The Missing Wendland Functions

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Abstract: The Wendland radial basis functions [8, 9] are piecewise polynomial compactly supported reproducing kernels in Hilbert spaces which are norm–equivalent to Sobolev spaces. But they only cover the Sobolev spaces

$$H^{d/2+k+1/2}(\mathbb{R}^d),\ k \in \mathbb{N} \tag{1}$$

and leave out the integer order spaces in even dimensions. We derive the missing Wendland functions working for half–integer k and even dimensions, reproducing integer–order Sobolev spaces in even dimensions, but they turn out to have two additional non–polynomial terms: a logarithm and a square root. To give these functions a solid mathematical foundation, a generalized version of the "dimension walk" is applied. While the classical dimension walk proceeds in steps of two space dimensions taking single derivatives, the new one proceeds in steps of single dimensions and uses "halved" derivatives of fractional calculus.

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1 Algorithm

First, we give a simple recipe for constructing generalized Wendland functions, provide a few examples, and leave details to the rest of the paper.

The basic tool is the "dimension walk" [9] dating back to Matheron [5], see also [7]. In its standard form [9], the dimension walk proceeds in steps of two dimensions using standard derivatives, while [6] has a generalized version capable of stepping in single dimensions using "halved" derivatives from a

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variation of fractional calculus. The formula (9.2.20) in [6] (see also Gneiting [4]) reads

$$\psi_{\mu,k}(x) = \int_x^1 r(1-r)^\mu \frac{(r^2-x^2)_+^{k-1}}{\Gamma(k)2^{k-1}} dr \text{ for all } x,\ \mu \geq 0, k > 0 \tag{2}$$

and it is normally used only for integer $k,\ \mu$ to get the polynomial representations of the Wendland functions. The connection to the $\phi_{d,k}$ notation in Wendland's monograph [9] to the above formula is via

$$\phi_{d,k} = \psi_{\lfloor d/2 \rfloor + k+1,k}, \tag{3}$$

because there are good reasons to pick the smallest known μ which guarantees $\psi_{\mu,k}$ to be positive definite in d dimensions for a given d, and this minimal μ is $\lfloor d/2 \rfloor + k + 1$ for integer k. The case of half–integer k or μ of the formula (2) was not treated so far, though it clearly generates functions with support in $[0,1]$. The following MAPLE snippet

```
wend:=proc(m,k,x)
local wend;
wend:=r*(1-r)^m*(r*r-x*x)^(k-1)/(GAMMA(k)*2^(k-1));
wend:=int(wend,r=x..1);
return factor(simplify(wend));
end proc:
```

calculates the above integral and runs for all reasonable and fixed choices of `m` and `k` where one half–integer is allowed, while it fails if both `m` and `k` are genuine half–integers. Since $\psi_{d/2+k+1/2,k} = \phi_{d,k}$ will be proven in Corollary 3.1 below to be reproducing in spaces norm–equivalent to $H^{d/2+k+1/2}(\mathbb{R}^d)$ for half–integer k and even d, the above recipe generates the missing Wendland functions for such k. The first interesting case is $\mu = d = 2,\ k = 1/2$ with

$$\psi_{2,1/2}(x) \quad = \quad \frac{\sqrt{2}}{3\sqrt{\pi}}\left(3x^2 \log\left(\frac{x}{1+\sqrt{1-x^2}}\right) + (2x^2+1)\sqrt{1-x^2}\right)$$

plotted in Figure 1. It is a reproducing kernel in an isomorphic copy of $H^2(\mathbb{R}^2)$. Using the abbreviations

$$\begin{array}{rcl} L(x) & := & \log\left(\dfrac{x}{1+\sqrt{1-x^2}}\right) \\ S(x) & := & \sqrt{1-x^2} \end{array} \tag{4}$$

the next cases are

$$\psi_{2,3/2}(x) \quad = \quad \frac{-\sqrt{2}}{60\sqrt{\pi}}\left(15x^4 L(x) + (8x^4+9x^2-2)S(x)\right),$$

