Limits of Bernstein–Bézier Curves for Periodic Control Nets

Dong-xu Qi and Robert Schaback

Abstract: If n given control points $b_0, \ldots, b_{n-1} \in I\!R^d$ are repeated periodically by $b_{j+kn} = b_j$ for all $k \in Z\!\!Z$, the uniform limit of the Bernstein–Bézier polynomial curves of degree r with control points $b_0, \ldots, b_r$ for $r \to \infty$ is a Poisson curve (after a suitable reparametrization). This fact reveals some interesting self–similar structures in case of regular n–gons in the plane.

1 Introduction

Let $n \geq 1$ control points $b_0, \ldots, b_{n-1} \in I\!R^d$ be given. These control points are repeated by

$$b_{j+kn} := b_j \ \text{ for } \ 0 \leq j \leq n-1, \ \text{ and all } \ k \in Z\!\!Z$$

to form an infinite periodic sequence. The centroid of the points is denoted by $\bar{b} := \frac{1}{n} \sum_{j=0}^{n-1} b_j$, and the Bernstein polynomials of degree r are

$$\beta_j^{(r)}(t) := \left(\begin{array}{c} r \\ j \end{array} \right) (1-t)^{r-j} t^j, \ \ 0 \leq j \leq r, \ t \in [0,1].$$

Then we consider the Bernstein–Bézier polynomials [1],[2],[3]

$$f_r(t) := \sum_{j=0}^{r} b_j \beta_j^{(r)}(t)$$

for large degrees r and investigate the behavior of $f_r(t)$ in the convex hull of the control points, when r tends to infinity. We want to characterize all limit points of the curves $f_r(t)$, as shown by figures 1 and 2 for $n = 4$ and $n = 7$ points forming a regular polygon in the plane.

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2 Convergence to the centroid

First we treat the case of a fixed argument $t \in (0,1)$.

Theorem. *For all $t \in (0,1)$ the centroid $\bar{b}$ is the limit*

$$\lim_{r\to\infty} f_r(t) = \bar{b}.$$

Proof: For fixed $t \in (0,1)$ we perform the de Casteljau construction:

$$\begin{aligned} b_j^{(0)} &:= b_j, & j \in \mathbb{Z} \\ b_j^{(r)}(t) &:= (1-t)b_j^{(r-1)}(t) + tb_{j+1}^{(r-1)}(t), & j \in \mathbb{Z},\ r \geq 1. \end{aligned}$$

Then, for fixed r and t, the $b_j^{(r)}(t)$ are also periodic with respect to j. Furthermore, any n subsequent points of the $b_j^{(r)}(t)$ will have the centroid $\bar{b}$.

