On the fractional derivatives of radial basis functions

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Abstract

The paper provides the fractional integrals and derivatives of the Riemann-Liouville and Caputo type for the five kinds of radial basis functions (RBFs), including the powers, Gaussian, multiquadric, Matern and thin-plate splines, in one dimension. It allows to use high order numerical methods for solving fractional differential equations. The results are tested by solving two fractional differential equations. The first one is a fractional ODE which is solved by the RBF collocation method and the second one is a fractional PDE which is solved by the method of lines based on the spatial trial spaces spanned by the Lagrange basis associated to the RBFs.

Keywords: Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, Radial basis functions

1. Introduction

Fractional calculus has gained considerable popularity and importance due to its attractive applications as a new modelling tool in a variety of scientific and engineering fields, such as viscoelasticity [13], hydrology [1], finance [5, 23], and system control [22]. These fractional models, described in the form of fractional differential equations, tend to be much more appropriate for the description of memory and hereditary properties of various materials and processes than the traditional integer-order models. In the last decade, a number of numerical methods have been developed to solve fractional differential equations. Most of them rely on the finite difference method to discretize both the fractional-order space and time derivative [6, 7, 15, 19, 27, 29]. Some numerical schemes using low-order finite elements [2, 24], matrix transfer technique [10, 11], and spectral methods [14, 16] have also been proposed.

Unlike traditional numerical methods for solving partial differential equations, meshless methods need no mesh generation, which is the major problem in finite difference, finite element and spectral methods [20, 26]. Radial basis function methods are truly meshless and simple enough to allow modelling of rather high dimensional problems [3, 4, 9, 12, 21]. These methods can be very efficient numerical schemes to discretize non-local operators like fractional differential operators.

In this paper, we provide the required formulas for the fractional integrals and derivatives of Riemann-Liouville and Caputo type for RBFs in one dimension. The rest of the paper is organized as follows. In section 2 we give some important definitions and theorems which are needed throughout the remaining sections of the paper. The corresponding formulas of the fractional integrals and derivatives of Riemann-Liouville and Caputo type for the five kinds of RBFs are given in section 3 . The results are applied to solve two fractional differential equations in section 4. The last section is devoted to a brief conclusion.

2. Preliminaries

In this section, we outline some important definitions, theorems and known properties of some special functions used throughout the remaining sections of the paper [17, 18, 28]. In all cases α denotes a

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non-integer positive order of differentiation and integration.

Definition 1. *The left-sided Riemann-Liouville fractional integral of order* α *of function* $f(x)$ *is defined as*

$$ {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad x > a. $$

Definition 2. *The right-sided Riemann-Liouville fractional integral of order* α *of function* $f(x)$ *is defined as*

$$ {}_xI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau) dt, \quad x < b. $$

Definition 3. *The left-sided Riemann-Liouville fractional derivative of order* α *of function* $f(x)$ *is defined as*

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-\tau)^{m-\alpha-1} f(\tau) d\tau, \quad x > a, $$

where $m = \lceil \alpha \rceil$.

Definition 4. *The right-sided Riemann-Liouville fractional derivative of order* α *of function* $f(x)$ *is defined as*

$$ {}_xD_b^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b (\tau-x)^{m-\alpha-1} f(\tau) d\tau, \quad x < b, $$

where $m = \lceil \alpha \rceil$.

Definition 5. *The Riesz space fractional derivative of order* α *of function* $f(x,t)$ *on a finite interval* $a \leq x \leq b$ *is defined as*

$$ \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} f(x,t) = -c_{\alpha} \left({}_aD_x^{\alpha} + {}_xD_b^{\alpha} \right) f(x,t), $$

where

$$ \begin{aligned} C_{\alpha} &= \frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}, \quad \alpha \neq 1, \\ {}_aD_x^{\alpha} f(x,t) &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-\tau)^{m-\alpha-1} f(\tau,t) d\tau, \\ {}_xD_b^{\alpha} f(x,t) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b (\tau-x)^{m-\alpha-1} f(\tau,t) d\tau. \end{aligned} $$

Definition 6. *The left-sided Caputo fractional derivative of order* α *of function* $f(x)$ *is defined as*

$$ {}_a^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad x > a, $$

where $m = \lceil \alpha \rceil$.

Definition 7. *The right-sided Caputo fractional derivative of order* α *of function* $f(x)$ *is defined as*

$$ {}_x^C D_b^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\tau-x)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad x < b, $$

where $m = \lceil \alpha \rceil$.

The definitions above hold for functions f with special properties depending on the situations. It is clear that

$$ \begin{aligned} &{}_aD_x^{\alpha} f(x) = D^m \left[{}_aI_x^{m-\alpha} f(x) \right], \\ &{}_xD_b^{\alpha} f(x) = (-1)^m D^m \left[{}_xI_b^{m-\alpha} f(x) \right], \\ &{}_a^C D_x^{\alpha} f(x) = {}_aI_x^{m-\alpha} \left[f^{(m)}(x) \right], \\ &{}_x^C D_b^{\alpha} f(x) = (-1)^m {}_xI_b^{m-\alpha} \left[f^{(m)}(x) \right]. \end{aligned} $$

Theorem 8. *For $\beta > -1$ and $x > a$ we have*

$$ {}_aI_x^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}. $$

Proof.

$$ {}_aI_x^{\alpha}(x-a)^{\beta} = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}(\tau-a)^{\beta}d\tau = \frac{1}{\Gamma(\alpha)}\int_0^{x-a} u^{\alpha-1}(x-u-a)^{\beta}du, $$

where $u = x - \tau$. Now with the change of variable $z = \frac{u}{x-a}$, we get

$$ {}_aI_x^{\alpha}(x-a)^{\beta} = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 z^{\alpha-1}(1-z)^{\beta}dz = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)}B(\alpha,\beta+1) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}. $$

Theorem 9. *For $\beta > \alpha - 1$ and $x > a$ we have*

$$ {}_aD_x^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. $$

Proof.

$$ \begin{aligned} {}_aD_x^{\alpha}(x-a)^{\beta} &= D^m\left[{}_aI_x^{m-\alpha}(x-a)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(m-\alpha+\beta+1)}D^m\left[(x-a)^{m-\alpha+\beta}\right] \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. \end{aligned} $$

Theorem 10. *For $\beta > \alpha - 1$ and $x > a$ we have*

$$ {}_a^CD_x^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. $$

Proof.

$$ \begin{aligned} {}_a^CD_x^{\alpha}(x-a)^{\beta} &= {}_aI_x^{m-\alpha}\left((x-a)^{\beta}\right)^{(m)} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-m+1)}{}_aI_x^{m-\alpha}(x-a)^{\beta-m} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. \end{aligned} $$

Theorem 11. *The following relations between the Riemann-Liouville and the Caputo fractional derivatives hold [8]:*

$$ {}_a^CD_x^{\alpha}f(x) = {}_aD_x^{\alpha}f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha}, $$

$$ {}_x^CD_b^{\alpha}f(x) = {}_xD_b^{\alpha}f(x) - \sum_{k=0}^{m-1}\frac{(-1)^kf^{(k)}(b)}{\Gamma(k+1-\alpha)}(b-x)^{k-\alpha}. $$

We now list some known properties of some special functions,

$$ \Gamma(2x) = \frac{2^{2x-1}}{\pi}\Gamma(x)\Gamma(x+\frac{1}{2}), \quad \text{for all } 2x \neq 0, -1, -2, \cdots \tag{1} $$

$$ b(\alpha,\beta;x) = \frac{x^{\alpha}}{\alpha}F_{21}(\alpha, 1-\beta; \alpha+1; x), \tag{2} $$

$$ F_{21}(a,b;c;x) = (1-x)^{c-a-b}F_{21}(c-a, c-b; c; x), \tag{3} $$

$$ \frac{d}{dx}\left(x^{\nu}K_{\nu}(x)\right) = -x^{\nu}K_{\nu-1}(x), $$

$$ (x)_{2n} = 2^{2n}\left(\frac{x}{2}\right)_n\left(\frac{1+x}{2}\right)_n, $$

$$ D^m x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)}x^{\lambda-m}, \quad m \in \mathbb{N}, $$

where $\Gamma(x)$, $b(\alpha,\beta;x)$, $F_{pq}(a_1,\ldots,a_p;b_1,\ldots,b_q;x)$, $K_\nu(x)$, and $(x)_n$ denote the Gamma function, lower incomplete Beta function, Hypergeometric series, modified Bessel function of the second kind, and Pochhammer symbol, respectively.

3. Fractional derivatives of RBFs in one dimension

Since RBFs are usually evaluated on Euclidean distances, we have to evaluate

$$\mathcal{D}^\alpha\phi(r) = \mathcal{D}^\alpha\phi(|x-y|), \quad \text{for all } x,y\in\mathbb{R},$$

where $\mathcal{D}^\alpha$ can be one of the notations used for fractional integrals and derivatives in section 2 and $\phi(r)$ is one of the RBFs listed in Table 1, [25]. The following theorems show that finding the fractional integrals and derivatives of $\phi(x)$ can lead to those of RBFs in one dimension.

