Solvability of Partial Differential Equations by Meshless Kernel Methods

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Abstract

This paper first provides a common framework for partial differential equation problems in both strong and weak form by rewriting them as generalized interpolation problems. Then it is proven that any well–posed linear problem in strong or weak form can be solved by certain meshless kernel methods to any prescribed accuracy.

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The fairly general statement made in the abstract needs some specification. We assume a problem to be posed that is solved by a function u in some Hilbert space U with inner product $(\cdot,\cdot)_U$. Note that this is satisfied for all problems that can be formulated in Sobolev spaces, for instance, but we also allow problems with strong solutions in Hilbert subspaces of differentiable or Hölder continuous functions. The elements of U are viewed as multivariate functions, and the elements $\lambda \in U^*$ are continuous linear functionals that we use to describe data $\lambda(u)$ of u, e.g. evaluations $u \mapsto \delta_x(u) := u(x)$ or $u \mapsto (\delta_x \circ \Delta)(u) = (\Delta u)(x)$.

The problems should be formulated by requiring that a (usually uncountable) set Λ of functionals, when applied to the solution u, attains certain prescribed values. This means that u solves

$$\lambda(u) = \varphi(\lambda) \text{ for all } \lambda \in \Lambda \tag{1}$$

where $\varphi : \Lambda \to I\!R$ is a given function. We do not care about assumptions on φ, but we assume that the functionals $\lambda \in \Lambda$ are continuous on U, i.e. they must be in the dual U^* of U. We call them *test functionals*, because they define the test criteria for a function u to be a solution of our problem. It is shown in the next section that plenty of strongly or weakly formulated linear problems of Applied Analysis have this form, because the test functionals λ can, for instance, describe point evaluations of u, its derivatives, or some differential or integral operator applied to u. We shall call a problem (1) *admissible*, if it is posed with $\Lambda \in U^*$, $\varphi : \Lambda \to I\!R$ and solvable by some function $u \in U$. An admissible problem will have a *unique* solution in U, if we know that the closed linear subspace of homogeneous solutions consists of the zero function only, but we need not assume unique solvability at this point.

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1 Strong and Weak Problems

As a model for a classically or *strongly* formulated problem, consider the Poisson problem

$$\begin{array}{rcl} \Delta u &=& f \text{ on } \Omega \\ u &=& g \text{ on } \partial\Omega, \end{array} \tag{2}$$

asking for a function u on a domain $\Omega \subset I\!\!R^d$ which is twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Here, the set Λ of test functionals consists of two parts, namely the functionals $\delta_x \circ \Delta$ for all $x \in \Omega$ and δ_y for all $y \in \partial\Omega$. The values $\lambda(u)$ are prescribed via function values of f in Ω and g on the boundary $\partial\Omega$, respectively. Note that one could take other linear partial differential operators and other types of boundary conditions, defining quite nonstandard mixed–type problems. Any Hilbert space U with $u \in U$ and $\Lambda \in U^*$ will do, since we shall always assume that there is a solution $u \in U$ to our problem, so that $f := \Delta u$ on Ω and $g := u$ on $\partial\Omega$ can be viewed as *definitions*, allowing us not to care about properties of f and g. Note that this allows a large variety of spaces, if the solution u is sufficiently regular.

A simple discretization of (2) proceeds via collocation. If we take a countable set of dense points $\{x_j\}_j \subset \Omega$ and $\{y_k\}_k \subset \partial\Omega$ and only use a total of n test functionals $\lambda_j(u) = \delta_{x_j}\Delta u$ and $\mu_k(u) = \delta_{y_k}(u)$, respectively, to produce a function u_n such that $\lambda(u_n) = \lambda(u)$ for this subset of n test functionals, we have a candidate for a sequence $\{u_n\}_{n\in I\!\!N} \subset U$ that hopefully converges to a solution $\tilde{u} \in U$ if n tends to infinity. It will be the purpose of the following sections to show that this works if we use reproducing kernels of certain Hilbert spaces to generate the collocation functions. Note that collocation just replaces (1) by a finite problem of the same form.

In the model situation of solving a Poisson problem (2) *weakly*, we move the boundary data prescribed by g into a function $u_0 \in W_2^1(\Omega)$ and consider the variational equation

$$\begin{array}{rcl} (u,v)_{1,\Omega} := \displaystyle\int_\Omega (\nabla^T u)(x)(\nabla v)(x)dx &=& (f,v)_{L_2(\Omega)} \text{ for all } v \in V_0 \subset W_2^1(\Omega) \\ u - u_0 &\in& V_0 \end{array} \tag{3}$$

where u should be in Sobolev space $W_2^1(\Omega)$. The space V_0 contains *test functions* and usually is the subspace of $W_2^1(\Omega)$ consisting of the W_2^1–closure of C^∞ functions with compact support inside the domain Ω. In comparison to the previous case, the crucial point here is that the space $W_2^1(\Omega)$ does not allow continuous point evaluations for dimensions $d > 1$. And, due to low regularity of g and "incoming corners" of the domain, the actual solution u does in general not lie in a space with functions of higher regularity.