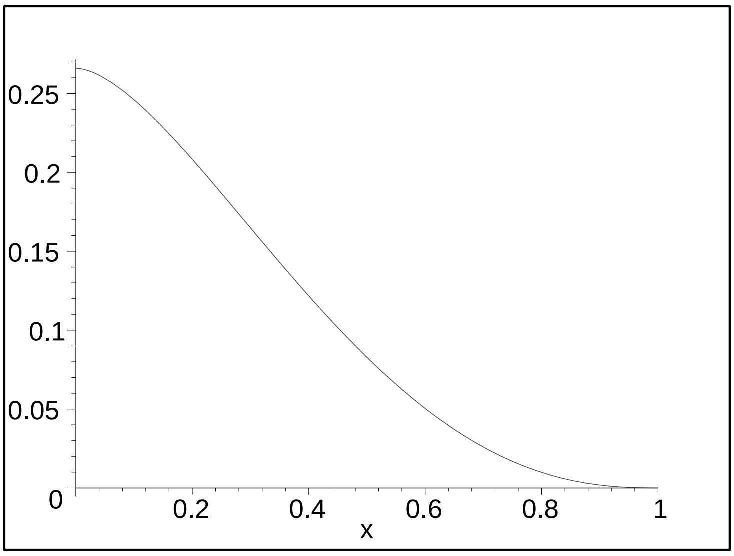

Figure 1: $\psi_{2,1/2}$

$$
\begin{aligned}
\psi_{2,5/2}(x) &= \frac{\sqrt{2}}{2520\sqrt{\pi}}\left(105x^6L(x)+(48x^6+87x^4-38x^2+8)S(x)\right).\\
\psi_{4,1/2}(x) &= \frac{\sqrt{2}}{30\sqrt{\pi}}\left((45x^4+60x^2)L(x)+(16x^4+83x^2+6)S(x)\right),\\
\psi_{4,3/2}(x) &= \frac{-\sqrt{2}}{420\sqrt{\pi}}\big((105x^6+210x^4)L(x)\\
&\quad +(32x^6+247x^4+40x^2-4)S(x)\big),\\
\psi_{4,5/2}(x) &= \frac{\sqrt{2}}{30240\sqrt{\pi}}\big((945x^8+2520x^6)L(x)\\
&\quad +(256x^8+263x^6+690x^4-136x^2+16)S(x)\big),\\
\psi_{6,1/2}(x) &= \frac{\sqrt{2}}{280\sqrt{\pi}}\big((525x^6+2100x^4+840x^2)L(x)\\
&\quad +(128x^6+1779x^4+1518x^2+40)S(x)\big).
\end{aligned}
$$

The general case will be proven below to be of the form

$$\psi_{2m,(2\ell-1)/2}(x) = x^{2\ell}p_{m,\ell}(x^2)L(x)+q_{m,\ell}(x^2)S(x) \tag{5}$$

where $p_{m,\ell}$ is of degree $m-1$ and $q_{m,\ell}$ is of degree $m-1+\ell$. These functions do not seem to be directly available via the technique of Buhmann [2].

Besides proving the above statements, the following background theory will touch a number of general issues concerning radial basis function construction. In particular, it will exhibit the important role of the Bessel radial basis functions.

2 Radial Transforms

To analyze the recursion and the positive definiteness of these functions, we now have to refer to the machinery of [7] and [6]. Proof details can be found there.

It is well–known that a radial basis function

$$\Phi(x) := \phi(\|x\|_2),\ x \in \mathbb{R}^d$$

has a radial d–variate Fourier transform

$$\hat{\Phi}(\omega) \quad = \quad \|\omega\|_2^{-(d-2)/2} \int_0^\infty \phi(r) r^{d/2} J_{(d-2)/2}(r\|\omega\|_2) dr \tag{6}$$

if the integral exists. It allows the Fourier transform of a radial function to be written as a univariate *Hankel transform*.

