Shift-invariant approximation

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1 Synonyms

Approximation by integer translates

2 Mathematics Subject Classification

41A25, 42C15, 42C25, 42C40

3 Short Definition

Shift–invariant approximation deals with functions f on the whole real line, e.g. *time series* and *signals*. It approximates f by shifted copies of a single *generator* φ, i.e.

$$f(x) \approx S_{f,h,\varphi}(x) := \sum_{k\in\mathbb{Z}} c_{k,h}(f)\varphi\left(\frac{x}{h}-k\right),\ x \in \mathbb{R}. \tag{1}$$

The functions $\varphi\left(\frac{\cdot}{h}-k\right)$ for $k \in \mathbb{Z}$ span a space that is *shift–invariant* wrt. integer multiples of h. Extensions [de Boor et al(1994a)de Boor, DeVore, and Ron, de Boor et al(1994b)de Boor, DeVore, and Ron] allow multiple generators and multivariate functions. Shift–invariant approximation uses only a single scale h, while *wavelets* use multiple scales and *refinable* generators.

4 Description

Nyquist–Shannon–Whittaker–Kotelnikov sampling provides the formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(kh)\,\mathrm{sinc}\left(\frac{x}{h}-k\right)$$

for *band–limited* functions with frequencies in $[-\pi/h, +\pi/h]$. It is basic in Electrical Engineering for AD/DA conversion of *signals* after *low–pass filtering*. Another simple example arises from the *hat function* or order two *B–spline*

$B_2(x) := 1 - |x|$ for $-1 \le x \le 1$ and zero elsewhere. Then the "connect–the–dots" formula

$$f(x) \approx \sum_{k \in \mathbb{Z}} f(kh) B_2 \left(\frac{x}{h} - k \right)$$

is a piecewise linear approximation of f by connecting the values $f(kh)$ by straight lines. These two examples arise from a generator φ satisfying the *cardinal* interpolation conditions $\varphi(k) = \delta_{0k},\ k \in \mathbb{Z}$, and then the right–hand side of the above formulas interpolates f at all integers. If the generator is a higher–order B–spline B_m, the approximation

$$f(x) \approx \sum_{k \in \mathbb{Z}} f(kh) B_m \left(\frac{x}{h} - k \right)$$

goes back to I.J. Schoenberg and is not interpolatory in general.

So far, these examples of (1) have very special coefficients $c_{k,h}(f) = f(kh)$ arising from *sampling* the function f at data locations $h\mathbb{Z}$. This connects shift–invariant approximation to *sampling* theory. If the shifts of the generator are orthonormal in $L_2(\mathbb{R})$, the coefficients in (1) should be obtained instead as $c_{k,h}(f) = (f, \varphi(\frac{\cdot}{h} - k))_2$ for any $f \in L_2(\mathbb{R})$ to turn the approximation into an optimal L_2 projection. Surprisingly, these two approaches coincide for the sinc case.

Analysis of shift–invariant approximation focuses on the error in (1) for various generators φ. and for different ways of calculating useful coefficients $c_{k,h}(f)$. Under special technical conditions, e.g. if the generator φ is compactly supported, the *Strang–Fix conditions* [Strang and Fix(1973)]

$$\hat{\varphi}^{(j)}(2\pi k) = \delta_{0k},\ k \in \mathbb{Z},\ 0 \le j < m$$

imply that the error of (1) is $\mathcal{O}(h^m)$ for $h \to 0$ in Sobolev space $W_2^m(\mathbb{R})$ if the coefficients are given via L_2 projection. This holds for B–spline generators of order m.

The basic tool for analysis of shift–invariant L_2 approximation is the *bracket product*

$$[\varphi, \psi](\omega) := \sum_{k \in \mathbb{Z}} \hat{\varphi}(\omega + 2k\pi)\overline{\hat{\psi}(\omega + 2k\pi)},\ \omega \in \mathbb{R}$$

which is a 2π–periodic function. It should exist pointwise, be in $L_2[-\pi, \pi]$ and satisfy a *stability property*

$$0 < A \le [\varphi, \varphi](\omega) \le B,\ \omega \in \mathbb{R}.$$

Then the L_2 projector for $h = 1$ has the convenient Fourier transform

$$\hat{S}_{f,1,\varphi}(\omega) = \frac{[f, \varphi](\omega)}{[\varphi, \varphi](\omega)} \hat{\varphi}(\omega),\ \omega \in \mathbb{R},$$

and if $[\varphi, \varphi](\omega) = 1/2\pi$ for all ω, the integer shifts $\varphi(\cdot - k)$ for $k \in \mathbb{Z}$ are orthonormal in $L_2(\mathbb{R})$.

Fundamental results on shift–invariant approximation are in [de Boor et al(1994a)de Boor, DeVore, and Ron, de Boor et al(1994b)de Boor, DeVore, and Ron], and the survey [Jetter and Plonka(2001)] gives a comprehensive account of the theory and the historical background.

References

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