In principle, it makes abolutely no sense to use numerical solutions of the above problem that are in the space $W_2^1(\Omega)$ and have no higher regularity. Those functions would have undefined function values, and one could only evaluate local means, for instance. The standard technique for solving weak problems, the method of finite elements, usually works with continuous piecewise linear functions, which also have a higher regularity than the functions in $W_2^1(\Omega)$. Therefore we feel free to reconstruct functions u of low regularity solving weak problems by numerical approximations of higher regularity.

To bring (3) in line with (1), we first rewrite (3) in the modified form

$$\begin{array}{rcl} (w,v)_{1,\Omega} &=& (f,v)_{L_2(\Omega)} - (u_0,v)_{1,\Omega} \text{ for all } v \in V_0 \subset W_2^1(\Omega) \\ w &\in& V_0 \end{array} \tag{4}$$

for $w := u - u_0 \in V_0$. This is a generalized interpolation problem of the form (1), if we define a *test functional* $\lambda_v(w) := (w,v)_{1,\Omega}$ for each *test function* v and require $w \in U := V_0$

to have the data

$$\lambda_v(w) = (f, v)_{L_2(\Omega)} - (u_0, v)_{1,\Omega} =: \varphi(\lambda_v) \text{ for all } v \in V_0.$$

Note that one needs a Poincaré type inequality to conclude that

- $(\cdot, \cdot)_{1,\Omega}$ is an inner product on V_0
- the test functionals λ_v are continuous on V_0 under this inner product

as required for (1). Clearly, the notion of *test functionals* generalizes the notion of *test functions* in such a way that the strong and weak problems both reduce to (1).

Problems in weak form can also be posed *locally* by splitting the integration into integrals over subdomains containing the supports of the test functions. Furthermore, the boundary values can be specified in weak form. This all amounts to using linear functionals in the form of (1) and is contained in the scope of this paper.

The standard technique for solving weak problems proceeds via finite element subspaces S_N of V_0 spanned by test functions $v_1, \ldots, v_N$ and posing the finite problem

$$(w_N, v_j)_{1,\Omega} \quad = \quad (f, v_j)_{L_2(\Omega)} - (u_0, v_j)_{1,\Omega} =: \varphi(\lambda_{v_j}),\ 1 \leq j \leq N \tag{5}$$

for some $w_N \in S_N$. Note that, quite as the collocation technique for "strong" problems, the finite element method for weak problems just replaces (1) by a finite problem of the same form. Consequently, there is no need to distinguish between strong and weak formulations for the next sections. We shall focus on discretizations of (1) that replace Λ by a finite subset $\Lambda_N := \{\lambda_1, \ldots, \lambda_N\} \subset \Lambda \subset U^*$.

But this does not mean that "strong" and "weak" formulations coincide. To avoid misunderstandings, the similarities and differences between "strong" and "weak" formulations should be pointed out more clearly:

- They share the problem form (1) in some Hilbert space.
- But usually they pick different Hilbert spaces. Weak formulations use only half of the smoothness of strong formulations, and thus the Hilbert space of weak formulations is larger than that of strong formulations.
- They have a different strategy for specifying the set Λ of test functionals. Strong formulations take point evaluations of the solution and its derivatives. Weak formulations take test functionals defined by inner products. They often require local integration, which strong formulations can avoid.

2 Kernels

If we want to apply meshless kernel methods to general admissible problems, we need a suitably general definition of a *kernel*. The standard way via reproducing kernel Hilbert spaces or positive definite functions is insufficient here, because we want to allow weak problems and Sobolev spaces like W_2^1 where point evaluation functionals are not continuous. We just take the canonical Riesz map $R : U^* \to U$ of the Hilbert space U with

$$\lambda(u) = (u, R(\lambda))_U = (R^{-1}u, \lambda)_{U^*} \text{ for all } u \in U,\ \lambda \in U^* \tag{6}$$

and use it as a kernel, because it maps functionals to functions. If $\Phi(x, y)$ is a "standard" kernel in a reproducing kernel Hilbert space [1, 18], the relation of the kernel to the Riesz map R is $R(\lambda) = \lambda^x \Phi(x, \cdot)$ where λ^x stands for the evaluation of λ with respect to the variable x. In fact, if point evaluations are continuous on a Hilbert space of functions, the standard kernel definition is

$$\Phi(x,y) := (\delta_x, \delta_y)_{U^*}$$

and the usual reproduction property is

$$u(x) = \delta_x u = (u, \Phi(x,\cdot))_U \text{ for all } u \in U$$

with its generalized form

$$\lambda(u) = (u, \lambda^x \Phi(x,\cdot))_U = (u, R(\lambda))_U \text{ for all } u \in U,\ \lambda \in U^*. \tag{7}$$

This shows that $R(\lambda) = \lambda^x \Phi(x,\cdot)$ is the connection between R and Φ.

Before we proceed, we need to associate certain subspaces of U and U^* with a set $\Lambda \subseteq U^*$ of test functionals:

$$\begin{array}{rclcl} U^*_\Lambda & := & \text{clos span } \Lambda & \subseteq & U^* \\ U_{R(\Lambda)} & := & \text{clos span } R(\Lambda) & \subseteq & U. \end{array}$$

We used clos to stand for Hilbert space closure. Finally, with a slight abuse of notation, we define

$$U^\perp_\Lambda := \{v \in U \ :\ \lambda(v) = 0 \text{ for all } \lambda \in \Lambda\} = U^\perp_{R(\Lambda)} \tag{8}$$

such that unique solvability of (1) is equivalent to $U^\perp_\Lambda = \{0\}$.