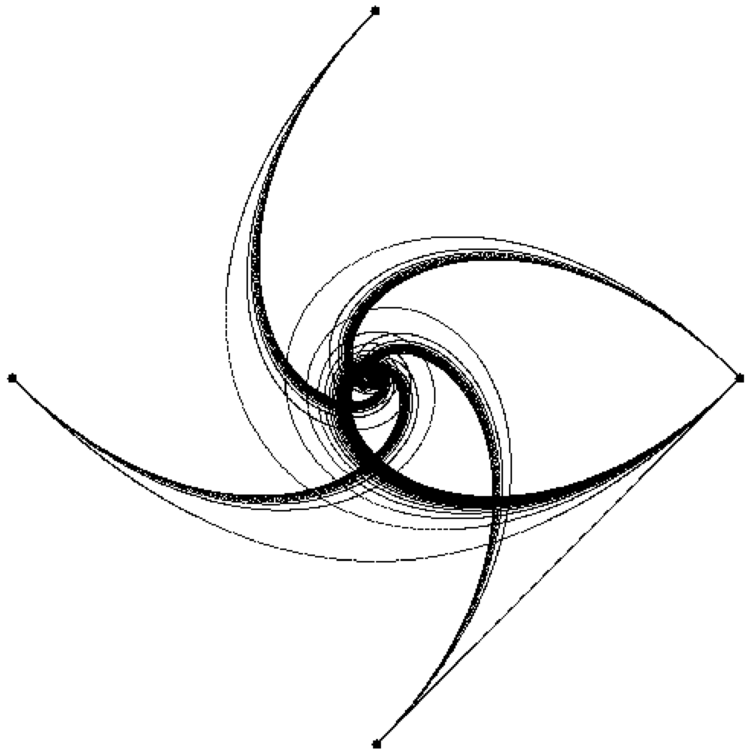

Figure 1: $n = 4$ points oriented clockwise

We now write the de Casteljau steps [5] in matrix notation [4]. If E is the $d \times d$ unit matrix, then the $(nd) \times (nd)$–matrix

$$T := \begin{pmatrix} tE & (1-t)E & 0 & \dots & 0 \\ 0 & tE & (1-t)E & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & tE & (1-t)E \\ (1-t)E & 0 & 0 & 0 & tE \end{pmatrix}$$

has the property

$$T \begin{pmatrix} b_0^{(r-1)} \\ \vdots \\ b_{n-1}^{(r-1)} \end{pmatrix} = \begin{pmatrix} b_0^{(r)} \\ \vdots \\ b_{n-1}^{(r)} \end{pmatrix} = T^r \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix}.$$

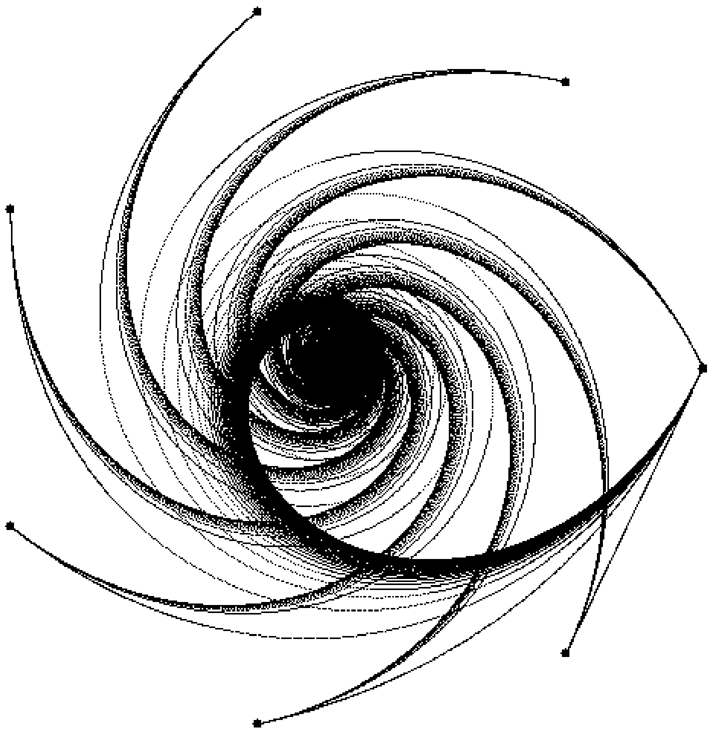

Figure 2: $n = 7$ points oriented clockwise

Now let S be the subspace of $I\!R^{nd}$ containing all sets of n vectors $c_0, \ldots, c_{n-1}$ in $I\!R^d$ with $\sum_{i=0}^{n-1} c_i = 0$. This subspace contains all vectors

$$\begin{pmatrix} b_0^{(r)}(t) - \bar{b} \\ \vdots \\ b_{n-1}^{(r)}(t) - \bar{b} \end{pmatrix} = T^r \begin{pmatrix} b_0 - \bar{b} \\ \vdots \\ b_{n-1} - \bar{b} \end{pmatrix}$$

for all $r \geq 0$. The whole de Casteljau process, when applied to the differences to the centroid, stays in the subspace S.

