Minicourse - PDE Techniques for Image Inpainting Part II

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Outline

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1. Cahn-Hilliard Inpainting
2. TV-H^{-1} Inpainting

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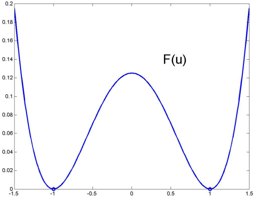
Cahn-Hilliard Inpainting Model

Following the approach of Bertozzi, Esedoglu and Gillette (2006) inpainting with Cahn-Hilliard is given by the evolution equation

$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u),$$

where g is a given binary image.

The Cahn-Hilliard equation is a pattern formation equation!



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The Cahn-Hilliard equation models phase separation and subsequent phase coarsening of binary alloys.

Figure: $\epsilon = 0.1$

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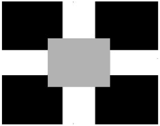
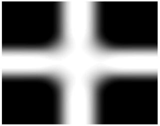
This is an interesting inpainting model since ...

- ...it provides us with a relatively simple fourth order PDE for the inpainting of binary images, rather than a more complex gradient flow to minimize a curvature functional
- ...its numerical solution is an order of magnitude or more faster than other competing PDE-based inpainting methods.

Inpainting examples $\lambda = 10^{-5}$

We apply Cahn-Hilliard inpainting in two steps:





[1]

[2]

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[1] u(1200) with $\epsilon = 0.1$

[2] u(2400) with $\epsilon = 0.01$

[3] Picture from Bertozzi et al.: steady states for $\epsilon = 0.01$ with and without previous $\epsilon = 0.1$ evolution.

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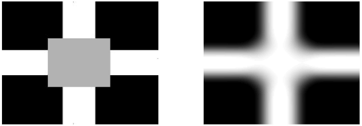
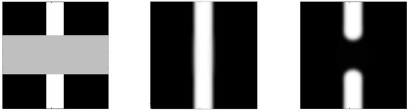

[1]


[2]

Note that steady states are not unique!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Inpainting examples $\lambda = 10^{-9}$

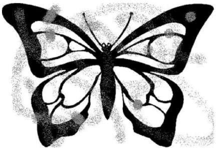
 [4]

 [5]

[4] u(200) with $\epsilon = 0.8$

[5] u(500) with $\epsilon = 0.01$

Inpainting examples $\lambda = 10^{-9}$



[6]

[7]

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[6] u(800) with $\epsilon = 0.8$
[7] u(1600) with $\epsilon = 0.01$

Rigorous results for the modified Cahn-Hilliard equation

Bertozzi, Esedoglu, Gillette 06 (BEG 06):

- Global existence for the evolution equation: For initial data $u_0 \in L^2(\Omega)$ and $\lambda \leq O(\epsilon^3)$:
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for g regular enough ($g \in C^2$).

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$$

for g regular enough ($g \in C^2$).
Continuation of the image gradient into the missing domain!

New result

Theorem (Existence of a stationary solution)

The stationary equation

$$\Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u) = 0 \quad \textit{in } \Omega,$$

has a unique weak solution in $H^1(\Omega)$ *if* $\lambda \leq O(\epsilon^3)$ *and* $|D| < \frac{2}{C}$ *for a constant* $C > 0$.

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The difficulty in dealing with the stationary equation: lack of an energy functional for the inpainting equation!

For the following illustration we replace $\frac{1}{\lambda}\chi_{\Omega\setminus D}$ by

$$\lambda(x) = \begin{cases} \lambda_0 \gg 1 & \text{in } \Omega \setminus D \\ 0 & \text{otherwise.} \end{cases}$$

Existence of a stationary solution - difficulties

In fact the most evident variational approach would be to minimize the functional

$$\int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \right) \, dx + \frac{1}{2} \left\| \lambda(u - f) \right\|_{-1}^2 .$$

This minimization problem exhibits the optimality condition

$$0 = -\epsilon \Delta u + \frac{1}{\epsilon} F'(u) + \lambda \Delta^{-1} \left(\lambda(u - f) \right),$$

which splits into

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Hence the minimization of the functional above translates into a second-order diffusion inside the inpainting domain D, whereas a stationary solution of CHI fulfills

$$0 = \Delta \left(-\epsilon \Delta u + \frac{1}{\epsilon} F'(u) \right) \qquad \text{in } D$$

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Existence of a stationary solution - Proof

Main ideas of the proof:

- Formulation of a fixed point equation;
- Existence of a fixed point with Schauder;
- Fixed point = stationary solution.