3 Symmetric Meshless Kernel Methods

To explain our basic numerical technique, we take a finite set $\Lambda_N := \{\lambda_1, \dots, \lambda_N\} \subset U^*$ of continuous linear functionals and fix a function $u \in U$. Then we define $\varphi(\lambda_j) := \lambda_j(u),\ 1 \le j \le N$ in (1) and construct a *trial function*

$$\tilde{u}_N = \sum_{k=1}^N \alpha_k R(\lambda_k) \tag{9}$$

in the space $U_{R(\Lambda_N)}$ by the interpolation or collocation requirement

$$\lambda_j(\tilde{u}_N) = \varphi(\lambda_j) := \lambda_j(u),\ 1 \le j \le N$$

leading to the system

$$\sum_{k=1}^N \alpha_k (\lambda_j, \lambda_k)_{U^*} = \lambda_j(u),\ 1 \le j \le N. \tag{10}$$

Note that we insist on a close link between the test functionals λ_k and the trial functions v_k via $v_k := R(\lambda_k)$ using the Riesz map R. The system has a positive semidefinite symmetric Gramian coefficient matrix. It is nonsingular and positive definite, if the test functionals are linearly independent. If not, the system is still solvable, because the right–hand side is in the range of the map

$$u \mapsto (\lambda_1(u), \dots, \lambda_N(u)) \in I\!R^N \text{ for all } u \in U,$$

and this range has the same dimension as the space $U_{R(\Lambda_N)}$, because the Riesz map is an isometry. Clearly, the resulting function $\tilde{u}_N$ is uniquely defined as the image of u under the Hilbert space projection $\Pi_{R(\Lambda_N)}$ of U onto the closed linear subspace $U_{R(\Lambda_N)}$, even in case its representation via (9) has nonunique coefficients. Furthermore, it satisfies the orthogonality relations

$$(u - \tilde{u}_N, R(\lambda_j))_U = \lambda_j(u - \tilde{u}_N) = 0,\ 1 \le j \le N \tag{11}$$

implying

$$u - \tilde{u}_N \in U^{\perp}_{R(\Lambda_N)},\ \|u\|_U^2 = \|u - \tilde{u}_N\|_U^2 + \|\tilde{u}_N\|_U^2. \tag{12}$$

Let us call $\tilde{u}_N$ the (symmetric) *projection approximation* of u with respect to the data $\lambda_1(u), \ldots, \lambda_N(u)$ or the set $\Lambda_N = \{\lambda_1, \ldots, \lambda_N\} \subset U^*$ of test functionals. Note that by (12) the function $\tilde{u}_N$ solves the minimization problem

$$\min\{\|v\|_U^2\ :\ v \in U,\ \lambda_j(v) = \lambda_j(u),\ 1 \le j \le N\}$$

because of (11).

For strong problems, this is a variation of the *symmetric collocation* technique of Z. Wu [26] used for the approximate recovery of u from its data $\lambda_j(u)$. This method has a solid theoretical basis including convergence orders [9, 10].

For weak problems, it may be surprising that the Rayleigh–Ritz technique, and in particular the finite element method arise just as special cases of symmetric projection methods. In fact, for (5) we took test functionals with

$$\lambda_{v_j}(w) = (w, v_j)_{1,\Omega} \text{ for all } w \in V_0$$

associated to test functions v_j. But since V_0 is a Hilbert space under $(\cdot,\cdot)_{1,\Omega}$, we have $R(\lambda_{v_j}) = v_j$ and the finite element solution coincides with the projection approximation. Due to

$$(\lambda_{v_j}, \lambda_{v_k})_{U^*} = (v_j, v_k)_U = (v_j, v_k)_{1,\Omega}$$

the system (10) has the standard stiffness matrix.

Though this paper will focus on symmetric projections, we should point out that

- unsymmetric collocation in the sense of Kansa [15] for strong problems and
- unsymmetric Petrov–Galerkin schemes for weak problems

formally coincide, too. Unlike (9), which closely relates trial functions v_k to test functionals λ_k via $v_k := R(\lambda_k)$, these methods define a new space W_N of *trial functions* $w_1, \ldots, w_N$ to approximate the solution. This space is unrelated to the test functionals, while the symmetric setup uses them directly to determine the solution space via the Riesz map. The unsymmetric case constructs

$$\tilde{u}_N = \sum_{k=1}^{N} \alpha_k w_k \tag{13}$$

in the space W_N by the interpolation or collocation requirement

$$\lambda_j(\tilde{u}_N) = \varphi(\lambda_j) := \lambda_j(u),\ 1 \le j \le N$$

leading to the system

$$\sum_{k=1}^{N} \alpha_k \lambda_j(w_k) = \lambda_j(u),\ 1 \le j \le N. \tag{14}$$

The arising matrix has coefficients

$$\lambda_j(w_k) = (R(\lambda_j), w_k)_U = (\lambda_j, R^{-1}(w_k))_{U^*}$$

and is unsymmetric. In addition, it may be singular, if the test functionals $\lambda_1, \ldots, \lambda_N$ are not linearly independent over W_N.