Now let λ be an eigenvalue of T with eigenvector $(c_0, \ldots, c_{n-1})^T$, and we again extend periodically by $c_{j+kn} := c_j$ for $0 \leq j \leq n-1$ and all $k \in Z\!\!\!Z$. Then

$$tc_i + (1-t)c_{i+1} = \lambda c_i,$$

and

$$c_{i+1} = \frac{\lambda - t}{1-t} c_i = \left(\frac{\lambda - t}{1-t} \right)^{i+1} c_0.$$

Because of periodicity, $c_n = c_0$ holds and implies

$$\left(\frac{\lambda - t}{1-t} \right)^n = 1.$$

The eigenvalue $\lambda = 1$ can occur only with eigenvectors satisfying $c_{i+1} = c_i = c_0$ for all i. This is not possible for nonzero vectors in the subspace S. The other eigenvalues are of the form

$$\lambda = t \cdot 1 + (1-t) \cdot \omega_n$$

with a complex n–th root of unity $\omega_n \neq 1$. They must necessarily lie in the interior of the unit circle, because they are nontrivial convex combinations of two different roots of unity.

This proves that T as a mapping on S has only eigenvalues λ with $|\lambda| < 1$. Therefore

$$\begin{pmatrix} b_0^{(r)}(t) - \bar{b} \\ \vdots \\ b_{n-1}^{(r)}(t) - \bar{b} \end{pmatrix} = T^r \begin{pmatrix} b_0 - \bar{b} \\ \vdots \\ b_{n-1} - \bar{b} \end{pmatrix}$$

converges to zero. The first component is $f_r(t) - \bar{b}$, and the assertion of the theorem follows. QED.

For later use, we prove a stronger result:

Theorem. *For points $t_r = \tau_r/r$ with $t_r \to 0$ and $\tau_r \to \infty$ for $r \to \infty$,*

$$\lim_{r\to\infty} f_r(t_r) = \bar{b}.$$

Proof. A direct refinement of the previous proof yields

$$\|f_r(t) - \bar{b}\| \le \max_{0\le i\le n-1} \|b_i - \bar{b}\| \cdot \|T(t)\|^r,$$

and, since the eigenvalues of $T^T(t)T(t)$ are

$$\lambda_k(t) = t^2 + (1-t)^2 + 2t(1-t)\cos(2\pi k/n), \quad 0 \le k \le n-1,$$

each eigenvalue occurring d times, the Euclidean norm of $T(t)$ is bounded by $1 - \alpha_n t$ for small values of t, where $\alpha_n = \mathcal{O}(1/n)$ for $n \to \infty$. Inserting t_r as defined above we get

$$\|f_r(t_r) - \bar{b}\| \le C(1 - \alpha_n \tau_r/r)^r$$

for all $r \ge 1$ with a constant C, and this bound tends to zero. QED.

3 Convergence to a Poisson curve

The previous section showed that variable arguments $t_r \le \tau/r$ for some fixed value of $\tau \in (0,\infty)$ should be considered next.

Theorem. *The "**Poisson**" curve*

$$p(\tau) := e^{-\tau} \sum_{j=0}^{\infty} b_j \tau^j / j!$$

is the limit of reparametrized Bernstein–Bézier curves, i.e.:

$$\lim_{r\to\infty} f_r(\tau/r) = p(\tau), \quad \tau \in [0,\infty).$$

Furthermore,

$$\lim_{\tau\to\infty} p(\tau) = \bar{b}.$$

Proof: Stirling's formula gives

$$\lim_{r\to\infty} \beta_j^{(r)}(\tau/r) = \lim_{r\to\infty} \frac{r!}{j!(r-j)!}\frac{(r-\tau)^{r-j}\tau^j}{r^r} = e^{-\tau}\tau^j/j!.$$

This proves $\lim_{r\to\infty} f_r(\tau/r) = p(\tau)$, because the b_j are uniformly bounded and the series for p converges nicely. This part of the proof resembles the fact that the binomial probability distribution, occurring as a weight in the Bernstein–Bézier polynomial curves, converges to the Poisson distribution.

