zeller (eds.), Springer, Berlin-Heidelberg-New York, 1977, 85–100
5. Duchon, J., Sur L'erreur d'interpolation des fonctions de plusieurs variables par les D^m- splines, *RAIRO Anal. Num.* **12** (1978), 325-334.
6. Dyn, N., On best nonlinear approximation in sign-monotone norms and in norms induced by inner products, *SIAM J. Numer. Anal.***16** (1979), 612–622
7. Golomb, W., and H. F. Weinberger, Optimal approximation and error bounds, in *Numerical Approximation*, R. E. Langer (ed.), Univ. of Wisconsin Press, 1959, 117-190
8. Hardy, R. L., Multiquadric equations of topography and other irregular surfaces, *J. Geophysical Res.* **76** (1971), 1905–1915
9. Larkin, F. M., Optimal Approximation in Hilbert Spaces with Reproducing Kernel Functions, *Math. Comp.* **24** (1970), 911–921
10. Laurent, P. J., Quadratic convex analysis and splines, in *Methods of Functional Analysis in Approximation Theory*, C. A. Micchelli, D. V. Pai, B. V. Limaye (eds.) ISNM **76**, Birkhäuser, Basel, 1986, 17–43
11. Madych, W. R., and Nelson, S. A., Multivariate Interpolation and conditionally positive definite functions, *Approx. Th. Appl.***4** (1988), 77–89
12. Madych, W. R., and Nelson, S. A., Multivariate Interpolation and conditionally positive definite functions II, *Math. Comp.***54** (1990), 211–230
13. Madych, W. R., and Nelson, S. A., Error Bounds for Multiquadric Interpolation, in *Approximation Theory VI: Vol 2*, C. K. Chui, L. L. Schumaker and J. D. Ward.(eds.), Acadamic Press, New York, 413–416
14. Micchelli, C. A., Interpolation of scattered data: distance matrices and conditionally positive definite functions, *Constr. Approx.***2** (1986), 11–22
15. Micchelli, C. A., and T. J. Rivlin, A survey of optimal recovery, in *Optimal Estimation in Approximation Theory*, C. A. Micchelli and T. J. Rivlin (eds.), Plenum Press, New York, 1977
16. Oberhettinger, F., *Fourier Transforms of Distributions and Their Inverses*, Academic Press, New York–London, 1973
17. Sard, A., *Optimal Approximation*, *J. Functional Anal.* **1** (1967), 222–224
18. Schumaker, L. L., *Spline Functions: Basic Theory*, Wiley, New York, 1981.
19. Stein, E. M., and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1971
20. Swartz, B. K., and R. S. Varga, Error bounds for spline and L-spline interpolation, *J. Approx. Th.* **6** (1972), 6–49
21. Wu, Zong–min, Die Kriging–Methode zur Lösung mehrdimensionaler Interpolationsprobleme, Dissertation, Göttingen 1986
22. Wu, Zong-min, and R. Schaback, Local Error Estimates for Radial Basis Function Interpolation *IMA J. of Numer. Anal.*, to appear
23. Yosida, *Functional Analysis*, Springer, Berlin–Heidelberg–New York 1968

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