Theorem 12. *For all $x,y\in\mathbb{R}$ and $x>a$ we have*

$${}_aI_x^{\alpha}\phi(|x-y|) = \xi^\alpha\left({}_{\xi(a-y)}I_x^{\alpha}\phi\right)(|x-y|),$$

where $\xi = sign(x-y)$.

Proof. We get

$${}_aI_x^{\alpha}\phi(x-y) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\phi(\tau-y)d\tau = \frac{1}{\Gamma(\alpha)}\int_{a-y}^{x-y}(x-y-u)^{\alpha-1}\phi(u)du,$$

where $u=\tau-y$. Then

$${}_aI_x^{\alpha}\phi(x-y) = \left({}_{(a-y)}I_x^{\alpha}\phi\right)(x-y). \tag{4}$$

Moreover,

$${}_aI_x^{\alpha}\phi(y-x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\phi(y-\tau)d\tau = \frac{(-1)^\alpha}{\Gamma(\alpha)}\int_{y-a}^{y-x}(y-x-u)^{\alpha-1}\phi(u)du,$$

where $u=y-\tau$. Then

$${}_aI_x^{\alpha}\phi(y-x) = (-1)^\alpha\left({}_{(y-a)}I_x^{\alpha}\phi\right)(y-x). \tag{5}$$

Then (4) and (5) give the result.

Remark 1. Similarly, one can show that for all $x,y\in\mathbb{R}$ and $x<b$

$${}_xI_b^{\alpha}\phi(|x-y|) = (-\xi)^\alpha\left({}_{\xi(b-y)}I_x^{\alpha}\phi\right)(|x-y|),$$

where $\xi = sign(x-y)$.

Theorem 13. *For all $x,y\in\mathbb{R}$ and $x>a$ we have*

$${}_aD_x^{\alpha}\phi(|x-y|) = \xi^{-\alpha}\left({}_{\xi(a-y)}D_x^{\alpha}\phi\right)(|x-y|),$$

where $\xi = sign(x-y)$.

Proof. We get

$${}_aD_x^{\alpha}\phi(x-y) = D^m[{}_aI_x^{m-\alpha}\phi(x-y)] = D^m[\left({}_{(a-y)}I_x^{m-\alpha}\phi\right)(x-y)] = \left({}_{(a-y)}D_x^{\alpha}\phi\right)(x-y). \tag{6}$$

Moreover,

$$\begin{aligned}{}_aD_x^{\alpha}\phi(y-x) &= D^m[{}_aI_x^{m-\alpha}\phi(y-x)] = D^m[(-1)^{m-\alpha}\left({}_{(y-a)}I_x^{m-\alpha}\phi\right)(y-x)] \\ &= (-1)^{2m-\alpha}D^m[\left({}_{(y-a)}I_x^{m-\alpha}\phi\right)(y-x)] = (-1)^{-\alpha}\left({}_{(y-a)}D_x^{\alpha}\phi\right)(y-x).\end{aligned} \tag{7}$$

Then (6) and (7) give the result.

Remark 2. Similarly, one can show that for all $x, y \in \mathbb{R}$ and $x < b$

$$ {}_{x}D_{b}^{\alpha}\phi(|x-y|) = (-\xi)^{-\alpha} \left({}_{\xi(b-y)}D_{x}^{\alpha}\phi \right)(|x-y|), $$

where $\xi = sign(x-y)$.

Theorem 14. *For all $x, y \in \mathbb{R}$ and $x > a$ we have*

$$ {}_{a}^{C}D_{x}^{\alpha}\phi(|x-y|) = \xi^{-\alpha} \left({}_{\xi(a-y)}^{C}D_{x}^{\alpha}\phi \right)(|x-y|), $$

where $\xi = sign(x-y)$.

Proof. We get

$$ \begin{aligned} {}_{a}^{C}D_{x}^{\alpha}\phi(x-y) &= {}_{a}I_{x}^{m-\alpha}[(\phi(x-y))^{(m)}] = {}_{a}I_{x}^{m-\alpha}[\phi^{(m)}(x-y)] = ({}_{(a-y)}I_{x}^{m-\alpha}\phi^{(m)})(x-y) \\ &= ({}_{(a-y)}^{C}D_{x}^{\alpha}\phi)(x-y). \end{aligned} \tag{8} $$

Moreover,

$$ \begin{aligned} {}_{a}^{C}D_{x}^{\alpha}\phi(y-x) &= {}_{a}I_{x}^{m-\alpha}[(\phi(y-x))^{(m)}] = (-1)^{m}{}_{a}I_{x}^{m-\alpha}[\phi^{(m)}(y-x)] \\ &= (-1)^{2m-\alpha}({}_{(y-a)}I_{x}^{m-\alpha}\phi^{(m)})(y-x) = (-1)^{-\alpha}({}_{(y-a)}^{C}D_{x}^{\alpha}\phi)(y-x). \end{aligned} \tag{9} $$

Then (8) and (9) give the result.

Remark 3. Similarly, one can show that for all $x, y \in \mathbb{R}$ and $x < b$

$$ {}_{x}^{C}D_{b}^{\alpha}\phi(|x-y|) = (-\xi)^{-\alpha} \left({}_{\xi(b-y)}^{C}D_{x}^{\alpha}\phi \right)(|x-y|), $$

where $\xi = sign(x-y)$.

In the sequel, we evaluate the Riemann-Liouville fractional integral and derivative, and also the Caputo fractional derivative of $\phi(x)$ corresponding to the five kinds of RBFs listed in Table 1.

3.1. Powers

For $\phi(x) = x^{\beta}$ the following results hold.

Theorem 15. *For $x > 0$ we have*

$$ \begin{aligned} {}_{0}I_{x}^{\alpha}x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}, \quad \beta > -1 \\ {}_{0}D_{x}^{\alpha}x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad \beta > \alpha - 1 \\ {}_{0}^{C}D_{x}^{\alpha}x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad \beta > \alpha - 1. \end{aligned} $$

Proof. Theorems 8, 9 and 10 for $a = 0$ give directly the results.

Theorem 16. *For $a \neq 0$, $n \in \mathbb{N}$ and $x > a$ we have*

$$ \begin{aligned} {}_{a}I_{x}^{\alpha}x^{n} &= n!(x-a)^{\alpha}\sum_{k=0}^{n}\frac{a^{n-k}(x-a)^{k}}{(n-k)!\Gamma(\alpha+k+1)}, \\ {}_{a}D_{x}^{\alpha}x^{n} &= n!(x-a)^{-\alpha}\sum_{k=0}^{n}\frac{a^{n-k}(x-a)^{k}}{(n-k)!\Gamma(k-\alpha+1)}, \\ {}_{a}^{C}D_{x}^{\alpha}x^{n} &= n!a^{-m}(x-a)^{m-\alpha}\sum_{k=0}^{n-m}\frac{a^{n-k}(x-a)^{k}}{(n-m-k)!\Gamma(m-\alpha+k+1)}. \end{aligned} $$

Proof. The Taylor expansion of x^n about the point $x=a$ gives

$$x^n = \sum_{k=0}^{n} \frac{n! a^{n-k}}{(n-k)!k!}(x-a)^k.$$

Now, according to the linearity of the Riemann-Liouville fractional integral and derivative, we have

$$\begin{aligned} {}_aI_x^{\alpha} x^n &= \sum_{k=0}^{n} \frac{n! a^{n-k}}{(n-k)!k!}\, {}_aI_x^{\alpha}(x-a)^k = \sum_{k=0}^{n} \frac{n! a^{n-k}}{(n-k)!k!}\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}(x-a)^{\alpha+k} \\ &= n!(x-a)^{\alpha}\sum_{k=0}^{n}\frac{a^{n-k}(x-a)^k}{(n-k)!\Gamma(\alpha+k+1)}. \end{aligned}$$

$$\begin{aligned} {}_aD_x^{\alpha} x^n &= \sum_{k=0}^{n} \frac{n! a^{n-k}}{(n-k)!k!}\, {}_aD_x^{\alpha}(x-a)^k = \sum_{k=0}^{n} \frac{n! a^{n-k}}{(n-k)!k!}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \\ &= n!(x-a)^{-\alpha}\sum_{k=0}^{n}\frac{a^{n-k}(x-a)^k}{(n-k)!\Gamma(k-\alpha+1)}. \end{aligned}$$

$$\begin{aligned} {}_a^C D_x^{\alpha} x^n &= {}_aI_x^{m-\alpha}(x^n)^{(m)} = \frac{n!}{(n-m)!}\, {}_aI_x^{m-\alpha} x^{n-m} \\ &= n! a^{-m}(x-a)^{m-\alpha}\sum_{k=0}^{n-m}\frac{a^{n-k}(x-a)^k}{(n-k)!\Gamma(m-\alpha+k+1)}. \end{aligned}$$

Remark 4. Similarly, one can show that for $x<b$ and $n\in\mathbb{N}$

$$\begin{aligned} {}_xI_b^{\alpha} x^n &= n!(b-x)^{\alpha}\sum_{k=0}^{n}\frac{b^{n-k}(x-b)^k}{(n-k)!\Gamma(\alpha+k+1)}, \\ {}_xD_b^{\alpha} x^n &= n!(b-x)^{-\alpha}\sum_{k=0}^{n}\frac{b^{n-k}(x-b)^k}{(n-k)!\Gamma(k-\alpha+1)}, \\ {}_x^C D_b^{\alpha} x^n &= (-1)^m n! b^{-m}(b-x)^{m-\alpha}\sum_{k=0}^{n-m}\frac{b^{n-k}(x-b)^k}{(n-m-k)!\Gamma(m-\alpha+k+1)}. \end{aligned}$$

3.2. Gaussian

For $\phi(x)=\exp(-x^2/2)$ the following results hold.