We now introduce $t = r^2/2$ as a new variable, writing a radial basis function ϕ "in f–form" as

$$\phi(\|\cdot\|_2) = f(\|\cdot\|_2^2/2). \tag{7}$$

Then (6) for $\nu = (d-2)/2$ turns into

$$(F_\nu f)(s) := \int_0^\infty f(t) t^\nu H_\nu(ts) dt \tag{8}$$

with the function

$$\left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = H_\nu(z^2/4) = \sum_{k=0}^\infty \frac{(-z^2/4)^k}{k!\Gamma(k+\nu+1)} \tag{9}$$

for $\nu \in \mathbb{C}$. Like (6), this is a generalization of the Fourier transform on radial functions, but now in f–form. Note that both transforms mimic Fourier transforms for spaces of fractal dimension, because $\nu = (d-2)/2$ need not be a half–integer.

The functions

$$\left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = H_\nu(z^2/4)$$

are called *oscillatory radial basis functions* by Fornberg et. al. [3]. The above derivation shows that their f–form is of central importance, because it guides the Fourier transform of general radial functions in f form. We shall use them frequently in what follows.

Using derivative formulae for the H_ν functions, one gets

$$-\frac{d}{ds}F_\nu(f)(s) = F_{\nu+1}(f)(s) \text{ and } (F_{\nu+1}(-f'))(s) = (F_\nu(f))(s).$$

Going back to $\nu = (d-2)/2$, these are the basic features of the dimension walk, but we shall need them later in steps of dimension one:

Theorem 2.1. *If the mentioned Fourier transforms and derivatives exist,*

- *the $(d+2)$-variate Fourier transform of a radial function after the f–form substitution (7) is the negative univariate derivative of the d-variate Fourier transform in f–form, and*
- *the d–variate Fourier transform of a radial function in f–form is the $(d+2)$–variate Fourier transform of the negative derivative of f.*

The dimension walk, expressed via derivatives, is extremely useful when programming with radial basis functions. It turns out that all the relevant classes of radial basis functions, when written in f form, are invariant under differentiation and integration, while the F_ν operators map the class to another one which is also closed under differentiation and integration. Implementing the general class in f form automatically yields an implementation of all derivatives. But we shall need fractional derivatives to generalize all of this.

To this end, [7] introduces a scale of integral operators

$$I_\alpha(f)(t) := \int_t^\infty f(s)\frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}ds \tag{10}$$

for all $\alpha > 0,\ t \geq 0$, defined on continuous functions on $[0,\infty)$ with compact support or exponential decay at infinity. The simplest special case is

$$I_1(f)(t) := \int_t^\infty f(s)ds$$

with the inverse

$$I_{-1}(f)(r) := -f'(r)$$

we already had above, doing the dimension walk. These operators satisfy

$$I_\alpha \circ I_\beta = I_{\alpha+\beta}$$

for all $\alpha,\ \beta > 0$. Thus the "semi-integration" operator

$$I_{1/2}(f)(t) \quad = \quad \int_t^\infty \frac{f(s)}{\sqrt{\pi(s-t)}}ds$$

satisfies $I_{1/2} \circ I_{1/2} = I_1$. These definitions can be extended to let

$$I_\alpha \circ I_\beta = I_{\alpha+\beta}$$

hold for all $\alpha, \beta \in \mathbb{R}$ and on suitable domains, but we refer to [7, 6] for details. Note that all of these operators are intended to work on f–forms of radial functions, not on their standard form. They are closely connected but not identical to what is used in the standard form of fractional calculus.