Kansa's collocation method [15] for strong problems takes $w_j := R(\delta_{x_j}) = \Phi(x_j, \cdot)$ for a set of points $x_1, \ldots, x_N$ in the context of continuous kernels. In the Petrov–Galerkin technique for weak problems, the test functionals λ_j have the form $\lambda_j = R^{-1}(v_j)$, where the v_j are called *test functions*, and the system then has the familiar coefficients $(v_j, w_k)_U$.

Analysis of unsymmetric problems is hard, because even the solvability [12] of the finite subproblems is not evident.

Furthermore, *local* methods can also be incorporated, in particular those who localize a weak setting. But we cannot go into details here. Candidates for further analysis are the weak meshless local Petrov–Galerkin method (MLPG, [3, 2, 4]) or the generalized finite element method [7] based on techniques using partitions of unity [17, 6]. Chances are good that such methods also work nicely for meshless kernel techniques, since they surely work for interpolation [25] and certain simple problems in weak form [24]. A new and promising development that incorporates multilevel techniques into meshless kernel methods introduces multiscale kernels [20].

Recent developments of meshless computational methods for solving real physical problems include the use of the fundamental solution as basis function (strong form) for solving inverse boundary determination and inverse heat conduction problems [14, 13]. A good review on the method of fundamental solutions (MFS) can be found from [11]. The advantages of meshless computational methods, in particular the weak form, have further been verified by the works on solving large deformation problems due to nonlinear structure [8, 30] and deformation behavior of smart material such as shape memory alloys [16]. Please refer to the survey paper [5] on the comparison between meshless method and traditional finite element and boundary element methods. To tackle the ill-conditioning of full coefficient matrix that may arise from these methods several techniques, such as domain decomposition, adaptive greedy, and preconditioning have been proposed. More effort in this direction is needed to enhance the performance and range of applicability of meshless methods.

4 Infinite Problems

The previous section defined our standard numerical method for the recovery of a function u from finitely many data $\lambda_1(u), \ldots, \lambda_N(u)$ via the image $\tilde{u}_N = \Pi_{R(\Lambda_N)}(u)$ of the projection onto the subspace $U_{R(\Lambda_N)}$. Since problems in Applied Analysis in the form (1) will usually have an uncountable number of prescribed data, and since sequences of finite problems deal with countably many data, we have to go over to the case of countable and uncountable data.

Assume first that $\Lambda_{I\!N} := \{\lambda_j\}_{j \in I\!N}$ is a countable set of test functionals. We can then form the sequence $\{\tilde{u}_N\}_{N \in I\!N}$ and use (11) to get

$$\tilde{u}_M - \tilde{u}_N \in U^{\perp}_{R(\Lambda_N)}, \ \|\tilde{u}_M\|_U^2 = \|\tilde{u}_M - \tilde{u}_N\|_U^2 + \|\tilde{u}_N\|_U^2 \leq \|u\|_U^2 \tag{15}$$

for all $M \geq N$. Thus the sequence $\{\|\tilde{u}_N\|_U^2\}_N$ is weakly monotonic and convergent. Furthermore, the above display implies that $\{\tilde{u}_N\}_{N \in I\!N}$ is a Cauchy sequence in U, and therefore convergent to some function $\tilde{u}_{I\!N} \in U_{R(\Lambda_{I\!N})}$. But then we have

$$\lambda_j(\tilde{u}_{I\!N}) = \lambda_j(\lim_{N \to \infty} \tilde{u}_N) = \lim_{N \to \infty} \lambda_j(\tilde{u}_N) = \lambda_j(u)$$

for all $j \in I\!N$, proving

Theorem 1. *For any admissible problem* (1) *with countably many data functionals, a solution* $\tilde{u}_{\mathbb{N}}$ *can be constructed via a sequence of finite projection approximations. It solves the minimization problem*

$$\min\{\|v\|_U^2 \;:\; \lambda_j(v) = \lambda_j(u),\; 1 \le j < \infty\}.$$

□

We now add a non–constructive result concerning general sets of test functionals.

Theorem 2. *Let an arbitrary nonempty set* $\Lambda \subseteq U^*$ *of linear test functionals from the dual* U^* *of a Hilbert space* U *be given, and fix an element* $u \in U$*. Then there is a unique element* $\tilde{u} \in U$ *with the properties*

$$\begin{array}{rcl} \tilde{u} & \in & U_{R(\Lambda)} \\ \lambda(\tilde{u}) & = & \lambda(u) \text{ for all } \lambda \in \Lambda \\ u - \tilde{u} & \perp & U_{R(\Lambda)} \\ \|\tilde{u}\|_U & = & \min\{\|v\|_U \;:\; v \in U,\; \lambda(v) = \lambda(u) \text{ for all } \lambda \in \Lambda\}. \end{array} \tag{16}$$

Proof: The space $U_{R(\Lambda)}$ is a closed subspace of U, and its orthogonal complement is $U_\Lambda^\perp$ from (8). Thus u has a unique decomposition $u = \tilde{u} + \tilde{u}^\perp$ with $\tilde{u} \in U_{R(\Lambda)}$ and $\tilde{u}^\perp \in U_\Lambda^\perp = U_{R(\Lambda)}^\perp$. This implies the first three properties of (16). If $v \in U$ is admissible for the infimum in the third property, we can write $v = v - \tilde{u} + \tilde{u}$ and use that $v - \tilde{u} \in U_\Lambda^\perp$ is orthogonal to $\tilde{u}$. Then $\|v\|_U^2 = \|v - \tilde{u}\|_U^2 + \|\tilde{u}\|_U^2$ proves the assertion. The uniqueness of $\tilde{u}$ with respect to the properties in (16) follows from the fact that the difference of two such functions must be in both $U_{R(\Lambda)}$ and $U_\Lambda^\perp$. □