Theorem 17. *For $x>0$ we have*

$$\begin{aligned} {}_0I_x^{\alpha} e^{-\frac{x^2}{2}} &= \frac{x^{\alpha}}{\Gamma(1+\alpha)} F_{22}(\frac{1}{2},1;\frac{1+\alpha}{2},\frac{2+\alpha}{2};-\frac{x^2}{2}), \\ {}_0D_x^{\alpha} e^{-\frac{x^2}{2}} &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{22}(\frac{1}{2},1;\frac{1-\alpha}{2},\frac{2-\alpha}{2};-\frac{x^2}{2}), \\ {}_0^C D_x^{\alpha} e^{-\frac{x^2}{2}} &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{22}(\frac{1}{2},1;\frac{1-\alpha}{2},\frac{2-\alpha}{2};-\frac{x^2}{2}). \end{aligned}$$

Proof. The Taylor expansion about the point $x=0$ gives

$$\mathrm{e}^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n}.$$

Therefore

$$\begin{aligned}
{}_0I_x^{\alpha}\mathrm{e}^{-\frac{x^2}{2}} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\,{}_0I_x^{\alpha}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\frac{\Gamma(2n+1)}{\Gamma(2n+\alpha+1)}x^{2n+\alpha}\\
&= \frac{x^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\frac{(1)_{2n}}{(1+\alpha)_{2n}n!}(-\frac{x^2}{2})^n = \frac{x^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(1)_n}{(\frac{1+\alpha}{2})_n(\frac{2+\alpha}{2})_n n!}(-\frac{x^2}{2})^n\\
&= \frac{x^{\alpha}}{\Gamma(1+\alpha)}F_{22}(\frac{1}{2},1;\frac{1+\alpha}{2},\frac{2+\alpha}{2};-\frac{x^2}{2}).
\end{aligned}$$

$$\begin{aligned}
{}_0D_x^{\alpha}\mathrm{e}^{-\frac{x^2}{2}} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\,{}_0D_x^{\alpha}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\frac{\Gamma(2n+1)}{\Gamma(2n-\alpha+1)}x^{2n-\alpha}\\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{(1)_{2n}}{(1-\alpha)_{2n}n!}(-\frac{x^2}{2})^n = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(1)_n}{(\frac{1-\alpha}{2})_n(\frac{2-\alpha}{2})_n n!}(-\frac{x^2}{2})^n\\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}F_{22}(\frac{1}{2},1;\frac{1-\alpha}{2},\frac{2-\alpha}{2};-\frac{x^2}{2}).
\end{aligned}$$

$${}_0^C\!D_x^{\alpha}\mathrm{e}^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\,{}_0^C\!D_x^{\alpha}x^{2n} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}F_{22}(\frac{1}{2},1;\frac{1-\alpha}{2},\frac{2-\alpha}{2};-\frac{x^2}{2}).$$

Theorem 18. *For $a \neq 0$ and $x > a$ we have*

$$\begin{aligned}
{}_aI_x^{\alpha}e^{-\frac{x^2}{2}} &= (x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(\alpha+k+1)}\right),\\
{}_aD_x^{\alpha}e^{-\frac{x^2}{2}} &= (x-a)^{-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)}\right),\\
{}_a^C\!D_x^{\alpha}e^{-\frac{x^2}{2}} &= a^{-m}(x-a)^{m-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}\left(\sum_{k=0}^{2n-m}\frac{a^{2n-k}(x-a)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)}\right).
\end{aligned}$$

Proof.

$$\begin{aligned}
{}_aI_x^{\alpha}\mathrm{e}^{-\frac{x^2}{2}} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\,{}_aI_x^{\alpha}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}(2n)!(x-a)^{\alpha}\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(\alpha+k+1)}\\
&= (x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(\alpha+k+1)}\right).
\end{aligned}$$

$$\begin{aligned}
{}_aD_x^{\alpha}\mathrm{e}^{-\frac{x^2}{2}} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\,{}_aD_x^{\alpha}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}(2n)!(x-a)^{-\alpha}\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)}\\
&= (x-a)^{-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)}\right).
\end{aligned}$$

$$\begin{aligned}
{}_a^C\!D_x^{\alpha}\mathrm{e}^{-\frac{x^2}{2}} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\,{}_a^C\!D_x^{\alpha}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}(2n)!a^{-m}(x-a)^{m-\alpha}\sum_{k=0}^{2n-m}\frac{a^{2n-k}(x-a)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)}\\
&= a^{-m}(x-a)^{m-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^n n!}\left(\sum_{k=0}^{2n-m}\frac{a^{2n-k}(x-a)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)}\right).
\end{aligned}$$

Remark 5. Similarly, one can show that for $x < b$

$$
\begin{aligned}
{}_xI_b^{\alpha} e^{-\frac{x^2}{2}} &= (b-x)^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} \left(\sum_{k=0}^{2n} \frac{b^{2n-k}(x-b)^k}{(2n-k)!\Gamma(\alpha+k+1)} \right), \\
{}_xD_b^{\alpha} e^{-\frac{x^2}{2}} &= (b-x)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} \left(\sum_{k=0}^{2n} \frac{b^{2n-k}(x-b)^k}{(2n-k)!\Gamma(k-\alpha+1)} \right), \\
{}_x^C D_b^{\alpha} e^{-\frac{x^2}{2}} &= (-1)^m b^{-m} (b-x)^{m-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n n!} \left(\sum_{k=0}^{2n-m} \frac{b^{2n-k}(x-b)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)} \right).
\end{aligned}
$$

3.3. Multiquadric

For $\phi(x) = (1 + x^2/2)^{\beta/2}$, $\beta \in \mathbb{R}$ the following results hold.

Theorem 19. *For $x > 0$ we have*

$$
\begin{aligned}
{}_0I_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \frac{x^{\alpha}}{\Gamma(1+\alpha)} F_{32}(\frac{1}{2}, 1, -\frac{\beta}{2}; \frac{1+\alpha}{2}, \frac{2+\alpha}{2}; -\frac{x^2}{2}), \\
{}_0D_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{22}(\frac{1}{2}, 1, -\frac{\beta}{2}; \frac{1-\alpha}{2}, \frac{2-\alpha}{2}; -\frac{x^2}{2}), \\
{}_0^C D_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{22}(\frac{1}{2}, 1, -\frac{\beta}{2}; \frac{1-\alpha}{2}, \frac{2-\alpha}{2}; -\frac{x^2}{2}).
\end{aligned}
$$

Proof. The Taylor expansion about the point $x = 0$ gives

$$
(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \sum_{n=0}^{\infty} \frac{(\frac{\beta}{2})^{(n)}}{2^n n!} x^{2n}.
$$

Therefore

$$
\begin{aligned}
{}_0I_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \sum_{n=0}^{\infty} \frac{(\frac{\beta}{2})^{(n)}}{2^n n!} {}_0I_x^{\alpha} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-\frac{\beta}{2})_n}{2^n n!} \frac{\Gamma(2n+1)}{\Gamma(2n+\alpha+1)} x^{2n+\alpha} \\
&= \frac{x^{\alpha}}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \frac{(1)_{2n}(-\frac{\beta}{2})_n}{(1+\alpha)_{2n} n!} (-\frac{x^2}{2})^n = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (1)_n (-\frac{\beta}{2})_n}{(\frac{1+\alpha}{2})_n (\frac{2+\alpha}{2})_n n!} (-\frac{x^2}{2})^n \\
&= \frac{x^{\alpha}}{\Gamma(1+\alpha)} F_{32}(\frac{1}{2}, 1, -\frac{\beta}{2}; \frac{1+\alpha}{2}, \frac{2+\alpha}{2}; -\frac{x^2}{2}).
\end{aligned}
$$

$$
\begin{aligned}
{}_0D_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \sum_{n=0}^{\infty} \frac{(-1)^n(-\frac{\beta}{2})_n}{2^n n!} {}_0D_x^{\alpha} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-\frac{\beta}{2})_n}{2^n n!} \frac{\Gamma(2n+1)}{\Gamma(2n-\alpha+1)} x^{2n-\alpha} \\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{(1)_{2n}(-\frac{\beta}{2})_n}{(1-\alpha)_{2n} n!} (-\frac{x^2}{2})^n = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (1)_n (-\frac{\beta}{2})_n}{(\frac{1-\alpha}{2})_n (\frac{2-\alpha}{2})_n n!} (-\frac{x^2}{2})^n \\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{32}(\frac{1}{2}, 1, -\frac{\beta}{2}; \frac{1-\alpha}{2}, \frac{2-\alpha}{2}; -\frac{x^2}{2}).
\end{aligned}
$$

$$
{}_0^C D_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n(-\frac{\beta}{2})_n}{2^n n!} {}_0^C D_x^{\alpha} x^{2n} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} F_{32}(\frac{1}{2}, 1, -\frac{\beta}{2}; \frac{1-\alpha}{2}, \frac{2-\alpha}{2}; -\frac{x^2}{2}).
$$