With these operators at hand, [7, 6] generalize the dimension walk to

$$\begin{array}{rcl} (F_\mu \circ F_\nu)(f) & = & I_{\nu-\mu}(f) \\ F_\mu & = & I_{\nu-\mu}F_\nu \\ F_\nu & = & F_\mu I_{\nu-\mu} \end{array} \tag{11}$$

as far as the operators are applicable, in particular for $\nu \geq \mu$ and on compactly supported functions, and this is what we need for generating the missing Wendland functions.

3 Application to Wendland Functions

Due to a result of Askey [1] the radial truncated power function

$$A_\mu(\cdot) := (1 - \|\cdot\|_2)_+^\mu$$

is positive definite on $\mathbb{R}^d$ for $\mu \geq \lfloor d/2 \rfloor + 1$, because it has a strictly positive radial Fourier transform in this case. Following [9], p. 81 we allow μ to be real. The f–form of Askey functions is

$$a_\mu(s) := (1 - \sqrt{2s})_+^\mu.$$

Since the I_α operators preserve compact supports and are applicable to a_μ for all $\alpha,\ \mu > 0$, the functions

$$\psi_{\mu,\alpha}(r) := (I_\alpha(a_\mu))(r^2/2)) \tag{12}$$

are well–defined and supported in $[0,1]$ for all $\alpha, \mu > 0$ and can be called *general Wendland functions*, and their f–form is

$$a_{\mu,\alpha} := I_\alpha(a_\mu) \text{ with } a_{\mu,0} = a_\mu.$$

At this point, we do not know for which parameters they are positive definite in which space dimension.

Let us evaluate their f–form as a finite integral

$$\begin{aligned}
(I_\alpha a_\mu)(u) &= \int_0^\infty (1-\sqrt{2s})_+^\mu \frac{(s-u)_+^{\alpha-1}}{\Gamma(\alpha)} ds \\
&= \int_u^{1/2} (1-\sqrt{2s})^\mu \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&= \int_{\sqrt{2u}}^1 t(1-t)^\mu \frac{(t^2/2-u)^{\alpha-1}}{\Gamma(\alpha)} dt \qquad (13) \\
&= \int_x^1 t(1-t)^\mu \frac{(t^2-x^2)^{\alpha-1}}{\Gamma(\alpha)2^{\alpha-1}} dt \\
&= \psi_{\mu,\alpha}(x)
\end{aligned}$$

for $0 \le u = x^2/2 \le 1/2$ or $0 \le x = \sqrt{2u} \le 1$. Note that this is (2) in the first section. If μ and α are integers, the resulting function is a single polynomial of degree $\mu + 2\alpha$ in the variable $x = \|\cdot\|_2$ on its support, but now we can construct the missing Wendland functions via half–integers α.

Theorem 3.1. *If $k \in \mathbb{N}/2$ and*

$$\mu \ge \lfloor d/2 + k \rfloor + 1, \qquad (14)$$

then the generalized Wendland function $\psi_{\mu,k}$ is positive definite on $\mathbb{R}^d$.

Proof: We use the identity $F_{\nu+\alpha} = F_\nu \circ I_\alpha$ from (11) for a_μ and get

$$F_{\nu+k} a_\mu = F_\nu(I_k(a_\mu)) \qquad (15)$$

which is valid for all $k,\ \mu > 0$ and all $\nu > -1/2$. But we restrict ourselves to the case

$$\nu + k \in \mathbb{Z}/2,\ \nu + k \ge -1/2,$$

and apply Askey's result for $d = 2\nu + 2k + 2$ to get that the left-hand side is strictly positive whenever

$$\mu \ge \lfloor \nu + k \rfloor + 2.$$

Looking at the right–hand side of (15) and introducing a new dimension with $(d-2)/2 = \nu$, we see that the function $I_k(A_\mu)$ is positive definite on $\mathbb{R}^d$ for (14). $\square$