Corollary 1. *In the sense of the above theorem, all admissible linear problems posed by some* $\Lambda \subseteq U^*$ *and having a solution* $u \in U$ *have a unique projection approximation solution* $\tilde{u}$*. The functions* u *and* $\tilde{u}$ *coincide, if there is no nontrivial homogeneous solution, i.e.* $U_\Lambda^\perp$ *from* (8) *is the null space.*

Proof: The assertion is an immediate consequence of the previous theorem. □

5 Density

In order to bridge the gap between Theorems 1 and 2, we now consider conditions under which we can replace an uncountable set Λ of data test functionals by a countable set $\tilde{\Lambda}$ of "dense" test functionals that can be handled via a sequence of finite problems. By the standard definition, a subset $\tilde{\Lambda} \subseteq \Lambda \subseteq U^*$ is *dense* in Λ, if all elements of Λ can be written as limits in U^* of elements of $\tilde{\Lambda}$. Then there are some easy observations to be made:

Theorem 3. *The following statements are equivalent:*

1. $\tilde{\Lambda} \subseteq \Lambda \subseteq U^*$ *is dense in* Λ
2. $U^*_{\tilde{\Lambda}}$ *is dense in* U^*_Λ
3. $R(\tilde{\Lambda}) \subseteq R(\Lambda) \subseteq U$ *is dense in* $R(\Lambda)$
4. $U_{R(\tilde{\Lambda})}$ *is dense in* $U_{R(\Lambda)}$
5. *For all* $u \in U$

$$\lambda(u) = 0 \text{ for all } \lambda \in \tilde{\Lambda} \text{ implies } \lambda(u) = 0 \text{ for all } \lambda \in \Lambda$$

6. $U^\perp_{\tilde\Lambda} = U^\perp_{\Lambda}$

□

Theorem 4. *An admissible linear problem posed by some $\Lambda \subseteq U^*$ can be solved by a convergent sequence of projection approximations, if Λ contains a dense countable subset.* □

Theorem 5. *An admissible linear problem posed by some $\Lambda \subseteq U^*$ has a unique solution, if Λ is dense in U^*.* □

Proving density will turn out to be dependent of the type of functionals. We thus have to be able to split sets of test functionals.

Theorem 6. *Let $\Lambda = \cup_{i\in I}\Lambda^i$ be a superposition of not necessarily disjoint sets Λ^i. If all Λ^i have a dense subset, so has Λ.* □

6 Continuity

We now focus on test functionals arising in strong problems, in particular point evaluations of functions or derivatives thereof. In such cases density of sets of functionals can be obtained from density of the related evaluation points together with continuity of the evaluated functions or derivatives. The simplest case is evaluation of plain function values.

Theorem 7. *Let Λ consist of all point evaluations on some set Ω, i.e. $\Lambda = \{\delta_x \ : \ x \in \Omega\}$, and let U consist of continuous functions. Then a subset of test functionals $\tilde\Lambda = \{\delta_x \ : x \in \tilde\Omega\} \subseteq \Lambda$ corresponding to a subset $\tilde\Omega \subseteq \Omega$ is dense in Λ if $\tilde\Omega$ is dense in Ω.* □

Since in this section we confine ourselves to strongly formulated problems, we assume U to be a *reproducing kernel Hilbert space* of functions on some set $\Omega \subseteq I\!R^d$ with a kernel function Φ and continuous point evaluations. Then the standard reproduction property (7) implies that continuity of all functions in U follows from continuity of the kernel:

Theorem 8. *If the kernel Φ of some reproducing kernel Hilbert space U is continuous, then U consists of continuous functions.*

Proof: Let $x, y \in \Omega$ and $u \in U$ be given, and use (7) and (6) for

$$\begin{array}{rcl}
(u(x) - u(y))^2 & = & (u, \Phi(x,\cdot) - \Phi(y,\cdot))^2_U \\
& \leq & \|u\|^2_U \|\Phi(x,\cdot) - \Phi(y,\cdot)\|^2_U \\
& \leq & \|u\|^2_U \left(\Phi(x,x) - \Phi(x,y) - \Phi(y,x) + \Phi(y,y)\right).
\end{array}$$

□

The next step concerns strongly formulated problems where data partially depend on a differential operator, e.g. a Poisson problem (2). If we take a countable set of dense points $\{x_j\}_j \subset \Omega$ on Ω and $\{y_k\}_k \subset \partial\Omega$ and use test functionals $\lambda_j(u) = \delta_{x_j}\Delta u$ and $\mu_k(u) = \delta_{y_k}(u)$, respectively, we want to infer that a function $u \in U$ with zero data must be identically zero. If the problem has enough regularity such that the solution lies in some reproducing kernel Hilbert space consisting of functions that are continuous on $\overline{\Omega}$, then Theorem 7 immediately yields $u = 0$ on $\partial\Omega$, but we still need something for the test functionals of the form $\lambda_j(u) = \delta_{x_j}\Delta u$ for $x_j \in \Omega$.