Theorem 20. *For $a \neq 0$ and $x > a$ we have*

$$ {}_aI_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \Gamma(\frac{\beta}{2}+1)(x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(\alpha+k+1)}\right), $$

$$ {}_aD_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \Gamma(\frac{\beta}{2}+1)(x-a)^{-\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)}\right), $$

$$ {}_a^CD_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \Gamma(\frac{\beta}{2}+1)a^{-m}(x-a)^{m-\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n-m}\frac{a^{2n-k}(x-a)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)}\right). $$

PROOF.

$$ \begin{aligned} {}_aI_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \sum_{n=0}^{\infty}\frac{(\frac{\beta}{2})^{(n)}}{2^n n!}\,{}_aI_x^{\alpha}x^{2n} \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{\beta}{2}+1)}{\Gamma(\frac{\beta}{2}-n+1)2^n n!}(2n)!(x-a)^{\alpha}\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(\alpha+k+1)} \\ &= \Gamma(\frac{\beta}{2}+1)(x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(\alpha+k+1)}\right). \end{aligned} $$

$$ \begin{aligned} {}_aD_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \sum_{n=0}^{\infty}\frac{(\frac{\beta}{2})^{(n)}}{2^n n!}\,{}_aD_x^{\alpha}x^{2n} \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{\beta}{2}+1)}{\Gamma(\frac{\beta}{2}-n+1)2^n n!}(2n)!(x-a)^{-\alpha}\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)} \\ &= \Gamma(\frac{\beta}{2}+1)(x-a)^{-\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)}\right). \end{aligned} $$

$$ \begin{aligned} {}_a^CD_x^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} &= \sum_{n=0}^{\infty}\frac{(\frac{\beta}{2})^{(n)}}{2^n n!}\,{}_a^CD_x^{\alpha}x^{2n} \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(\frac{\beta}{2}+1)}{\Gamma(\frac{\beta}{2}-n+1)2^n n!}(2n)!a^{-m}(x-a)^{m-\alpha}\sum_{k=0}^{2n-m}\frac{a^{2n-k}(x-a)^k}{(2n-m+k)!\Gamma(m-\alpha+k+1)} \\ &= \Gamma(\frac{\beta}{2}+1)a^{-m}(x-a)^{m-\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n-m}\frac{a^{2n-k}(x-a)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)}\right). \end{aligned} $$

Remark 6. Similarly, one can show that for $x < b$

$$ {}_xI_b^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \Gamma(\frac{\beta}{2}+1)(b-x)^{\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n}\frac{b^{2n-k}(x-b)^k}{(2n-k)!\Gamma(\alpha+k+1)}\right), $$

$$ {}_xD_b^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = \Gamma(\frac{\beta}{2}+1)(b-x)^{-\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n}\frac{b^{2n-k}(x-b)^k}{(2n-k)!\Gamma(k-\alpha+1)}\right), $$

$$ {}_x^CD_b^{\alpha}(1+\frac{x^2}{2})^{\frac{\beta}{2}} = (-1)^m\Gamma(\frac{\beta}{2}+1)b^{-m}(b-x)^{m-\alpha}\sum_{n=0}^{\infty}\frac{(2n)!}{2^n n!\Gamma(\frac{\beta}{2}-n+1)}\left(\sum_{k=0}^{2n-m}\frac{b^{2n-k}(x-b)^k}{(2n-m-k)!\Gamma(m-\alpha+k+1)}\right). $$

3.4. Thin-plate splines

For $\phi(x) = x^{2n}\ln(x)$, $n \in \mathbb{N}$ the following results hold.

Theorem 21. *For $x > 0$ we have*

$$
\begin{aligned}
{}_0I_x^{\alpha} x^{2n}\ln(x) &= \frac{\Gamma(2n+1)}{\Gamma(2n+1+\alpha)} x^{\alpha+2n}\left[\ln(x) + \Psi(2n+1) - \Psi(2n+1+\alpha)\right],\\
{}_0^{C}D_x^{\alpha} x^{2n}\ln(x) &= \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha)} x^{2n-\alpha}[\ln(x) + \Psi(2n-m+1) - \Psi(2n+1-\alpha)\\
&\quad + m!\Gamma(2n-m+1)\sum_{r=1}^{m}\frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}],\\
{}_0D_x^{\alpha} x^{2n}\ln(x) &= \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha)} x^{2n-\alpha}[\ln(x) + \Psi(2n-m+1) - \Psi(2n+1-\alpha)\\
&\quad + m!\Gamma(2n-m+1)\sum_{r=1}^{m}\frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}],
\end{aligned}
$$

where $\Psi(x)$ is the logarithmic derivative of the Gamma function.

Proof.

$$
\begin{aligned}
{}_aI_x^{\alpha} x^{2n}\ln(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\tau^{2n}\ln(\tau)d\tau = \frac{1}{2}\frac{d}{dn}\left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\tau^{2n}d\tau\right)\\
&= \frac{1}{2}\frac{d}{dn}\left({}_aI_x^{\alpha} x^{2n}\right).
\end{aligned}
$$

Then it suffices to find the derivative of the Riemann-Liouville fractional integral of the powers RBF x^{2n} with respect to n. Therefore

$$
{}_0I_x^{\alpha} x^{2n}\ln(x) = \frac{1}{2}\frac{d}{dn}\left(\frac{\Gamma(2n+1)}{\Gamma(\alpha+2n+1)} x^{\alpha+2n}\right),
$$

which in turn gives

$$
{}_0I_x^{\alpha} x^{2n}\ln(x) = \frac{\left(\Gamma'(2n+1)x^{\alpha+2n} + \Gamma(2n+1)x^{\alpha+2n}\ln(x)\right)\Gamma(2n+1+\alpha) - \Gamma'(2n+1+\alpha)\Gamma(2n+1)x^{\alpha+2n}}{\left(\Gamma(2n+1+\alpha)\right)^2}.
$$

Now by substituting

$$
\begin{aligned}
\Gamma'(2n+1) &= \Psi(2n+1)\Gamma(2n+1),\\
\Gamma'(2n+1+\alpha) &= \Psi(2n+1+\alpha)\Gamma(2n+1+\alpha),
\end{aligned}
$$

we have

$$
{}_0I_x^{\alpha} x^{2n}\ln(x) = \frac{\Gamma(2n+1)}{\Gamma(2n+1+\alpha)} x^{\alpha+2n}\left[\ln(x) + \Psi(2n+1) - \Psi(2n+1+\alpha)\right]. \tag{10}
$$

Since it is well known that for the thin-plate splines and their derivatives at $x = 0$ the limiting value 0 is considered avoiding the singularity, according to theorem 11 we have

$$
{}_0D_x^{\alpha} x^{2n}\ln(x) = {}_0^{C}D_x^{\alpha} x^{2n}\ln(x).
$$

Now for the Caputo fractional derivative, we have

$$
{}_0^{C}D_x^{\alpha} x^{2n}\ln(x) = {}_0I_x^{m-\alpha}\left(x^{2n}\ln(x)\right)^{(m)}.
$$

But for $x \neq 0$

$$\begin{aligned}
\left(x^{2n}\ln(x)\right)^{(m)} &= \sum_{r=0}^{m} \left(\begin{array}{c} m \\ r \end{array} \right) \frac{d^{m-r}}{dx^{m-r}} x^{2n} \frac{d^r}{dx^r} \ln(x) \\
&= \frac{\Gamma(2n+1)}{\Gamma(2n-m+1)} x^{2n-m} \ln(x) + \sum_{r=1}^{m} \left(\begin{array}{c} m \\ r \end{array} \right) \frac{d^{m-r}}{dx^{m-r}} x^{2n} \frac{d^r}{dx^r} \ln(x) \\
&= \frac{\Gamma(2n+1)}{\Gamma(2n-m+1)} x^{2n-m} \ln(x) + \sum_{r=1}^{m} \left(\begin{array}{c} m \\ r \end{array} \right) \frac{\Gamma(2n+1)x^{2n-m+r}}{\Gamma(2n-m+r+1)} \frac{(-1)^{r-1}(r-1)!}{x^r} \\
&= \frac{\Gamma(2n+1)}{\Gamma(2n-m+1)} x^{2n-m} \ln(x) + m!\Gamma(2n+1)x^{2n-m} \sum_{r=1}^{m} \frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
{}_0^C\!D_x^{\alpha} x^{2n} \ln(x) &= \frac{\Gamma(2n+1)}{\Gamma(2n-m+1)} {}_0I_x^{m-\alpha} x^{2n-m} \ln(x) \\
&\quad + m!\Gamma(2n+1) \left(\sum_{r=1}^{m} \frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)} \right) {}_0I_x^{m-\alpha} x^{2n-m} \\
&= \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha)} x^{2n-\alpha} [\ln(x) + \Psi(2n-m+1) - \Psi(2n+1-\alpha) \\
&\quad + m!\Gamma(2n-m+1) \sum_{r=1}^{m} \frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}].
\end{aligned}$$

