Minicourse - PDE Techniques for Image Inpainting Part I

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GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Outline

1. Digital Image Processing
   - Examples
   - What is a Digital Image and how do we Process it?
   - Mathematical Image Models
   - Gaussian Filtering, the Heat Equation and Nonlinear Diffusion

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   - Examples
   - What is a Digital Image and how do we Process it?
   - Mathematical Image Models
   - Gaussian Filtering, the Heat Equation and Nonlinear Diffusion
2. Image Inpainting
   - State of the Art Methods
   - The Variational/PDE Approach
   - Second- Versus Higher-Order PDEs for Inpainting

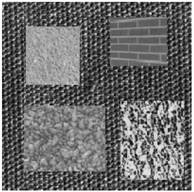
Outline

Imaging tasks[1]

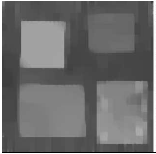
Image Denoising

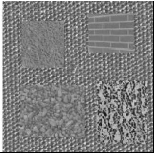
⇓

Cartoon/Texture Decomposition



⇓

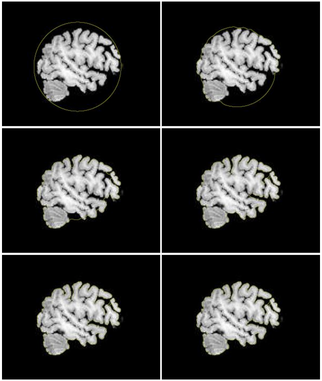




[1] Examples from L. Vese, S. Osher (2004)

Imaging tasks (cont.)

Image Segmentation[a]



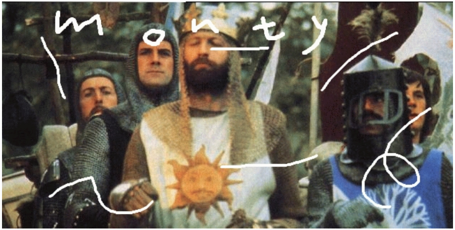
[a]Example from C. Guyader, L. Vese (2008)

Image Inpainting[a]

⇓

[a]Example from Bertalmio et al (1998)

Image inpainting - restore images of different kind

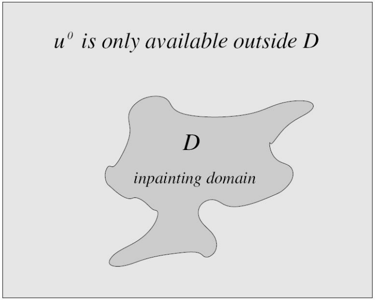


1

Image inpainting - restore images of different kind (cont.)


[1]

Image inpainting - in general the task is . . .



[a]

[a]source: Chan and Shen 2005

Inpainting = Image interpolation
Reconstruct the ideal image u on the missing domain D based on the data of u available outside D.

Some applications . . .

Digital Restoration of Frescoes:

Fife Senses Project on *Mathematical Methods for Image Analysis and Processing in the Visual Arts* sponsored by the Viennese Technology Fund (WWTF).
Collaboration between

- Faculty of Mathematics, University of Vienna (Peter A. Markowich, Massimo Fornasier):
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  Physcial Restoration of the Frescoes $\Rightarrow$ Comparison with the digital result

Some applications . . . (cont.)

Preliminary Results:

Given image:

Restored Image:

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. . . we need more sophisticated algorithms to restore the frescoes!

Some applications . . . (cont.)

Results with Cahn-Hilliard inpainting:

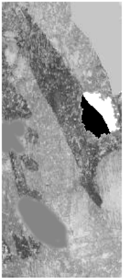
[2]

[2] Cahn-Hilliard inpainting after 200 timesteps with $\epsilon = 3$ and additional 800 timesteps with $\epsilon = 0.01$

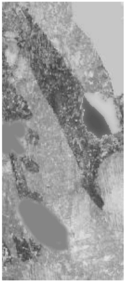
Some applications . . . (cont.)

Binary based total variation inpainting:
Based on the so recovered binary structure the fresco is colorized.