Now we still have to prove convergence of the Poisson curve $p(\tau)$ to the centroid for $\tau \to \infty$. For this we define the shifted Poisson curves

$$p_j(\tau) := e^{-\tau}\sum_{m=0}^{\infty} b_{j+m}\tau^m/m!$$

for all $j \in \mathbb{Z}$, using periodicity with respect to j. Then, by easy calculation,

$$p_j'(\tau) = p_{j+1}(\tau) - p_j(\tau)$$

for all $j \in \mathbb{Z}$, and

$$p_n(\tau) = p_0(\tau).$$

With the differential operator $D := d/d\tau$ we find $(D+1)p_j = p_{j+1}$ and

$$\begin{aligned}(D+1)^n p_j &= (D+1)^j(D+1)^{n-j}p_j = (D+1)^j p_n\\ &= (D+1)^j p_0 = p_j\end{aligned}$$

for all j. Thus all p_j satisfy the same linear constant coefficient differential equation of order n with characteristic polynomial

$$P_n(x) = (x+1)^n - 1.$$

The roots of P_n are of the form $x_k = -1 + \omega_n^k$, where ω_n is a n-th root of unity, i.e.:

$$\omega_n^k := \exp\frac{2\pi i k}{n}, \quad 0 \le k \le n-1.$$

With certain complex coefficients α_{jk} the functions p_j have the form

$$\begin{aligned} p_j(\tau) &= \sum_{k=0}^{n-1} \alpha_{jk} \exp((-1+\omega_n^k)\tau) \\ &= e^{-\tau} \sum_{k=0}^{n-1} \alpha_{jk} \exp(\tfrac{2\pi i k}{n}\tau), \end{aligned}$$

and all terms except the one for $k=0$ must go to zero for $\tau \to \infty$, because $-1+\omega_n^k$ has a negative real part for $k \neq 0$.

This implies $\lim_{\tau\to\infty} p_j(\tau) = \alpha_{j0}$ and

$$\begin{aligned} \lim_{\tau\to\infty} p_j'(\tau) &= 0 = \lim_{\tau\to\infty} p_{j+1}(\tau) - \lim_{\tau\to\infty} p_j(\tau) \\ &= \alpha_{j+1,0} - \alpha_{j,0}. \end{aligned}$$

Because of $0 = \sum_{j=0}^{n-1} p_j'(\tau)$ and $n \cdot \bar{b} = \sum_{j=0}^{n-1} p_j(0)$ we know that $n \cdot \bar{b} = \sum_{j=0}^{n-1} p_j(\tau)$ holds for all τ. But in the limit $\tau \to \infty$ all the values $p_j(\infty) = \alpha_{j,0}$ are equal, which proves the assertion. QED.

Theorem *If $z \in I\!R^d$ is an accumulation point of a sequence $f_r(t_r)$ with $t_r \in [0, 1/2]$, then either $z = \bar{b}$ or $z = p(\tau)$ for some $\tau \in [0, \infty)$.*

Proof: If we rule out the trivial case $z = \bar{b}$, we can assume $t_r = \tau_r/r$ with $\tau_r \leq \tau > 0$. On $[0, \tau]$, the curves $g_r(t) = f_r(t/r)$ are continuously differentiable with uniformly bounded derivatives, because the norm of

$$g_r'(t) = \frac{1}{r} r \sum_{j=0}^{\infty} \beta_j^{(r-1)}(t)(b_{j+1} - b_j)$$

is bounded by $\max_{0\leq j<n} \|b_{j+1} - b_j\|$. The convergence of g_r to the Poisson curve p on $[0, \tau]$ thus is uniform, and the assertion follows. QED.