Theorem 22. *For $a \neq 0$ and $x > a$ we have*

$$\begin{aligned}
{}_aI_x^{\alpha} x^{2n} \ln(x) &= \frac{x^{2n+\alpha}(x-a)^{\alpha}}{a^{\alpha}\Gamma(1+\alpha)} \Bigg(F_{21}(\alpha, \alpha+2n+1; \alpha+1; \frac{a-x}{a}) (\ln(x) - \Psi(\alpha+2n+1)) \\
&\quad + \alpha \sum_{k=0}^{\infty} \frac{(\alpha+2n+1)_k \Psi(\alpha+2n+k+1)(a-x)^k}{(\alpha+k)a^k k!} \Bigg), \\
{}_a^C\!D_x^{\alpha} x^{2n} \ln(x) &= \frac{\Gamma(2n+1)x^{2n-\alpha}(x-a)^{m-\alpha}}{a^{m-\alpha}\Gamma(1+m-\alpha)} \Bigg(F_{21}(m-\alpha, 2n-\alpha+1; m-\alpha+1; \frac{a-x}{a}) \\
&\quad \left(\frac{\ln(x) - \Psi(2n-\alpha+1)}{\Gamma(2n-m+1)} + m! \sum_{r=1}^{m} \frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)} \right) \\
&\quad + \frac{(m-\alpha)}{\Gamma(n-m+1)} \sum_{k=0}^{\infty} \frac{(2n-\alpha+1)_k \Psi(2n-\alpha+k+1)}{(m-\alpha+k)a^k k!} (a-x)^k \Bigg).
\end{aligned}$$

Proof. We know that

$${}_aI_x^{\alpha} x^{\beta} = \frac{a^{\beta}(x-a)^{\alpha}}{\Gamma(1+\alpha)} F_{21}(1, -\beta; \alpha+1; \frac{a-x}{a}).$$

Then by using (3), we get

$${}_aI_x^{\alpha} x^{\beta} = \frac{x^{\alpha+\beta}(x-a)^{\alpha}}{a^{\alpha}\Gamma(1+\alpha)} F_{21}(\alpha, \alpha+\beta+1; \alpha+1; \frac{a-x}{a}). \tag{11}$$

Thus

$$
\begin{aligned}
{}_aI_x^{\alpha} x^{2n}\ln(x) &= \frac{1}{2}\frac{d}{dn}\left(\frac{x^{\alpha+2n}(x-a)^{\alpha}}{a^{\alpha}\Gamma(1+\alpha)}F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a})\right)\\
&= \frac{x^{\alpha}(x-a)^{\alpha}}{2a^{\alpha}\Gamma(1+\alpha)}\frac{d}{dn}\left(x^{2n}F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a})\right)\\
&= \frac{x^{\alpha}(x-a)^{\alpha}}{2a^{\alpha}\Gamma(1+\alpha)}\left(2x^{2n}\ln(x)F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a})\right.\\
&\quad \left.+\,x^{2n}\frac{d}{dn}(F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a}))\right). \qquad (12)
\end{aligned}
$$

Moreover,

$$
\frac{d}{dn}\left(F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a})\right) = \frac{d}{dn}\left(\sum_{k=0}^{\infty}\frac{(\alpha)_k(\alpha+2n+1)_k}{(\alpha+1)_k}\frac{(a-x)^k}{a^k k!}\right),
$$

and so

$$
\begin{aligned}
&\frac{d}{dn}\left(F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a})\right)\\
&= \sum_{k=0}^{\infty}\frac{(\alpha)_k(a-x)^k}{(\alpha+1)_k a^k k!}\left(\frac{2\Gamma'(\alpha+2n+k+1)\Gamma(\alpha+2n+1)-2\Gamma'(\alpha+2n+1)\Gamma(\alpha+2n+k+1)}{\Gamma(\alpha+2n+1)^2}\right),\\
&= \sum_{k=0}^{\infty}\frac{2(\alpha)_k(a-x)^k}{(\alpha+1)_k a^k k!}\left(\frac{\Gamma(\alpha+2n+k+1)\left(\Psi(\alpha+2n+k+1)-\Psi(\alpha+2n+1)\right)}{\Gamma(\alpha+2n+1)}\right),\\
&= \sum_{k=0}^{\infty}\frac{2(\alpha)_k(a-x)^k}{(\alpha+1)_k a^k k!}\left((\alpha+2n+1)_k\left(\Psi(\alpha+2n+k+1)-\Psi(\alpha+2n+1)\right)\right),\\
&= 2\alpha\sum_{k=0}^{\infty}\frac{(\alpha+2n+1)_k\Psi(\alpha+2n+k+1)(a-x)^k}{(\alpha+k)a^k k!} - 2\Psi(\alpha+2n+1)F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a}).
\end{aligned}
$$

Then by substituting the relation above into (12) and simplifying the expressions, we obtain

$$
\begin{aligned}
{}_aI_x^{\alpha} x^{2n}\ln(x) &= \frac{x^{2n+\alpha}(x-a)^{\alpha}}{a^{\alpha}\Gamma(1+\alpha)}\left(F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{a-x}{a})\left(\ln(x)-\Psi(\alpha+2n+1)\right)\right.\\
&\quad \left.+\alpha\sum_{k=0}^{\infty}\frac{(\alpha+2n+1)_k\Psi(\alpha+2n+k+1)(a-x)^k}{(\alpha+k)a^k k!}\right). \qquad (13)
\end{aligned}
$$

Now for the Caputo fractional derivative, we have

$$
{}_a^C D_x^{\alpha} x^{2n}\ln(x) = {}_aI_x^{m-\alpha}\left(x^{2n}\ln(x)\right)^{(m)}.
$$

Then by using (13) and (11), and after simplifying the expressions, we have

$$
\begin{aligned}
{}_a^C D_x^{\alpha} x^{2n}\ln(x) &= \frac{\Gamma(2n+1)}{\Gamma(2n-m+1)}\,{}_aI_x^{m-\alpha}x^{2n-m}\ln(x)\\
&\quad +m!\Gamma(2n+1)\left(\sum_{r=1}^{m}\frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}\right){}_aI_x^{m-\alpha}x^{2n-m}\\
&= \frac{\Gamma(2n+1)x^{2n-\alpha}(x-a)^{m-\alpha}}{a^{m-\alpha}\Gamma(m-\alpha+1)}\left(F_{21}(m-\alpha,2n-\alpha+1;m-\alpha+1;\frac{a-x}{a})\right.\\
&\quad \left(\frac{\ln(x)-\Psi(2n-\alpha+1)}{\Gamma(2n-m+1)}+m!\sum_{r=1}^{m}\frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}\right)\\
&\quad \left.+\frac{(m-\alpha)}{\Gamma(2n-m+1)}\sum_{k=0}^{\infty}\frac{(2n-\alpha+1)_k\Psi(2n-\alpha+k+1)}{(m-\alpha+k)a^k k!}(a-x)^k\right).
\end{aligned}
$$

Remark 7. Similarly, one can show that for $x < b$

$$
\begin{aligned}
{}_xI_b^{\alpha} x^{2n}\ln(x) &= \frac{x^{2n+\alpha}(b-x)^{\alpha}}{b^{\alpha}\Gamma(1+\alpha)}\Bigg(F_{21}(\alpha,\alpha+2n+1;\alpha+1;\frac{b-x}{b})\left(\ln(x)-\Psi(\alpha+2n+1)\right)\\
&\quad +\alpha\sum_{k=0}^{\infty}\frac{(\alpha+2n+1)_k\Psi(\alpha+2n+k+1)(b-x)^k}{(\alpha+k)b^k k!}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
{}_x^C D_b^{\alpha} x^{2n}\ln(x) &= \frac{(-1)^m\Gamma(2n+1)x^{2n-\alpha}(b-x)^{m-\alpha}}{b^{m-\alpha}\Gamma(m-\alpha+1)}\Bigg(F_{21}(m-\alpha,2n-\alpha+1;m-\alpha+1;\frac{b-x}{b})\\
&\quad \left(\frac{\ln(x)-\Psi(2n-\alpha+1)}{\Gamma(2n-m+1)}+m!\sum_{r=1}^{m}\frac{(-1)^{r-1}}{r(m-r)!\Gamma(2n-m+r+1)}\right)\\
&\quad +\frac{(m-\alpha)}{\Gamma(2n-m+1)}\sum_{k=0}^{\infty}\frac{(2n-\alpha+1)_k\Psi(2n-\alpha+k+1)}{(m-\alpha+k)a^k k!}(b-x)^k\Bigg).
\end{aligned}
$$

3.5. Matern

For $\phi(x) = x^{\nu}K_{\nu}(x)$ with non-integer $\nu > 0$, the following results hold.