Given image f = initial condition u_0



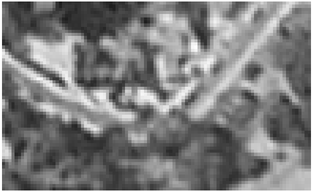
Inpainting result after 5000 iterations



Some applications . . . (cont.)

Road Reconstruction:[3]

Our data consists of satellite images of roads in Los Angeles of the following kind . . .



[3]Joint work with Andrea Bertozzi from UCLA

Some applications . . . (cont.)

Road Recontruction with Bitwise Cahn-Hilliard:

Goal: Remove objects like trees that cover the road and recover the picture of the plain road.

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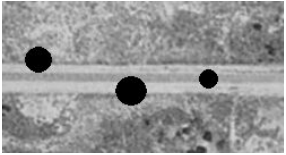

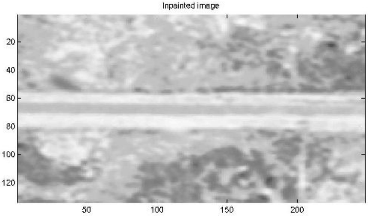

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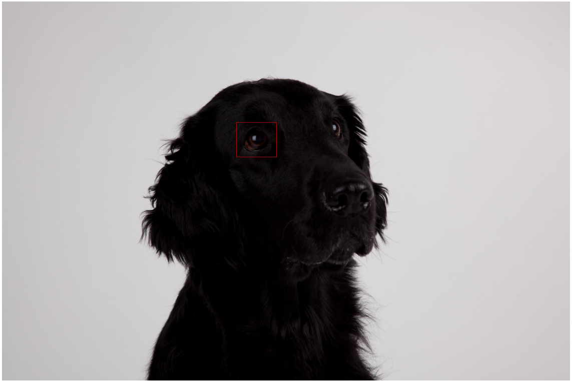
- . . . connects edges/objects even across large gaps (cf. road examples)
- . . . works for all numbers and shapes of missing domains
- . . . performs the restoration process in an automated way
- . . . and all this with a reliable and efficient numerical scheme.

Digital images

A digital image is obtained from an analogue image (representing the continuous world) by sampling and quantization . . .



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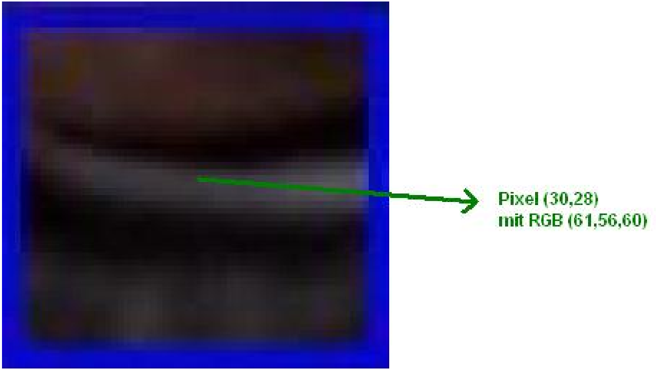
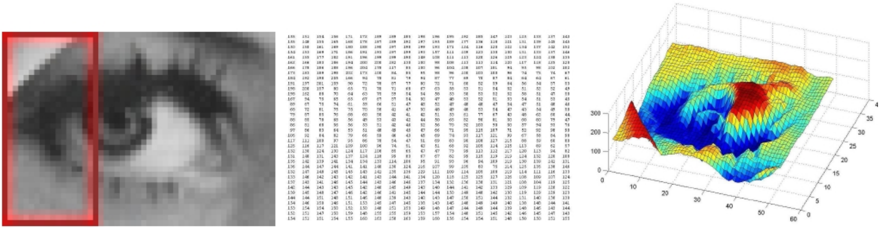


Digital images (cont.)

. . . it consists of pixels (grid element, matrix positions) which are assigned with the mean grayscale or colour information within this element . . .

- grayscale image
  $u : \Omega = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \rightarrow I = \{0, 1, \ldots, 255\}$
- colour image $u : \Omega \rightarrow I^3$, where $u(x, y) = (r, g, b)$ = (red, green, blue).