Remark: The limit points of $f_{j+kn}(1 - \tau/(j+kn))$ for $k \to \infty$ and $0 \leq j \leq n-1$ fixed are points of the "backward" and "shifted" Poisson curves $\hat{p}_j$ defined by

$$\hat{p}_j(\tau) := e^{-\tau} \sum_{m=0}^{\infty} b_{j-m} \tau^m / m!,$$

$$p_j(\tau) := e^{-\tau} \sum_{m=0}^{\infty} b_{j+m} \tau^m / m!,$$

where we used the periodicity and added the shifted Poisson curves p_j. The union of these two sets of Poisson curves, together with their limit $\bar{b}$, make up the set of all limit points of the backward and shifted Bernstein–Bézier curves

$$\hat{b}_{r,j}(t) := \sum_{m=0}^{\infty} b_{j-m}\beta_m^{(r)}(t), \quad 0 \le j < n,$$

$$b_{r,j}(t) := \sum_{m=0}^{\infty} b_{j+m}\beta_m^{(r)}(t), \quad 0 \le j < n,$$

for $r \to \infty$.

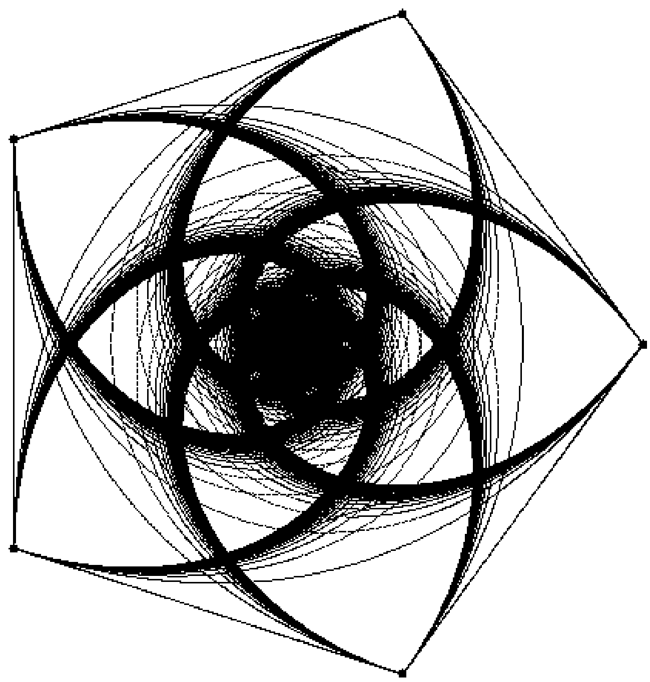

Figure 3: $n = 5$ points, all of the curves

Figure 3 shows these curves for a regular pentagon in the plane. To avoid numerical instabilities for large degrees r, we use the de Casteljau construction in the form

$$b_{r,j}(t) = (1-t)b_{r-1,j}(t) + tb_{r-1,j+1}(t)$$

with three loops over j, t, and r (innermost to outermost). This requires storage of $n+1$ discretized curve images of equal degree, starting with the constant curves b_j of degree zero for $j = 0, \dots, n$.

4 Regular polygons in the plane

Now let $b_0, \dots, b_{n-1}$ be the vertices of the standard regular n–gon in the complex plane, i.e.: $b_j = \omega_n^j = \exp(2\pi i j/n)$ for $0 \le j < n$. By easy calculation, the Poisson curves for this configuration are the logarithmic spirals

$$p_j(\tau) = \omega_n^j e^{\tau(\omega_n - 1)},$$
$$\hat{p}_j(\tau) = \omega_n^j e^{\tau(1/\omega_n - 1)}.$$