Theorem 3.2. *For $k \in \mathbb{N}/2$, the d–variate Fourier transform $\mathcal{F}_d(\psi_{\mu,k})$ of $\psi_{\mu,k}$ with*

$$\mu = \lfloor d/2 + k \rfloor + 1 \tag{16}$$

satisfies

$$\mathcal{F}_d(\psi_{\mu,k})(r) = \Theta(r^{-(d+2k+1)}) \text{ for } r \to \infty. \tag{17}$$

Proof: Using (15) again, we see that the d–variate Fourier transform of $\psi_{\mu,k}$ in f form is identical to the $(d+2k)$–variate Fourier transform of $\psi_{\mu,0}$ in f form. From section 10.5 of [9] we cite

$$\begin{array}{rcl} (\mathcal{F}_{2\mu-1}\phi_{\mu,0}(\cdot^2/2))(r) & = & \Theta(r^{-2\mu}) \\ (\mathcal{F}_{2\mu-2}\phi_{\mu,0}(\cdot^2/2))(r) & = & \Theta(r^{-2\mu+1}) \end{array} \tag{18}$$

for integer μ and the usual Fourier transform $\mathcal{F}_d$ in d dimensions. But if $d+2k$ is an integer, we can choose μ by (16) and get a Fourier transform with behavior (17). $\square$

To generalize (1) for half–integers k and even–dimensional spaces, this implies

Corollary 3.1. *For integer m and n, the generalized Wendland function $\psi_{m+n+1,n+1/2}$ taken for even dimensions $d = 2m$ is reproducing in a Hilbert space which is isomorphic to Sobolev space $H^{m+n+1}(\mathbb{R}^{2m}) = H^{d/2+k+1/2}(\mathbb{R}^d)$ where $k = n + 1/2$.* $\square$

4 Inductive Construction

This chapter proves the representation (5) for the missing Wendland functions $\psi_{2m,(2\ell-1)/2}$ for all ℓ and m. We start with $\ell = 1$ and general m.

Theorem 4.1. *The general Wendland functions $a_{2m,1/2}(t)$ have the form*

$$\begin{array}{rcl} a_{2m,1/2}(t) & := & \displaystyle\int_t^{1/2} \frac{(1-\sqrt{2s})^{2m}}{\sqrt{\pi(s-t)}} ds \\ & = & P_{m,0}(t)L(\sqrt{2t}) + Q_{m,0}(t)S(\sqrt{2t}) \end{array}$$

with polynomials $P_{m,0}$, $Q_{m,0}$ of degree m and

$$\begin{array}{rcl} P_{m,0}(1/2) & = & Q_{m,0}(1/2) \\ P_{m,0}(0) & = & 0 \end{array} \tag{19}$$

for all $m \geq 1$.

Proof: We can also consider

$$\begin{aligned} a_{2m,1/2}(x^2/2) &= \frac{\sqrt{2}}{\sqrt{\pi}}\int_x^1 \frac{r(1-r)^{2m}}{\sqrt{r^2-x^2}}dr \\ &= P_{m,0}(x^2/2)L(x)+Q_{m,0}(x^2/2)S(x) \end{aligned} \tag{20}$$

and transform this by $z := \sqrt{r^2-x^2}$ into

$$a_{2m,1/2}(x^2/2) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{\sqrt{1-x^2}} (1-\sqrt{x^2+z^2})^{2m}dz.$$

Applying the binomial formula leads to terms

$$g_k(x) := \int_0^{\sqrt{1-x^2}} (x^2+z^2)^k dz$$

for all $k \in \mathbb{Z}/2$ with $0 \le k \le m$ combining into

$$a_{2m,1/2}(x^2/2) = \frac{\sqrt{2}}{\sqrt{\pi}}\sum_{j=0}^{2m}(-1)^j\binom{2m}{j}g_{j/2}(x).$$