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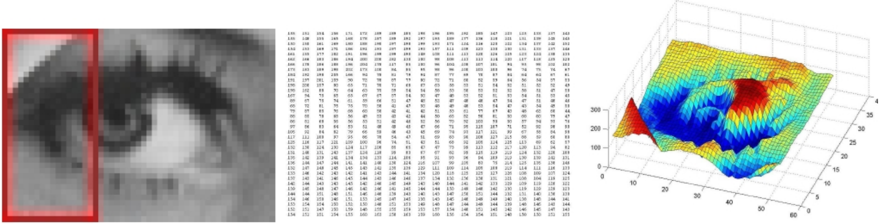

Image models and representations

Image function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, Ω is open and bounded (eventually with Lipschitz boundary).

- Deterministic image models, e.g., generalized functions, L^p functions, Sobolev images H^k, BV images.

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Deterministic image models - Distributions

Let Ω be a two-dimensional Lipschitz domain, i.e., open, bounded with Lipschitz boundary. Then we define the set of sensors on Ω as

$$D(\Omega) = \{\phi \in C^\infty(\Omega), \mathrm{supp}(\phi) \subseteq \Omega\}\,.$$

An image u on Ω is then treated as a distribution, i.e., a linear functional on $D(\Omega)$:

$$u : \phi \to (u, \phi)\,.$$

Deterministic image models - Distributions (cont.)

Properties:

- sensing is linear
- with additional positivity constraint we have that the Riesz representation theorem is valid, i.e., u is a positive distribution, then for any sensor $\phi \in D(\Omega)$ there exists a Radon measure μ on Ω s.t.

$$(u, \phi) = \int_{\Omega} \phi(x)\, d\mu.$$

- notion of derivatives, i.e., distributional derivative $v = \partial^{\alpha}\partial^{\beta}u$ is defined as a new distribution such that

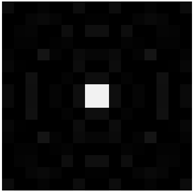
$$(v, \phi) = (-1)^{\alpha+\beta}\left(u, \partial^{\alpha}\partial^{\beta}\phi\right), \quad \forall \phi \in D(\Omega)$$

- sensing of distributions mimics the digital sensor devices in CCD cameras
- very general class of functions . . .

. . . to capture intrinsic visual features in an image we have to impose more information into the image model ⇒ Sobolev spaces, bounded variation, . . .

Deterministic image models - Distributions (cont.)

Example 1 - Bright spot as delta distribution



Here, u is a bright spot concentrated at the origin, i.e., $u(x) = \delta(x)$, where δ stands for the Dirac delta function. Then

$$(u, \phi) = \phi(0) \quad \text{for any sensor } \phi \in D(\Omega)$$

Deterministic image models - Distributions (cont.)

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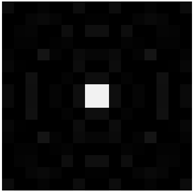

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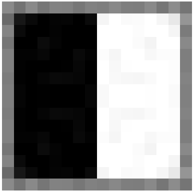
Example 2 - Step edge as Heaviside function



Here, u describes a step edge from 0 to 255, i.e., $u(x) = u(x_1, x_2) = H(x_1)$, where $H(t)$ is the Heaviside $0 - 255$ step function, i.e.,

$$H(t) = \begin{cases} 255 & t \geq t_1 \\ 0 & t < t_1. \end{cases}$$

Deterministic image models - L^p functions

For any $p \in [0, \infty)$, the Lebesgue L^p function space is defined as

$$L^p(\Omega) = \left\{ u : \int_\Omega |u(x)|^p \, dx < \infty \right\}.$$

For $p = \infty$ an L^∞ image is understood as an essentially bounded function. Properties:

- Banach spaces with norms

$$\|u\|_p = \left[\int_\Omega |u(x)|^p \, dx \right]^{1/p}.$$

- L^p images are naturally distributional images (Riesz respresentation theorem for $L^p + D(\Omega)$ is dense in L^{p*}, for $1 \leq p^* < \infty$) ...
- ...but they carry more structures than general distributions, cf. layer-cake representation.