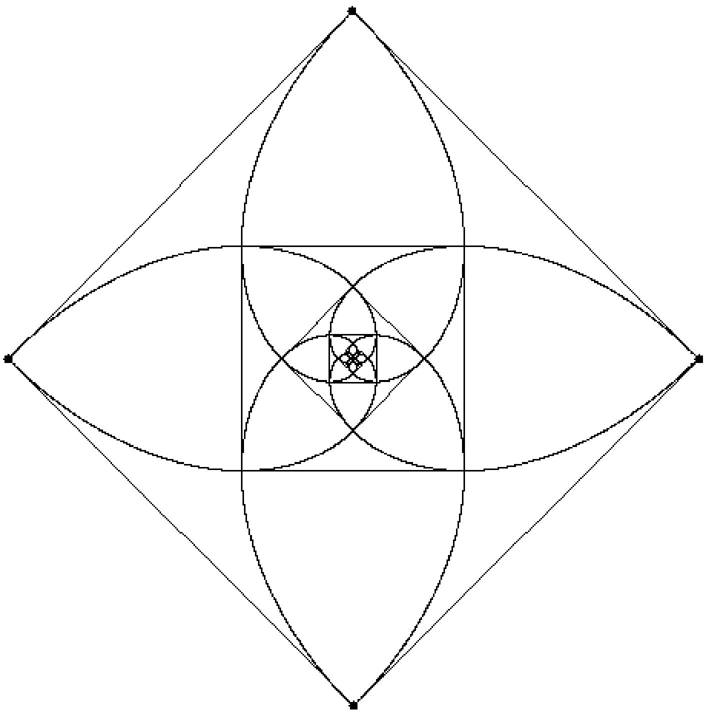

Figure 4: $n = 4$ points, Poisson curves plus squares at intersection points

The curves p_j and $\hat{p}_{j+1}$ first intersect in $z_j = p_j(\tau_n) = \hat{p}_{j+1}(\tau_n)$ with

$$\tau_n = \frac{1}{2} \frac{2\pi/n}{\sin(2\pi/n)},$$

and $z_0, \ldots, z_{n-1}$ form another regular n–gon. This smaller polygon contains a complete scaled copy of the contents of the original n–gon, including the Poisson curves on $[\tau_n, \infty)$, because these satisfy simple functional equations like

$$p_j(\tau + \sigma) = p_j(\tau) \cdot p_0(\sigma).$$

This gives a full account of the self–similarity of the structures in figures 4 and 5, showing the set of Poisson curves for $n = 4$ and $n = 12$, together with the polygons obtained by connecting the k-th intersection points of Poisson curves.

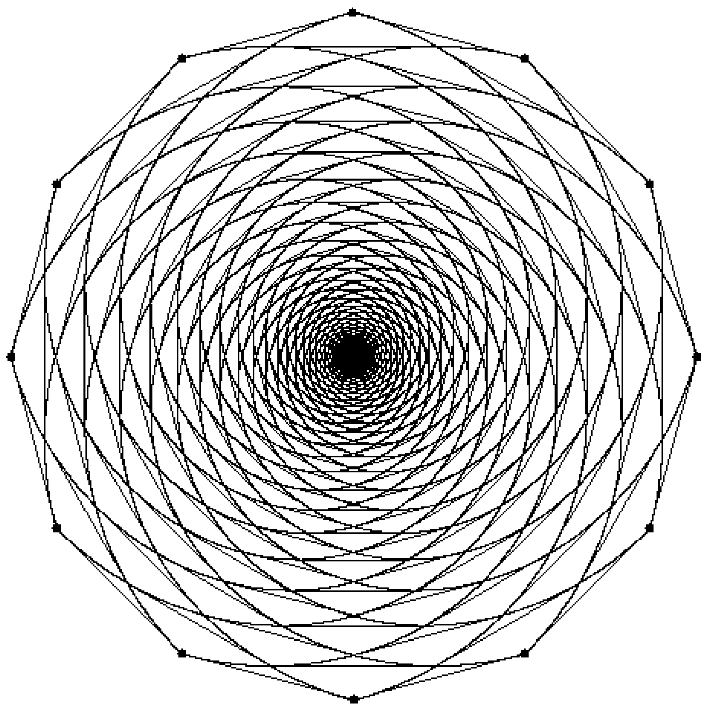

Figure 5: $n = 12$ points, Poisson curves plus polygons at intersection points

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Author's addresses:

Dong-xu Qi, Computer and CAD Research Center, North China University of Technology, Beijing 100041, People's Republic of China.

Robert Schaback, Institut für Numerische und Angewandte Mathematik der Universität Göttingen, Lotzestraße 16–18, D–3400 Göttingen, Federal Republic of Germany.