We get

$$\begin{aligned} g_{k+1}(x) &= \int_0^{\sqrt{1-x^2}} (x^2+z^2)(x^2+z^2)^k dz \\ &= x^2 g_k(x) + \int_0^{\sqrt{1-x^2}} z^2(x^2+z^2)^k dz \\ &= x^2 g_k(x) + \frac{\sqrt{1-x^2}}{2(k+1)} - \frac{1}{2(k+1)}\int_0^{\sqrt{1-x^2}} (x^2+z^2)^{k+1} dz \\ &= x^2 g_k(x) + \frac{\sqrt{1-x^2}}{2(k+1)} - \frac{1}{2(k+1)} g_{k+1}(x) \end{aligned}$$

such that

$$\begin{aligned} g_{k+1}(x)\left(1+\frac{1}{2(k+1)}\right) &= x^2 g_k(x) + \frac{\sqrt{1-x^2}}{2(k+1)} \\ g_{k+1}(x) &= \frac{2k+2}{2k+3}x^2 g_k(x) + \frac{S(x)}{2k+3} \end{aligned}$$

is a useful recursion that boils everything down to

$$\begin{aligned} g_0(x) &= \sqrt{1-x^2} &&= S(x) \\ g_{1/2}(x) &= -\frac{1}{2}x^2 L(x) + \frac{\sqrt{1-x^2}}{2} &&= \frac{1}{2}\left(S(x)-x^2L(x)\right). \end{aligned}$$

This gives polynomials $p_j, q_j,\ r_j$ of degree at most j with

$$\begin{aligned} g_j(x) &= S(x)p_j(x^2) \\ g_{j-1/2}(x) &= L(x)q_j(x^2) + S(x)r_{j-1}(x^2) \end{aligned}$$

such that

$$\begin{aligned} \frac{\sqrt{\pi}}{\sqrt{2}} a_{2m,1/2}(x^2/2) &= \sum_{i=0}^{m} \binom{2m}{2i} g_i(x) \\ &\quad - \sum_{i=1}^{m} \binom{2m}{2i-1} g_{i-1/2}(x) \\ &= S(x)\left(\sum_{i=0}^{m} \binom{2m}{2i} p_i(x^2) - \sum_{i=1}^{m} \binom{2m}{2i-1} r_{i-1}(x^2) \right) \\ &\quad - L(x) \sum_{i=1}^{m} \binom{2m}{2i-1} q_i(x^2) \end{aligned}$$

is of the required form.

We have to check the additional conditions (19). Since q_j has no constant term, we get $P_{m,0}(0) = 0$. To prove the conditions at $1/2$, we remark that evaluation of an f form at $1/2$ means evaluation of the standard form at 1. We rewrite the representation (20) in terms of $z = \sqrt{1 - x^2}$ as

$$\begin{aligned} a_{2m,1/2}((1 - z^2)/2) &= P_{m,0}((1 - z^2)/2)L(\sqrt{1 - z^2}) \\ &\quad + Q_{m,0}((1 - z^2)/2)S(\sqrt{1 - z^2}) \end{aligned}$$

and now evaluation at $x = 1$ means evaluation at $z = 0$. We expand the terms at $z = 0$ to get

$$\begin{aligned} L(\sqrt{1 - z^2}) &= -z + \mathcal{O}(z^3) \\ S(\sqrt{1 - z^2}) &= z \end{aligned}$$

to see that

$$a_{2m,1/2}(1/2) = -P_{m,0}(1/2) + Q_{m,0}(1/2)$$

which vanishes due to the support of the f form ending at $1/2$. □

Theorem 4.2. *The representation*

$$a_{2m,(2\ell+1)/2}(s) = P_{m,\ell}(s)L(\sqrt{2s}) + Q_{m,\ell}(s)S(\sqrt{2s}) \tag{21}$$

with polynomials $P_{m,\ell}$, $Q_{m,\ell}$ of degree $m + \ell$ and

$$\begin{aligned} P_{m,\ell}(1/2) &= Q_{m,\ell}(1/2) \\ P_{m,0}(0) &= 0 \end{aligned}$$

holds for all $m \geq 1$ and all $\ell \geq 0$.