Deterministic image models - H^k functions

- An image $u \in L^2(\Omega)$ whose (distributional) gradient ∇u is in $L^2 \times L^2$ is said to be Sobolev H^1.

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- $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, v)_{H^1} = (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2 \times L^2},$$

and corresponding norm

$$\|u\|_{H^1} = \sqrt{\|u\|_{L^2}^2 + \|\nabla u\|_{L^2 \times L^2}^2}.$$

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  Measure image information!
- Higher-order Sobolev spaces $H^k(\Omega)$ defined in similar fashion (higher-order derivatives!).

Deterministic image models - The total variation (TV)

For $u \in L^1_{loc}(\Omega)$

$$V(u,\Omega) := \sup\left\{\int_\Omega u\nabla\cdot\varphi\,dx : \varphi \in \left[C_c^1(\Omega)\right]^2, \|\varphi\|_\infty \leq 1\right\}$$

is the variation of u. Further

$u \in BV(\Omega)$ **(the space of bounded variation functions)**

$$\Leftrightarrow$$

$$V(u,\Omega) < \infty.$$

In such a case,

$$|Du|(\Omega) = V(u,\Omega),$$

where $|Du|(\Omega)$ is the **total variation** of the finite Radon measure Du, the derivative of u in the sense of distributions.

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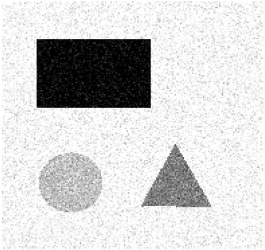
where $|Du|(\Omega)$ is the **total variation** of the finite Radon measure Du, the derivative of u in the sense of distributions.
For a function $u \in C^1(\Omega)$, the total variation of u is equivalent to

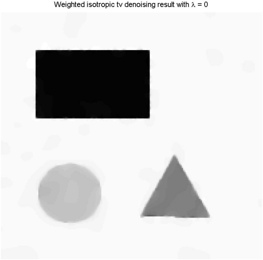
$$|\mathbf{Du}|(\boldsymbol{\Omega}) = \int_{\boldsymbol{\Omega}} |\nabla\mathbf{u}|\,\mathbf{dx}.$$

Deterministic image models - The total variation (cont.)

Properties:

- BV images allow edges (in contrast to $W^{1,1}$, i.e., H^1 images).
- The total variation penalizes small irregularities/oscillations while respecting instrinsic image features such as edges.





Deterministic image models - The total variation (cont.)

What does it measure?

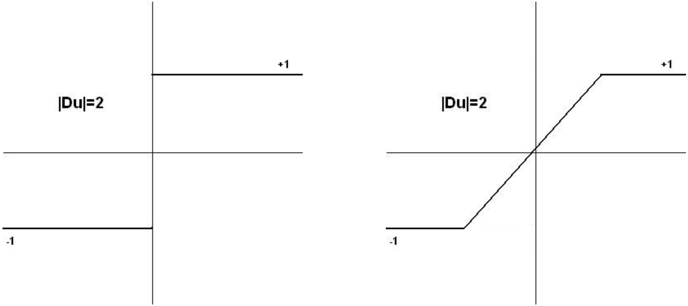


Figure: The total variation measures the size of the jump.

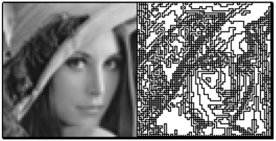
Level lines & coarea formula

Let $u = u(x)$ be a grayscale image on Ω. For each real value λ we define the λ-level line of u to be

$$\Gamma_\lambda = \{x \in \Omega : u(x) = \lambda\} .$$

Classical level line representation is the one-parameter family of all the level lines, i.e.,

$$\{\Gamma_\lambda : \ \lambda \in \mathbb{R}\}$$



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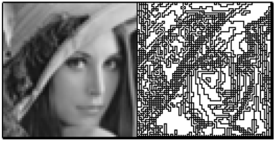

Coarea formula (weak version): For a smooth image u we have

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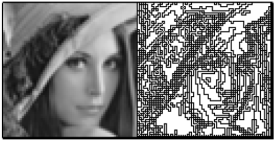

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…a similar relation holds for functions of bounded variation (length has to be redefined).