Theorem 9. *Let U be a reproducing kernel Hilbert space with kernel Φ defined on some set Ω, and let $L : U \to S$ be a linear operator from U onto a space $S = L(U)$ of functions on Ω. It should have the properties*

$$\begin{array}{rcl} L_x^s\Phi(s,\cdot) & \in & U \textit{ for all } x \in \Omega \\ (Lu)(x) & = & (u, L_x^s\Phi(s,\cdot))_U \textit{ for all } u \in U,\ x \in \Omega \\ L_x^s L_y^t \Phi(s,t) & =: & \Phi_L(x,y) \textit{ is continuous in } x, y \in \Omega \end{array} \tag{17}$$

where $L_x^t(u(t)) = (Lu)(x)$ means evaluation of L with respect to the variable t at the point x. Then S consists of continuous functions. In particular, for all dense countable sets $\{x_j\}_j$ in Ω and for all functions $u \in U$ with $(Lu)(x_j) = 0$ for all j one has $Lu = 0$ on Ω.

Proof: We repeat the proof of Theorem 8 with a slight variation:

$$\begin{array}{rcl} ((Lu)(x) - (Lu)(y))^2 & = & (u, L_x^s\Phi(s,\cdot) - L_y^s\Phi(s,\cdot))_U^2 \\ & \leq & \|u\|_U^2 \|L_x^s\Phi(s,\cdot) - L_y^s\Phi(s,\cdot)\|_U^2 \\ & \leq & \|u\|_U^2 \left(L_x^sL_x^t\Phi(s,t) - L_x^sL_y^t\Phi(s,t) - L_y^sL_x^t\Phi(s,t) + L_y^sL_y^t\Phi(s,t)\right) \\ & \leq & \|u\|_U^2 \left(\Phi_L(x,x) - \Phi_L(x,y) - \Phi_L(y,x) + \Phi_L(y,y)\right) \end{array}$$

where we used that (17) implies

$$(L_x^s\Phi(s,\cdot), L_y^s\Phi(s,\cdot))_U = L_x^sL_y^t\Phi(s,t) = \Phi_L(x,y) \text{ for all } x, y \in \Omega.$$

□

7 Strong Problems

Assume now that we have a general strongly formulated problem with countably many linear operators L_i on domains $\Omega_i \subseteq \Omega$ such that we have to recover a function $u \in U$ from its values $L_i(u)$ on each Ω_i. Note how the Poisson problem fits into this. If we take countable dense subsets of the Ω_i and use test functionals of the form $\delta_{x_j}(L_iu)$ there, we see that under the hypotheses of Theorem 9 we can always find a solution to the generalized interpolation problem that is based on a countable subset of the data functionals and obtainable as the limit of a convergent sequence of approximants. In addition, we can reconstruct the true solution u from countably many data uniquely if there is no nonzero function $v \in U$ that simultaneously satisfies all homogeneous equations $L_iv = 0$ on Ω_i for all i. This reduces the problem of unique numerical reconstruction of u to the uniqueness of the analytical problem itself. In the special case of the Poisson problem, the uniqueness of the analytical problem follows from the maximum principle.

In general, we can summarize our results so far roughly by saying that unique reconstruction of a solution of a strongly formulated generalized interpolation problem is possible, if

1. the kernel Φ and the linear operators L_i satisfy (17) on domains $\Omega_i \subseteq \Omega$,
2. there is a function $u \in U$ that solves the problem defined by data $L_i(u)$,
3. the discretizations of the Ω_i are dense,
4. there is no nonzero solution of the homogeneous problem in U.

We now show how to check the conditions (17) for linear operators L in the standard case of Hilbert spaces on $I\!R^d$ with smooth symmetric translation–invariant and Fourier–transformable kernels. Then differential operators L are definable via Fourier transforms as

$$(Lu)(x) = (2\pi)^{-d/2} \int_{I\!R^d} \hat{u}(\omega)\hat{L}(\omega)e^{ix^T\omega}d\omega \text{ for all } x \in I\!R^d.$$

Standard reproducing kernel Hilbert spaces U on all of $I\!R^d$ with kernels $\Phi(x-y)$ (instead of $\Phi(x,y)$, due to translation invariance) consist of the functions u with

$$(u,u)_U := (2\pi)^{-d/2}\int_{I\!R^d}\frac{|\hat{u}(\omega)|^2}{\hat{\Phi}(\omega)}d\omega < \infty$$

where the Fourier transform of Φ is positive. Now the first property of (17) means

$$\int_{I\!R^d}\frac{\hat{\Phi}(\omega)^2|\hat{L}(\omega)|^2}{\hat{\Phi}(\omega)}d\omega = \int_{I\!R^d}\hat{\Phi}(\omega)|\hat{L}(\omega)|^2 d\omega < \infty.$$