Theorem 23. *For $x > a$ we have*

$$
\begin{aligned}
{}_aI_x^{\alpha} x^{\nu}K_{\nu}(x) &= \frac{\pi}{2\sin\pi\nu}\Bigg(\frac{2^{\nu}x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)}F_{23}(\frac{1}{2},1;1-\nu,\frac{\alpha+1}{2},\frac{\alpha+2}{2};\frac{x^2}{4})\\
&\quad -\frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)}x^{\alpha+2\nu}F_{12}(\nu+\frac{1}{2};\frac{\alpha+2\nu+1}{2};\frac{\alpha}{2}+\nu+1;\frac{x^2}{4})\\
&\quad -\frac{a2^{\nu}x^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{2k}}{(2k+1)\Gamma(-\nu+k+1)4^k k!}F_{21}(2k+1,1-\alpha;2k+2;\frac{a}{x})\\
&\quad +\frac{a^{2\nu+1}}{2^{\nu}\Gamma(\alpha)}x^{\alpha-1}\sum_{k=0}^{\infty}\Bigg(\frac{a^{2k}}{4^k k!(2\nu+k+1)\Gamma(\nu+k+1)}\\
&\quad F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{x^2}{4})\Bigg)\Bigg).
\end{aligned}
$$

Proof. We know that

$$
{}_aI_x^{\alpha} x^{\nu}K_{\nu}(x) = \frac{\pi}{2\sin(\pi\nu)}\left({}_aI_x^{\alpha} x^{\nu}J_{-\nu}(x) - {}_aI_x^{\alpha} x^{\nu}J_{\nu}(x)\right),
$$

where

$$
J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{x^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}.
$$

Now

$$
\begin{aligned}
{}_aI_x^{\alpha} x^{\nu}J_{\nu}(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\tau^{\nu}J_{\nu}(\tau)d\tau = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\tau^{\nu}\left(\frac{\tau}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{\tau^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}d\tau\\
&= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha)2^{\nu}4^k k!\Gamma(\nu+k+1)}\int_a^x (x-\tau)^{\alpha-1}\tau^{2\nu+2k}d\tau.
\end{aligned}
$$

By the change of variable $u = \frac{\tau}{x}$, we have

$$\begin{aligned}\int_a^x (x-\tau)^{\alpha-1}\tau^{2\nu+2k}d\tau &= x^{\alpha+2\nu+2k}\int_{\frac{a}{x}}^1 (1-u)^{\alpha-1}u^{2\nu+2k}du \\ &= x^{\alpha+2\nu+2k}\left(\frac{\Gamma(2\nu+2k+1)\Gamma(\alpha)}{\Gamma(\alpha+2\nu+2k+1)} - b(2\nu+2k+1,\alpha;\frac{a}{x})\right).\end{aligned}$$

Therefore

$$\begin{aligned}{}_aI_x^{\alpha} x^{\nu}J_{\nu}(x) &= \sum_{k=0}^{\infty}\frac{x^{\alpha+2\nu+2k}}{2^{\nu}4^k k!\Gamma(\alpha)\Gamma(\nu+k+1)}\left(\frac{\Gamma(2\nu+2k+1)\Gamma(\alpha)}{\Gamma(\alpha+2\nu+2k+1)} - b(2\nu+2k+1,\alpha;\frac{a}{x})\right) \\ &= \sum_{k=0}^{\infty}\frac{\Gamma(2\nu+2k)x^{\alpha+2\nu+2k}}{2^{\nu-1}4^k k!\Gamma(\nu+k)\Gamma(\alpha+2\nu+2k+1)} - \sum_{k=0}^{\infty}\frac{b(2\nu+2k+1,\alpha;\frac{a}{x})x^{\alpha+2\nu+2k}}{2^{\nu}4^k k!\Gamma(\alpha)\Gamma(\nu+k+1)}.\end{aligned}$$

Now by using (1) and (2) we get

$$\begin{aligned}{}_aI_x^{\alpha} x^{\nu}J_{\nu}(x) &= \sum_{k=0}^{\infty}\frac{2^{\nu}\Gamma(\nu+k+\frac{1}{2})x^{\alpha+2\nu+2k}}{\sqrt{\pi}\Gamma(\alpha+2\nu+2k+1)k!} \\ &\quad - \frac{a^{2\nu+1}x^{\alpha-1}}{2^{\nu}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{2k}}{4^k k!(2\nu+2k+1)\Gamma(\nu+k+1)}F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{a}{x}) \\ &= \frac{x^{\alpha+2\nu}2^{\nu}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(\nu+k+\frac{1}{2})x^{2k}}{\Gamma(\alpha+2\nu+2k+1)k!} \\ &\quad - \frac{a^{2\nu+1}x^{\alpha-1}}{2^{\nu}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{2k}}{4^k k!(2\nu+2k+1)\Gamma(\nu+k+1)}F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{a}{x}) \\ &= \frac{x^{\alpha+2\nu}2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)}\sum_{k=0}^{\infty}\frac{(\nu+\frac{1}{2})_k x^{2k}}{(\alpha+2\nu+1)_{2k}k!} \\ &\quad - \frac{a^{2\nu+1}x^{\alpha-1}}{2^{\nu}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{2k}}{4^k k!(2\nu+2k+1)\Gamma(\nu+k+1)}F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{a}{x}) \\ &= \frac{x^{\alpha+2\nu}2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)}\sum_{k=0}^{\infty}\frac{(\nu+\frac{1}{2})_k x^{2k}}{(\frac{\alpha}{2}+\nu+\frac{1}{2})_k(\frac{\alpha}{2}+\nu+1)_k 4^k k!} \\ &\quad - \frac{a^{2\nu+1}x^{\alpha-1}}{2^{\nu}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{2k}}{4^k k!(2\nu+2k+1)\Gamma(\nu+k+1)}F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{a}{x}) \\ &= \frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)}x^{\alpha+2\nu}F_{12}(\nu+\frac{1}{2};\frac{\alpha}{2}+\nu+\frac{1}{2},\frac{\alpha}{2}+\nu+1;\frac{x^2}{4}) \\ &\quad - \frac{a^{2\nu+1}x^{\alpha-1}}{2^{\nu}\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{2k}}{4^k k!(2\nu+2k+1)\Gamma(\nu+k+1)}F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{a}{x}).\end{aligned}$$

Moreover,

$$\begin{aligned}
{}_aI_x^{\alpha} x^{\nu} J_{-\nu}(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} \tau^{\nu} J_{-\nu}(\tau) d\tau = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} \tau^{\nu} \left(\frac{\tau}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{\tau^2}{4}\right)^k}{k!\Gamma(-\nu+k+1)} d\tau \\
&= \sum_{k=0}^{\infty} \frac{2^{\nu}}{\Gamma(\alpha) 4^k k! \Gamma(-\nu+k+1)} \int_a^x (x-\tau)^{\alpha-1} \tau^{2k} d\tau \\
&= \sum_{k=0}^{\infty} \frac{2^{\nu} x^{\alpha+2k}}{\Gamma(\alpha) 4^k k! \Gamma(-\nu+k+1)} \left(\frac{\Gamma(2k+1)\Gamma(\alpha)}{\Gamma(\alpha+2k+1)} - b(2k+1, \alpha; \frac{a}{x}) \right) \\
&= 2^{\nu} x^{\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(2k+1) x^{2k}}{4^k k! \Gamma(-\nu+k+1)\Gamma(\alpha+2k+1)} - \sum_{k=0}^{\infty} \frac{2^{\nu} x^{\alpha+2k} b(2k+1, \alpha; \frac{a}{x})}{4^k k! \Gamma(\alpha)\Gamma(-\nu+k+1)} \\
&= \frac{2^{\nu} x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)} \sum_{k=0}^{\infty} \frac{(1)_{2k} x^{2k}}{(1-\nu)_k (1+\alpha)_{2k} 4^k k!} \\
&- \frac{2^{\nu} x^{\alpha-1} a}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k (2k+1) \Gamma(-\nu+k+1) k!} F_{21}(2k+1, 1-\alpha; 2k+2; \frac{a}{x}) \\
&= \frac{2^{\nu} x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (1)_k x^{2k}}{(1-\nu)_k (\frac{1+\alpha}{2})_k (\frac{2+\alpha}{2})_k 4^k k!} \\
&- \frac{2^{\nu} x^{\alpha-1} a}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k (2k+1) \Gamma(-\nu+k+1) k!} F_{21}(2k+1, 1-\alpha; 2k+2; \frac{a}{x}) \\
&= \frac{2^{\nu} x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)} F_{23}(\frac{1}{2}, 1; 1-\nu, \frac{1+\alpha}{2}, \frac{2+\alpha}{2}; \frac{x^2}{4}) \\
&- \frac{2^{\nu} x^{\alpha-1} a}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{a^{2k}}{4^k (2k+1) \Gamma(-\nu+k+1) k!} F_{21}(2k+1, 1-\alpha; 2k+2; \frac{a}{x}).
\end{aligned}$$