Gaussian filtering

Let f be a grayscale image, represented by a real-valued mapping $f \in L^1(\mathbb{R}^2)$ (zero-expansion of image from Ω to $\mathbb{R}^2$). Gaussian smoothing of f by convolution

$$u = (K_\sigma * f)(x) := \int_{\mathbb{R}^2} K_\sigma(x-y) f(y)\, dy,$$

where K_σ denotes the 2D Gaussian of width $\sigma > 0$, i.e.,

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Properties:

- Since $K_\sigma \in C^\infty(\mathbb{R}^2)$ we get $u = K_\sigma * f \in C^\infty(\mathbb{R}^2)$ (even if f is only absolutely integrable).

Gaussian filtering (cont.)

- Behaviour in the frequency domain:

$$\begin{aligned}(\mathcal{F}(K_\sigma * f))(w) &= (\mathcal{F}K_\sigma)(w) \cdot (\mathcal{F}f)(w) \\ &= e^{-|w|^2/(2\sigma^2)} \cdot (\mathcal{F}f)(w).\end{aligned}$$

Low-pass filter that attenuates high frequencies.

Gaussian filtering (cont.)

- Behaviour in the frequency domain:

$$\begin{aligned}(\mathcal{F}(K_\sigma * f))(w) &= (\mathcal{F}K_\sigma)(w) \cdot (\mathcal{F}f)(w) \\ &= e^{-|w|^2/(2\sigma^2)} \cdot (\mathcal{F}f)(w).\end{aligned}$$

Low-pass filter that attenuates high frequencies.

Equivalence to linear diffusion filtering ...

The heat equation for image smoothing

For a given $f \in C(\mathbb{R}^2)$ we consider the linear diffusion prozess

$$\begin{aligned} u_t &= \Delta u \\ u(x,0) &= f(x), \end{aligned}$$

with solution

$$u(x) = \begin{cases} f(x) & t = 0 \\ (K_{\sqrt{2t}} * f)(x) & t > 0. \end{cases}$$

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Properties:

- Solution is unique in the class of functions u with
$$|u(x,t)| \leq M \cdot e^{a|x|^2}, \quad (M, a > 0).$$
- The solution depends continuously on the initial image f w.r.t. $\|\cdot\|_{L^\infty}$.
- The solution fulfills the max-min principle
$$\inf_{\mathbb{R}^2} f \leq u(x,t) \leq \sup_{\mathbb{R}^2} f \quad \text{on } \mathbb{R}^2 \times [0,\infty).$$

The heat equation for image smoothing (cont.)

- Gaussian smoothing structures of order σ requires to stop the diffusion process at time

$$T = \frac{1}{2}\sigma^2.$$

Nonlinear diffusion filtering

Perona-Malik model: Replace linear diffusion process by a nonlinear one which reduces the diffusivity at edges:

$$u_t = \operatorname{div}\left(g(|\nabla u|^2)\nabla u\right),$$

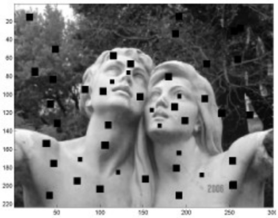
where g is called the diffusivity of the process, e.g.,

$$g(s^2) = \frac{1}{1 + s^2/\delta^2} \quad (\delta > 0).$$

Interplay of forward- and backward diffusion $\Rightarrow$ image smoothing & edge detection in a single process.

The heat equation for image inpainting

Construction of grayvalues inside the holes by ”averaging“ grayvalues around the holes!



The heat equation for image inpainting (cont.)