If $\hat{\Phi}(\omega)$ decays at infinity at least like $\|\omega\|_2^{-\beta}$ and if L is a differential operator of order at most m, the above integral is bounded if

$$\beta > 2m + d. \tag{18}$$

The second property then follows from

$$\begin{array}{rcl} L_y^s\Phi(s,t) & = & (2\pi)^{-d/2}\int_{I\!R^d}\Phi(\cdot - t)^\wedge(\omega)\hat{L}(\omega)e^{iy^T\omega}d\omega \\ & = & (2\pi)^{-d/2}\int_{I\!R^d}\hat{\Phi}(\omega)\hat{L}(\omega)e^{i(y-t)^T\omega}d\omega \\ (L_y^s\Phi(s-\cdot))^\wedge(\omega) & = & \hat{\Phi}(-\omega)\hat{L}(-\omega)e^{-iy^T\omega} \\ (u, L_y^s\Phi(s-\cdot))_U & = & (2\pi)^{-d/2}\int_{I\!R^d}\frac{\hat{u}(\omega)\overline{(L_y^s\Phi(s-\cdot))^\wedge(\omega)}}{\hat{\Phi}(\omega)}d\omega \\ & = & (2\pi)^{-d/2}\int_{I\!R^d}\frac{\hat{u}(\omega)\hat{L}(\omega)\hat{\Phi}(\omega)e^{iy^T\omega}}{\hat{\Phi}(\omega)}d\omega \\ & = & (Lu)(y). \end{array}$$

The continuity of $L_x^sL_y^t\Phi(s,t) =: \Phi_L(x-y)$ will usually follow from a direct calculation of this new positive semidefinite kernel, if the original kernel Φ is smooth enough. But since the Fourier transform of Φ_L is $\hat{\Phi}|\hat{L}|^2$, the continuity of Φ_L follows already from (18) by Sobolev space embedding arguments. Altogether, (18) implies (17), and is satisfied if the kernel Φ is sufficiently smooth for the problem to be solved.

8 Weak Problems

To reduce uncountable sets of test functionals for *weak* problems to countable subsets, we first observe that the test functionals for weak problems have the form $\lambda_v = R^{-1}(v)$ where the functions v vary in the subspace $V = R(\Lambda)$ of U that arises in the variational equation to be solved. By property 4 of Theorem 3, the full set Λ of such test functionals contains a countable dense subset, iff V contains a dense subspace with a countable basis. This is the standard background for proving convergence of the Rayleigh–Ritz and in particular the finite element method.

But the framework we developed here allows other subspaces of $V = R(\Lambda)$. In particular, we can use meshless kernel methods to generate an extremely large variety of dense subspaces with countably many generators. The first technique is to use positive definite functions with compact and arbitrarily small support that are contained in U. Such functions are provided by Wu [27] and Wendland [23]. Placing such functions at rational centers and using rational support radii will yield dense subspaces of U with countable bases. Another technique may take shifts of the (possibly singular) kernel of U

and convolve these functions with smooth locally supported functions to improve smoothness and remove singularities. A third variation can apply scaled partitions of unity with rather arbitrary local spaces of functions, e.g. those generated by moving least squares techniques. All of these variations will generate subspaces with countable bases, and they will yield convergent algorithms by the theory developed here.

A detailed analysis of spaces for meshless kernel methods solving weak problems is still to be done. In particular, one can follow the proof technique for finite element methods up to and including Cea's lemma, and then one has to prove approximation orders for kernel–based subspaces of Sobolev spaces.

9 Overcoming Low Regularity

The previous sections ignored the difficulty arising when the solution $u \in \mathcal{U}$ of the given analytic problem (1) posed in a normed linear space $\mathcal{U}$ does not have enough regularity to be in a suitable Hilbert space U with a useful positive definite reproducing kernel Φ. Plenty of authors report good convergence of meshless methods in such cases, and the standard examples are numerical techniques using multiquadrics, where U consists of analytic functions. This is a serious problem for proving convergence, error bounds, and convergence orders, and it systematically arises when the user wants to work with some "nice" kernel Φ, but ignores that the solution of the given strongly or weakly formulated problem does not have sufficient regularity to lie in the "native" Hilbert space U for the chosen kernel.

However, since native spaces of positive definite kernels usually are dense in various other, much larger spaces, functions u from those spaces can be approximated by functions from native spaces to arbitrary accuracy. Thus there can be approximants by meshless kernel methods that actually converge towards u, but not in the topology of the native space U, but only in the topology of the larger space $\mathcal{U}$. It is a problem of Numerical Analysis to show that certain algorithms actually produce such approximants. Standard examples are in [21, 19, 28, 29]. Here, we are satisfied with pointing out that such approximants exist under very weak conditions.

Theorem 10. *Assume that the generalized interpolation problem* (1) *posed in some normed linear space $\mathcal{U}$ of functions on a domain $\Omega \subseteq I\!R^d$ has a solution $u \in \mathcal{U}$. Assume further that there is a countable subset of test functionals $\lambda_j \in \Lambda,\ j \in I\!N$ such that there is no nontrivial function $v \in \mathcal{U}$ with $\lambda_j(v) = 0$ for all $j \in I\!N$, i.e. the problem is well–posed in $\mathcal{U}$ even for a dense countable subset of functionals. Let a sequence of functions $u_k \in \mathcal{U},\ k \in I\!N$ be constructed to satisfy*

$$\lambda_j(u) = \lambda_j(u_k),\ 1 \le j \le k$$

by any method whatsoever. Then the functions u_k converge towards u in a norm on $\mathcal{U}$ that is bounded above by $\|.\|_{\mathcal{U}}$.