Proof: We know that (21) holds for $\ell = 0$ and all $m \geq 1$, and thus we proceed by induction on ℓ. By the standard dimension walk rules (11) we have to construct $P_{m,\ell},\ Q_{m,\ell}$ from $P_{m,\ell-1},\ Q_{m,\ell-1}$ to satisfy

$$-a_{2m,2\ell-1}(s) = a_{2m,(2\ell+1)/2}(s)'. \tag{22}$$

The induction recipe will be to define $P_{m,\ell}$ by

$$\begin{aligned} P_{m,\ell}(s)' &= -P_{m,\ell-1}(s) \\ P_{m,\ell}(0) &= 0 \end{aligned} \tag{23}$$

and to define $Q_{m,\ell}$ via

$$\begin{aligned} & Q_{m,\ell}(s)'(1-2s) - Q_{m,\ell}(s) \\ = & -\frac{P_{m,\ell}(s)}{2s} - Q_{m,\ell-1}(s)(1-2s). \end{aligned}$$

It can easily be shown that the above equation is uniquely solvable for $Q_{m,\ell}$ of degree $m+\ell$, and it implies

$$Q_{m,\ell}(1/2) = P_{m,\ell}(1/2).$$

Now we have to evaluate both sides of (22) in order to finish the induction. We need the derivatives

$$\begin{aligned} L'(x) &= \frac{1}{x\sqrt{1-x^2}} \\ S'(x) &= \frac{-x}{\sqrt{1-x^2}} \\ L(\sqrt{2s})' &= \frac{1}{2s\sqrt{1-2s}} \\ S(\sqrt{2s})' &= \frac{-1}{\sqrt{1-2s}} \end{aligned}$$

and get

$$\begin{aligned} & a_{2m,(2\ell+1)/2}(s)' \\ = & P_{m,\ell}(s)'L(\sqrt{2s}) + P_{m,\ell}(s)L(\sqrt{2s})' \\ & +Q_{m,\ell}(s)'S(\sqrt{2s}) + Q_{m,\ell}(s)S(\sqrt{2s})' \\ = & P_{m,\ell}(s)'L(\sqrt{2s}) + P_{m,\ell}(s)\frac{1}{2s\sqrt{1-2s}} \\ & +Q_{m,\ell}(s)'S(\sqrt{2s}) - Q_{m,\ell}(s)\frac{1}{\sqrt{1-2s}}. \end{aligned}$$

Focusing on the log terms above and in $a_{2m,(2\ell-1)/2}(s)$, we see that they are correct due to our choice of $P_{m,\ell}$. Now we are left with

$$\begin{array}{rl} & -Q_{m,\ell-1}(s)S(\sqrt{2s}) \\ = & -Q_{m,\ell-1}(s)\sqrt{1-2s} \\ \stackrel{?}{=} & P_{m,\ell}(s)\dfrac{1}{2s\sqrt{1-2s}} \\ & +Q_{m,\ell}(s)'\sqrt{1-2s} - Q_{m,\ell}(s)\dfrac{1}{\sqrt{1-2s}} \\ = & \dfrac{P_{m,\ell}(s) + 2s(1-2s)Q_{m,\ell}(s)' - 2sQ_{m,\ell}(s)}{2s\sqrt{1-2s}}. \end{array}$$

There, we introduce $z := \sqrt{1-2s}$ to get

$$\begin{array}{rl} & -Q_{m,\ell-1}\left(\dfrac{1-z^2}{2}\right)z \\ \stackrel{?}{=} & \dfrac{1}{z(1-z^2)}\left(P_{m,\ell}\left(\dfrac{1-z^2}{2}\right)\right. \\ & \left. + Q'_{m,\ell}\left(\dfrac{1-z^2}{2}\right)z^2(1-z^2) - Q_{m,\ell}\left(\dfrac{1-z^2}{2}\right)(1-z^2)\right). \end{array}$$

Since our construction yields

$$\begin{array}{rcl} P_{m,\ell}(1/2) & = & Q_{m,\ell}(1/2) \\ P_{m,\ell}(0) & = & 0, \end{array}$$

the critical denominator cancels, and our definition of $Q_{m,\ell}$ does the job. $\square$.