Let $D \subset \Omega$ be a hole in the image domain. We solve

$$\begin{aligned} u_t &= \Delta u && \text{in } D \\ u(x,0) &= 0 && \text{in } D \\ u|_{\partial D} &= f|_{\partial D}. \end{aligned}$$

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Let $D \subset \Omega$ be a hole in the image domain. We solve

$$\begin{aligned} u_t &= \Delta u \qquad && \text{in } D \\ u(x,0) &= 0 \qquad && \text{in } D \\ u|_{\partial D} &= f|_{\partial D}. \end{aligned}$$

... or we solve for the union of all holes D ...

$$\begin{aligned} u_t = \lambda \Delta u + \chi_{\Omega \setminus D}(u - f) \qquad && \text{in } \Omega \\ u(x,0) = 0 \qquad && \text{in } \Omega, \end{aligned}$$

where

$$\chi_{\Omega \setminus D}(x) = \begin{cases} 1 & x \in \Omega \setminus D \\ 0 & \text{otherwise,} \end{cases}$$

and for $1 \gg \lambda$.

Outline

Some history and classifications

- The term inpainting was invented by art restoration workers [G. Emile-Male 76, S. Walden 85] and first appeared in the framework of digital restoration in the work of [Bertalmio et al. 00] (PDE approach).
- Before this: works of engineers and computer scientists: statistical and algorithmic approaches in the context of image interpolation, e.g., [A. C. Kokaram et al. 95], image replacement, e.g., [H. Igehy, and L. Pereira 97], error concealment [K.-H. Jung et al. 94], and image coding, e.g., [J. R. Casas 96]. Some of these coding techniques already used PDEs for this task, e.g., [J. R. Casas 96].
- Mathematics community got involved in image restoration, using partial differential equations and variational methods for this task, e.g., [M. Nitzberg, D. Mumford, and T. Shiota 93, Masnou, and Morel 98, V. Caselles, J.-M. Morel, and C. Sbert 98, Bertalmio et al. 00].

Some history and classifications (cont.)

- PDE- and variational methods are local inpainting methods, i.e., restore structure (geometry) but are not applicable for texture restoration/synthesis.
- Global inpainting methods: texture synthesis & exemplar based inpainting, e.g., [Efros and Leung 99]
- Inpainting in transform spaces, e.g., [Eldad, Starck, Sapiro]
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Let us consider some examples . . .

Bertalmio et al.'s approach 2000

Bertalmio et al.'s model: based on observations about the work of museum artists, who restorate old paintings.

- Principle: prolongating the image intensity in the direction of the level lines (sets of image points with constant grayvalue) arriving at the hole. This results in solving a discrete approximation of the PDE

$$u_t = \nabla^{\perp} u \cdot \nabla \Delta u,$$

  solved within the hole D extended by a small strip around its boundary.
- To avoid the crossing of level lines: apply intermediate steps of anisotropic diffusion.

Masnou & Morel's disocclusion 1998

Masnou and Morel presented a variational technique for removing occlusions of objects with the goal of image segmentation.

- Idea: connect T-junctions at the occluding boundaries of objects with Euler elastica minimizing curves! (A curve is said to be Eulers elastica if it is the equilibrium curve of the Euler elastica energy

$$E(\gamma) = \int_\gamma (a + b\kappa^2)\, ds,$$

where ds denotes the arc length element, $\kappa(s)$ the scalar curvature, and a, b two positive constants.)
- Basic principle: prolongate level lines by minimizing their length and curvature.

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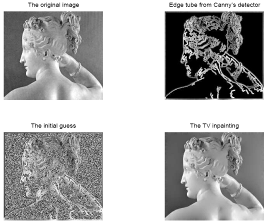
Chan & Shen: TV inpainting 2001

Chan & Shen's approach is based on the most famous model in image processing, the total variation (TV) model.

- Principle: action of anisotropic diffusion inside the inpainting domain $\Rightarrow$ preserves edges and diffuses homogeneous regions and small oscillations like noise.
- $$u_t = \lambda \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + \chi_{\Omega \setminus D}(f - u).$$
- Disadvantage: level lines are interpolated linearly.