Thus

$$\begin{aligned}
{}_aI_x^{\alpha} x^{\nu} K_{\nu}(x) &= \frac{\pi}{2\sin\pi\nu} \left(\frac{2^{\nu} x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)} F_{23}(\frac{1}{2}, 1; 1-\nu, \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \frac{x^2}{4}) \right. \\
&- \frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)} x^{\alpha+2\nu} F_{12}(\nu+\frac{1}{2}; \frac{\alpha+2\nu+1}{2}; \frac{\alpha}{2}+\nu+1; \frac{x^2}{4}) \\
&- \frac{2^{\nu} x^{\alpha-1} a}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k+1)\Gamma(-\nu+k+1) 4^k k!} F_{21}(2k+1, 1-\alpha; 2k+2; \frac{a}{x}) \\
&+ \frac{a^{2\nu+1}}{2^{\nu}\Gamma(\alpha)} x^{\alpha-1} \sum_{k=0}^{\infty} \left(\frac{a^{2k}}{4^k k! (2\nu+k+1)\Gamma(\nu+k+1)} \right. \\
&\left.\left. F_{21}(2\nu+2k+1, 1-\alpha; 2\nu+2k+2; \frac{x^2}{4}) \right) \right).
\end{aligned}$$

Theorem 24. *For $x > a$ we have*

$$ {}_a^C D_x^{\alpha} x^{\nu} K_{\nu}(x) = (-1)^m {}_aI_x^{m-\alpha} x^{\nu-m} K_{\nu-m}(x). $$

Proof. By definition, we have

$$ {}_a^C D_x^{\alpha} x^{\nu} K_{\nu}(x) = {}_aI_x^{m-\alpha} (x^{\nu} K_{\nu}(x))^{(m)} = (-1)^m {}_aI_x^{m-\alpha} x^{\nu-m} K_{\nu-m}(x). $$

Remark 8. For the special case $a = 0$ and $x > 0$ we have

$$\begin{aligned} {}_0I_x^{\alpha} x^{\nu} K_{\nu}(x) &= \frac{\pi}{2\sin \pi\nu}\left(\frac{2^{\nu}x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)}F_{23}(\frac{1}{2},1;1-\nu,\frac{\alpha+1}{2},\frac{\alpha+2}{2};\frac{x^2}{4})\right. \\ &\left. -\frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)}x^{\alpha+2\nu}F_{12}(\nu+\frac{1}{2};\frac{\alpha+2\nu+1}{2};\frac{\alpha}{2}+\nu+1;\frac{x^2}{4})\right). \end{aligned}$$

$${}_0^C D_x^{\alpha} x^{\nu} K_{\nu}(x) = (-1)^m {}_0I_x^{m-\alpha} x^{\nu-m} K_{\nu-m}(x).$$

Remark 9. Similarly, one can show that for $x < b$

$$\begin{aligned} {}_xI_b^{\alpha} x^{\nu} K_{\nu}(x) &= \frac{(-1)^{\alpha}\pi}{2\sin \pi\nu}\left(\frac{2^{\nu}x^{\alpha}}{\Gamma(1-\nu)\Gamma(1+\alpha)}F_{23}(\frac{1}{2},1;1-\nu,\frac{\alpha+1}{2},\frac{\alpha+2}{2};\frac{x^2}{4})\right. \\ &- \frac{2^{\nu}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+2\nu+1)}x^{\alpha+2\nu}F_{12}(\nu+\frac{1}{2};\frac{\alpha+2\nu+1}{2};\frac{\alpha}{2}+\nu+1;\frac{x^2}{4}) \\ &- \frac{2^{\nu}x^{\alpha-1}b}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{b^{2k}}{(2k+1)\Gamma(-\nu+k+1)4^k k!}F_{21}(2k+1,1-\alpha;2k+2;\frac{b}{x}) \\ &+ \frac{b^{2\nu+1}}{2^{\nu}\Gamma(\alpha)}x^{\alpha-1}\sum_{k=0}^{\infty}\left(\frac{b^{2k}}{4^k k!(2\nu+k+1)\Gamma(\nu+k+1)}\right. \\ &\left.\left. F_{21}(2\nu+2k+1,1-\alpha;2\nu+2k+2;\frac{x^2}{4})\right)\right), \end{aligned}$$

$${}_x^C D_b^{\alpha} x^{\nu} K_{\nu}(x) = (-1)^{-\alpha} {}_a^C D_x^{\alpha} x^{\nu} K_{\nu}(x).$$

4. Application

In this section we apply the results of the previous section to solve two fractional differential equations. The first one is a fractional ODE which is solved by the RBF collocation method and the second one is a fractional PDE which is solved by the method of lines based on the spatial trial spaces spanned by the Lagrange basis associated to the RBFs. In both cases, we work with scaled RBFs, i.e.

$$\phi\left(\frac{|x-y|}{c}\right),$$

where the RBF scale c controls their flatness. The infinite sums appearing in the previous formulas are truncated once the terms are smaller in magnitude than machine precision.

4.1. Test problem 1

Consider the following fractional ODE [22]:

$$\begin{aligned} &{}_0D_t^{3/2}u(t) + u(t) = f(t), \quad t \in (0,T], \\ &u(0) = u'(0) = 0. \end{aligned}$$

Let $t_i,\ 1 \le i \le n$ be the equidistant discretization points in the interval $[0,T]$ such that $t_1 = 0$ and $t_n = T$. Then the approximate solution can be written as

$$u(t) = \sum_{j=0}^{n}\lambda_j \phi(|t-t_j|),$$

where t_j's are known as centers. The unknown parameters λ_j are to be determined by the collocation method. Therefore, we get the following equations for the ODE

$$\sum_{j=0}^{n} \lambda_j \; {}_0D_t^{3/2} \phi(|t_i - t_j|) + \sum_{j=0}^{n} \lambda_j \phi(|t_i - t_j|) = f(t_i), \quad i = 2, \ldots n-1, \tag{14}$$

and the following equations for the initial conditions

$$\sum_{j=0}^{n} \lambda_j \; \phi(|t_1 - t_j|) = 0, \tag{15}$$

$$\sum_{j=0}^{n} \lambda_j \; \phi'(|t_1 - t_j|) = 0. \tag{16}$$

Then (14)-(16) lead to the following system of equations:

$$\begin{bmatrix} {}_0D_t^{3/2} \boldsymbol{\phi} + \boldsymbol{\phi} \\ \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_1' \end{bmatrix} [\boldsymbol{\lambda}] = \begin{bmatrix} \boldsymbol{F} \\ 0 \\ 0 \end{bmatrix}.$$

The necessary matrices and vectors are

$$\begin{aligned} \boldsymbol{\phi} &= (\phi(|t_i - t_j|))_{2 \le i \le n-1, 1 \le j \le n}, \\ {}_0D_t^{3/2} \boldsymbol{\phi} &= \left({}_0D_t^{3/2} \phi(|t_i - t_j|) \right)_{2 \le i \le n-1, 1 \le j \le n}, \\ \boldsymbol{\phi}_1 &= (\phi(|t_1 - t_j|))_{1 \le j \le n}, \\ \boldsymbol{\phi}_1' &= (\phi'(|t_1 - t_j|))_{1 \le j \le n}, \\ \boldsymbol{\lambda} &= (\lambda_j, 1 \le j \le n)^T, \\ \boldsymbol{F} &= (f(t_i), 2 \le i \le n-1)^T. \end{aligned}$$

Now, we take 501 points in the interval $0 \le t \le 50$ and work with the powers RBF $\phi(x) = x^3$ with $c = 10^{-4}$. The numerical solutions are plotted with different right-hand side functions $f(t) = 1$, $f(t) = t\mathrm{e}^{-t}$, and $f(t) = \mathrm{e}^{-t} \sin(0.2t)$ in Figures 1, 2, and 3, respectively. The results are in agreement with the results of [22].