Note that Theorems 4.1 and 4.2 imply the representation (5). The special form of the $p_{m,\ell}$ part is due to the fact that (23) does not change the number of monomial terms, which is fixed at startup in Theorem 4.1, but only blows the degree up by one.

5 Open Problems

Readers will have noticed that we did not deal with the two remaining cases of generalized Wendland functions $\psi_{\mu,k}$: those with integer k and half–integer μ and those with both indices being half–integer. We did not focus on these because they are less promising. This is based on some hypotheses, confirmed for special cases, which we now formulate.

First, there is quite some evidence that

$$F_\nu(a_{\mu,0})(t) = \Theta\left(t^{-\nu-3/2}\right) \text{ for } t \to \infty \tag{24}$$

holds in full generality, in particular independent of μ, and is positive for all

$$\mu \geq \lceil \nu + \frac{3}{2} \rceil, \tag{25}$$

not only in the special cases related to (18). Note here that the standard case is recovered by

$$\lceil \nu + 3/2 \rceil = \lceil d/2 + 1/2 \rceil = \lfloor d/2 \rfloor + 1 \text{ if } \nu = (d-2)/2,$$

but we allow more general ν. The above assertions should follow from a very thorough inspection of chapters 6 and 10 of [9], and they generalize Theorems 3.1 and 3.2. Since large μ do not pay off, Wendland's notation (3) based on the smallest μ yielding positive definiteness for dimension d makes a lot of sense.

If the above is assumed, the minimal μ for $\psi_{\mu,\alpha}$ to be positive definite in generalized dimension d is

$$\mu = \lceil \alpha + d/2 + 1/2 \rceil.$$

Then (24) can be applied for $\nu = \alpha + (d-2)/2$, proving that the generalized Wendland function $\psi_{\lceil \alpha + d/2 + 1/2 \rceil, \alpha}$ is reproducing in Hilbert spaces isomorphic to $H^m(\mathbb{R}^d)$ for $m = \alpha + d/2 + 1/2$. This should be expected for all real d and α, leading to compactly supported reproducing kernels also in fractional–order Sobolev spaces.

To generate the integer–order Hilbert spaces in all dimensions, it therefore suffices to use the classical Wendland functions and those we described here. The μ parameter is not of central importance. However, fractional Sobolev spaces will require fractional α, but our integral operators (10) will allow to generate these either directly via (12) and (13), if the integral can be calculated, or from a polynomial Wendland function via

$$I_\alpha a_\mu = I_{\alpha - \lfloor \alpha \rfloor} I_{\lfloor \alpha \rfloor} a_\mu = I_{\alpha - \lfloor \alpha \rfloor} a_{\mu, \lfloor \alpha \rfloor}$$

if the operator $I_{\alpha - \lfloor \alpha \rfloor}$ can be explicitly evaluated on monomials.

The case of $\psi_{\mu,\alpha}$ with half–integer $\mu \in (\mathbb{N}/2) \setminus \mathbb{N}$ and integer α can easily be handled with the methods of Section 1 and the MAPLE program presented there. It generates polynomials times $\sqrt{1-x}$. But under the above assertions, genuinely half–integer μ do not seem to be minimally chosen. Things would be different if there were a half–integer leeway in the condition (25), but $F_{1/2} a_{7/4,0}$ is not positive, thanks to MAPLE.

Finally we remark that there are good chances to put all of this into a uniform theory based on hypergeometric functions. But we leave these things open.

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