The variational approach

$$\mathcal{J}(u) = R(u) + \frac{1}{2\lambda} \left\| \chi_{\Omega \setminus D}(u - g) \right\|_{B_1}^2 \to \min_{u \in B_2},$$

where

$$\chi_{\Omega \setminus D}(x) = \begin{cases} 1 & \Omega \setminus D \\ 0 & D, \end{cases}$$

is the characteristic function of $\Omega \setminus D$.

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Now for $B_1 = L^2(\Omega)$ we also have ...

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…in other situations we encounter equations that do not come from variational principles, such as Cahn-Hilliard- and TV-H^{-1} inpainting. Then the image processing approach is directly given by an (evolutionary) PDE.

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In the setting of such global inpainting approaches . . .

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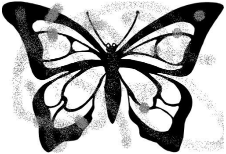
These properties make such inpainting approaches applicable to all kinds of damaged images . . .

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... so why do we consider higher-order inpainting models?

Second order approaches

Example: **TV-inpainting**

$$|Du|(\Omega) \approx \int_\Omega |\nabla u| \, dx$$

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- Penalizes length of edges $\Longrightarrow$ cannot connect contours across very large distances −
- Can result in corners of the level lines across the inpainting domain −

Higher order approaches

Example: **Euler's elastica inpainting**

$$\min_u \int_\Omega (a + b\left(\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right)\right)^2)|\nabla u|\, dx$$

$$\Longleftrightarrow$$

$$\min_{\Gamma_\lambda} \int_{-\infty}^{\infty} (a \,\text{length}(\Gamma_\lambda) + b \,\text{curvature}^2(\Gamma_\lambda))\, d\lambda.$$

[4] Pictures taken from Chan, Kang, Shen 2002

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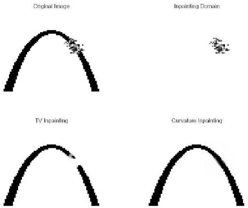
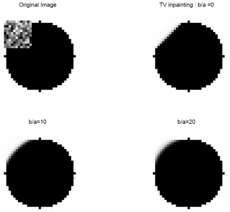


(c) Connectivity principle



(d) Smooth continuation [4]

[4] Pictures taken from Chan, Kang, Shen 2002

Higher order approaches (cont.)

Example: **Cahn-Hilliard inpainting**
The inpainted version u of $f \in L^2(\Omega)$ assumed with any (trivial) extension to the inpainting domain is constructed by following the evolution of

$$u_t = \Delta(-\epsilon \Delta u + \frac{1}{\epsilon} F'(u)) + \chi_{\Omega \setminus D}(f - u) \quad \text{in } \Omega,$$

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where $F(u)$ is a so called double-well potential, e.g., $F(u) = (u^2 - 1)^2$. [Bertozzi et al. 06] proved that in the limit $\lambda_0 \to \infty$ a stationary solution solves

$$\begin{aligned} \Delta(\epsilon \Delta u - \frac{1}{\epsilon} F'(u)) &= 0 && \text{in } D \\ u &= f && \text{on } \partial D \\ \nabla u &= \nabla f && \text{on } \partial D, \end{aligned}$$

for f regular enough ($f \in C^2$).

Higher order approaches

Challenges

- Higher order equations are very new and little is known about them.
- Often they do not possess a maximum principle or comparison principle.
- For the proof of well-posedness of higher order inpainting models variational methods are often not applicable.
- Need of simple but effective models.
- Need of stable and fast numerical solvers.

Next time ...

- Cahn-Hilliard inpainting:

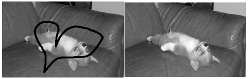
$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(f-u) \quad \text{in } \Omega.$$

(existence, stationary solutions, characterization of solutions, boundary conditions.)

- TV-H^{-1} inpainting: (generalization of Cahn-Hilliard inpainting for grayvalue images!)

$$u_t = \Delta p + \frac{1}{\lambda}\chi_{\Omega\setminus D}(f-u), \quad p \in \partial TV(u).$$

(Stationary solution, cooperation of transport and diffusion, error analysis.)



End of Part I

Thank you for your kind attention!

write to: c.b.schonlieb@damtp.cam.ac.uk