4.2. Test problem 2

Consider the following Riesz fractional differential equation [28]:

$$\begin{aligned} &\tfrac{\partial u(x,t)}{\partial t} = -K_\alpha \tfrac{\partial^\alpha}{\partial |x|^\alpha} u(x,t), \qquad x \in [0, \pi], \; t \in (0, T], \\ &u(x, 0) = u_0(x), \\ &u(0, t) = u(\pi, t) = 0, \end{aligned} \tag{17}$$

where u is, for example, a solute concentration and K_α represents the dispersion coefficient. Let x_i, $1 \le i \le n$ be the equidistant discretization points in the interval $[0, \pi]$ such that $x_1 = 0$ and $x_n = \pi$. Now, we construct the Lagrange basis $L_1(x), \ldots, L_n(x)$ of the span of the functions $\phi(|x - x_j|)$, $1 \le j \le n$ via solving the system

$$\boldsymbol{L}(x) = \boldsymbol{\phi}(x) \boldsymbol{A}^{-1},$$

where

$$\boldsymbol{L}(x) = (L_j(x), 1 \le j \le n), \quad \boldsymbol{\phi}(x) = (\phi(|x - x_j|), 1 \le j \le n), \quad \boldsymbol{A} = (\phi(|x_i - x_j|))_{1 \le i \le n, 1 \le j \le n}.$$

If $\mathcal{L}$ is a differential operator, and if the RBF ϕ is sufficiently smooth to to allow application of $\mathcal{L}$, the required derivatives $\mathcal{L} L_j$ of the Lagrange basis L_j come via solving

$$(\mathcal{L} L)(x) = (\mathcal{L} \boldsymbol{\phi})(x) \boldsymbol{A}^{-1}.$$

Due to the standard Lagrange conditions, the zero boundary conditions at $x_0 = 0$ and $x = \pi$ are satisfied if we use the span of the functions $L_2, \ldots, L_{n-1}$ as our trial space. Then the approximate solution can be written as

$$u(x,t) = \sum_{j=2}^{n-1} \beta_j(t) L_j(x),$$

with the unknown vector

$$\boldsymbol{\beta}(t) = (\beta_j(t), 2 \leq j \leq n-1).$$

We now write the PDE at a point $x_i,\ 2 \leq i \leq n-1$ as follows:

$$\sum_{j=2}^{n-1} \beta_j'(t) L_j(x_i) = -K_\alpha \sum_{j=2}^{n-1} \beta_j(t) \frac{\partial^\alpha}{\partial |x|^\alpha} L_j(x_i).$$

The initial conditions also provide

$$\beta_j(0) = u_0(x_j), \quad 2 \leq j \leq n-1.$$

Thus we get the following system of ODEs

$$\boldsymbol{\beta}'(t) = -K_\alpha \left(\frac{\boldsymbol{\partial^\alpha}}{\boldsymbol{\partial |x|^\alpha}} \boldsymbol{L} * \boldsymbol{\beta}(t) \right),$$

with the initial conditions

$$\boldsymbol{\beta}(0) = U_0,$$

where

$$\frac{\boldsymbol{\partial^\alpha}}{\boldsymbol{\partial |x|^\alpha}} \boldsymbol{L} = \left(\frac{\partial^\alpha}{\partial |x|^\alpha} L_j(x_i) \right)_{2 \leq i \leq n-1, 2 \leq j \leq n-1},$$

$$\boldsymbol{U}_0 = (u_0(x_i), 2 \leq i \leq n-1)^T.$$

Now, consider problem (17) with the parameters $\alpha = 1.8$, $K_\alpha = 0.25$, $T = 0.4$, and $u_0(x) = x^2(\pi - x)$. The numerical solution is plotted by using the Gaussian RBF with $c = 1$, and taking 101 discretization points, in Figure 4. In the second experiment, we use parameters $\alpha = 1.5$, $K_\alpha = -0.25$, $T = 0.5$, and $u_0(x) = \sin(4x)$. The numerical solution is plotted by using the Gaussian RBF with $c = 1$, and taking 101 discretization points, in Figure 5. The results are in agreement with the results of [28]. It should be noted that using the multiquadric RBF with $\beta = 1$ and $c = 1$ also gives the same results.

5. Conclusion

The Riemann-Liouville fractional integral and derivative and also the Caputo fractional derivative of the five kinds of RBFs including the powers, Gaussian, multiquadric, Matern and thin-plate splines, in one dimension, are obtained. These formulas allow to use new fractional variations of numerical methods based on RBFs. Two examples of such techniques are given. The first one is a fractional ODE which is solved by the RBF collocation method and the second one is a fractional PDE which is solved by the method of lines based on the spatial trial spaces spanned by the Lagrange basis associated to RBFs.

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Table 1: Definition of some types of RBFs, where β, ν, and n are RBF parameters.

Name	Definition
Gaussian	$\exp(-r^2/2)$
Multiquadric	$(1 + r^2/2)^{\beta/2}$
Powers	r^β
Matern/Sobolev	$K_\nu(r)r^\nu$
Thin-plate splines	$r^{2n}\ln(r)$

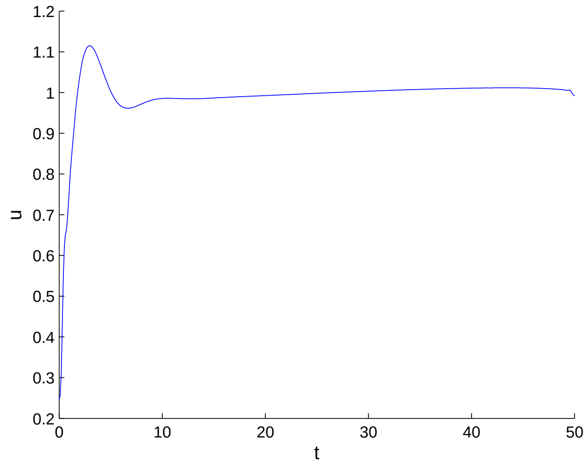

Figure 1: Solution of the test problem 1 for $f(t) = 1$.

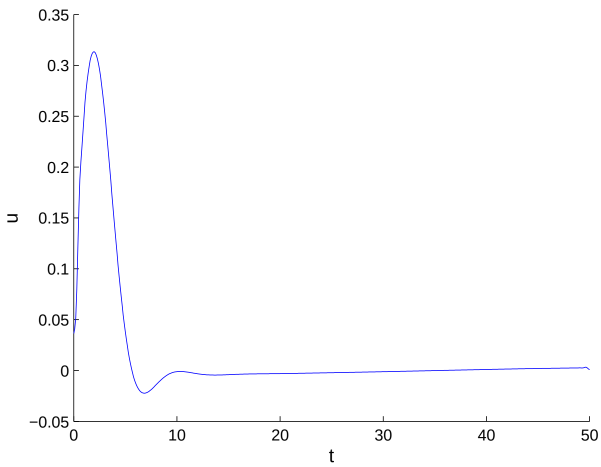

Figure 2: Solution of the test problem 1 for $f(t) = te^{-t}$.

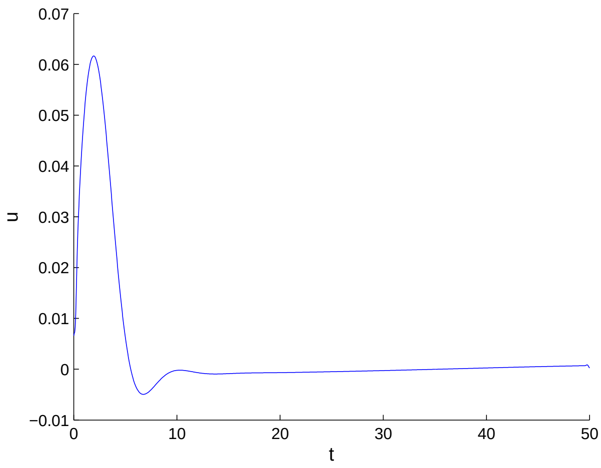

Figure 3: Solution of the test problem 1 for $f(t) = \mathrm{e}^{-t}\sin(0.2t)$.

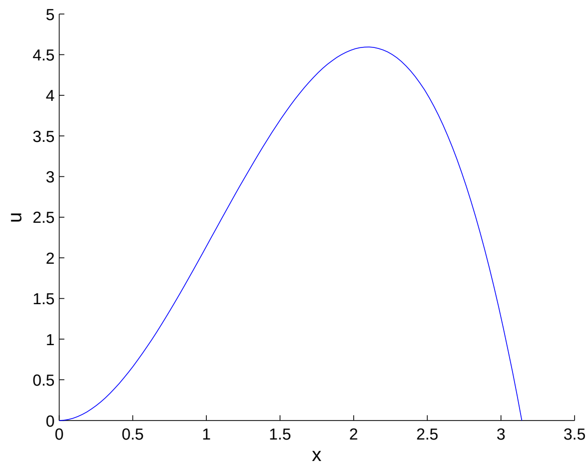

Figure 4: Solution of the test problem 2 with $\alpha = 1.8$, $K_\alpha = 0.25$, and $T = 0.4$.

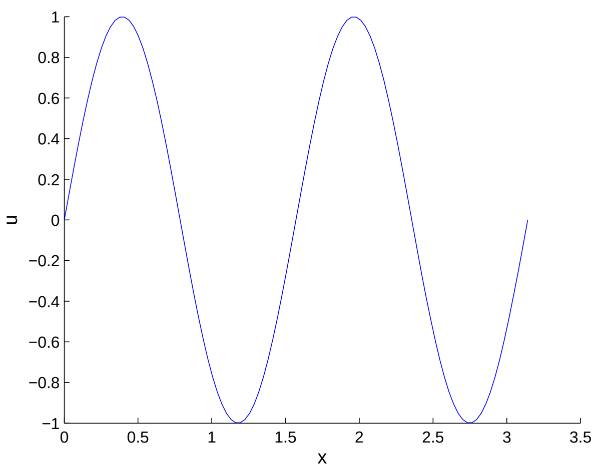

Figure 5: Solution of the test problem 2 with $\alpha = 1.5$, $K_\alpha = -0.25$, and $T = 0.5$.