Proof: Take any sequence of positive real numbers ρ_j such that $\sum_j \rho_j \|\lambda_j\|^2_{\mathcal{U}^*}$ converges. Then

$$(u,v)_\rho := \sum_{j=1}^{\infty} \rho_j \lambda_j(u)\lambda_j(v),\ u,v \in \mathcal{U} \tag{19}$$

is an inner product on $\mathcal{U}$, and the corresponding norm has the bound

$$\|u\|^2_\rho = \sum_{j=1}^{\infty} \rho_j \lambda_j^2(u) \le \|u\|^2_{\mathcal{U}} \sum_{j=1}^{\infty} \rho_j \|\lambda_j\|^2_{\mathcal{U}^*}.$$

We remark that (19) is a variation of a technique for generating special–purpose kernels, as used in [22, 20]. Define $c_k := \|u - u_k\|_{\mathcal{U}}$ and observe that the calculation stops after a finite number of steps if one of the c_k vanishes. Assume now that all c_k are positive. We recursively define a sequence of positive numbers ϵ_k such that $2c_k^2\epsilon_{k+1} \leq c_{k+1}^2\epsilon_k$ holds for all k and the ϵ_k converge to zero. Then we pick the numbers ρ_k such that $2c_k^2\rho_{k+1}\|\lambda_{k+1}\|_{\mathcal{U}^*}^2 \leq \epsilon_k$ holds for all k and the aforementioned sum converges.

Now by construction

$$\begin{array}{rcl}\|u - u_k\|_\rho^2 & = & \displaystyle\sum_{j=k+1}^{\infty} \rho_j\lambda_j^2(u - u_k) \\ & \leq & \|u - u_k\|_{\mathcal{U}}^2 \displaystyle\sum_{j=k+1}^{\infty} \rho_j\|\lambda_j\|_{\mathcal{U}^*}^2 \\ & \leq & c_k^2 \displaystyle\sum_{j=k+1}^{\infty} \frac{1}{2}\frac{\epsilon_{j-1}}{c_{j-1}^2} \\ & \leq & \dfrac{1}{2}c_k^2\dfrac{\epsilon_k}{c_k^2}(1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots) \\ & = & \epsilon_k. \end{array}$$

□

Here is a non–constructive related result:

Theorem 11. *Assume that the generalized interpolation problem* (1) *posed in some normed linear space $\mathcal{U}$ of functions on a domain $\Omega \subseteq I\!R^d$ has a solution $u \in \mathcal{U}$. Assume further that Φ is a reproducing symmetric positive definite kernel on $I\!R^d$ for a Hilbert space U that is continuously embedded in $\mathcal{U}$. Finally, there should be a countable subset of test functionals $\lambda_j \in \Lambda$, $j \in I\!N$ such that there is no nontrivial function $v \in \mathcal{U}$ with $\lambda_j(v) = 0$ for all $j \in I\!N$. Then there is s sequence $\{v_k\}_k$ of functions in U that converges to u in $\mathcal{U}$. This sequence consists of solutions of finite subproblems with data close to the data of u.*

Proof: Note first that functionals in $\mathcal{U}^*$ are in U^*. We thus can formulate the problem in U, but it has no solution there. Furthermore, we can extract finite subsets of test functionals and work in finite–dimensional subspaces of $U \subset \mathcal{U}$ to generate candidates for convergence towards u.

Now assume that all test functionals are normalized to have norm 1 in U. The solution $u \in \mathcal{U}$ to (1) can be approximated by functions $u_k \in U \subset \mathcal{U}$ to any prescribed accuracy, and we assume that the sequence $\|u - u_k\|_U$ tends to zero for $k \to \infty$. Due to unique solvability of the countable subproblem in U we can pick for each k an index $n_k \in I\!N$ such that the solution $v_k \in U$ of the problem

$$\lambda_j(u_k) = \lambda_j(v_k),\ 1 \leq j \leq n_k$$

satisfies $\|u_k - v_k\|_U \leq \|u - u_k\|_{\mathcal{U}}$, because for large n_k the left–hand side can be made arbitrarily small. We then have

$$\begin{array}{rcl}\|u - v_k\|_{\mathcal{U}} & \leq & \|u - u_k\|_{\mathcal{U}} + \|u_k - v_k\|_{\mathcal{U}} \\ & \leq & \|u - u_k\|_{\mathcal{U}} + C\|u_k - v_k\|_U \\ & \leq & \|u - u_k\|_{\mathcal{U}} + C\|u - u_k\|_U \\ & \leq & (1 + C)\|u - u_k\|_U \end{array}$$

and get convergence $v_k \to u$ in $\mathcal{U}$. The data of the functions v_k are close to those of u due to

$$\begin{array}{rcl}|\lambda_j(v_k) - \lambda_j(u)| & = & |\lambda_j(u_k) - \lambda_j(u)| \\ & \leq & 1 \cdot \|u_k - u\|_U. \end{array}$$

□

10 Final Remarks

We fulfill the purpose if this paper was

1. to provide a unified view of strongly or weakly formulated problems in Applied Analysis,
2. to define a general meshless kernel technique for their solution and
3. to prove that this technique is successful under weak assumptions on the problem background.

Among many other open problems, we suggest to devote future research to

1. replace the density arguments of this paper by error bounds and convergence rates,
2. find efficient variations of the algorithm described here,
3. extend the framework to nonsymmetric techniques like Kansa's collocation method [15] or the Meshless Local Petrov–Galerkin technique of Atluri et.al. [3, 2, 4].

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