Existence of a stationary solution - Proof (cont.)

We define a weak solution of the stationary equation as a function $u \in H = \{u \in H^1(\Omega),\ u|_{\partial\Omega} = f|_{\partial\Omega}\}$ that fulfills

$$\langle \epsilon \nabla u, \nabla \phi \rangle_2 + \left\langle \frac{1}{\epsilon} F'(u), \phi \right\rangle_2 - \langle \lambda(f-u), \phi \rangle_{-1} = 0, \quad \forall \phi \in H_0^1(\Omega).$$

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Dirichlet boundary conditions!

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Dirichlet boundary conditions!

$\mathbf{H^{-1}}$: We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ with corresponding norm $\|.\|_{-1}$. For a function $f \in H^{-1}(\Omega)$ the norm is defined as

$$\|f\|_{-1}^2 = \left\|\nabla \Delta^{-1} f\right\|_2^2 = \int_\Omega \left(\nabla \Delta^{-1} f\right)^2 \, dx.$$

Thereafter the operator Δ^{-1} denotes the inverse to the negative Dirichlet Laplacian, i.e., $u = \Delta^{-1} f$ is the unique solution to

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Existence of a stationary solution - Proof (cont.)

Fixed point argument:
We consider the fixed point operator $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ where $\mathcal{A}(v) = u$ fulfills for a given $v \in L^2(\Omega)$ the equation

$$\begin{cases} \frac{1}{\tau}\Delta^{-1}(u-v) = \epsilon\Delta u - \frac{1}{\epsilon}F'(u) + \Delta^{-1}\left[\lambda(f-u) + (\lambda_0-\lambda)(v-u)\right] & \text{in } \Omega, \\ u = f, \quad \Delta^{-1}\left(\frac{1}{\tau}(u-v) - \lambda(f-u) - (\lambda_0-\lambda)(v-u)\right) = 0 & \text{on } \partial\Omega, \end{cases}$$

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We define a weak solution of the fixed point equation as before ...
A fixed point of the operator $\mathcal{A}$, provided it exists, then solves the stationary equation with Dirichlet boundary conditions.

Existence of a stationary solution - Proof (cont.)

For the fixed point equation we can state a variational formulation. This is, for a given $v \in L^2(\Omega)$ the fixed point equation is the Euler-Lagrange equation of the minimization problem

$$u^* = \text{argmin}_{u \in H^1(\Omega), u|_{\partial\Omega} = f|_{\partial\Omega}} \mathcal{J}^\epsilon(u, v)$$

with

$$\mathcal{J}^\epsilon(u, v) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \right) dx + \frac{1}{2\tau} \|u - v\|_{-1}^2 + \frac{\lambda_0}{2} \left\| u - \frac{\lambda}{\lambda_0} f - \left(1 - \frac{\lambda}{\lambda_0}\right) v \right\|_{-1}^2 .$$

Existence of a stationary solution - Proof (cont.)

Lemma (Existence & uniqueness for fixed point equation)

The fixed point equation admits a weak solution in $H^1(\Omega)$. For $\tau \leq C\epsilon^3$, where C is a positive constant depending on $|\Omega|, |D|,$ and F only, the weak solution is unique.

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Further we prove that the operator $\mathcal{A}$ admits a fixed point under certain conditions.

Lemma (Existence of a fixed point)

Set $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$, $\mathcal{A}(v) = u$, where $u \in H^1(\Omega)$ is the unique weak solution of the fixed point equation. Then $\mathcal{A}$ admits a fixed point $\hat{u} \in H^1(\Omega)$ if $\tau \leq C\epsilon^3$ and $\lambda_0 \geq \frac{C}{\epsilon^3}$ for a positive constant C depending on $|\Omega|, |D|$, and F only.

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Hence the existence of a stationary solution follows under the condition $\lambda_0 \geq C/\epsilon^3$.

Existence of a stationary solution - Proof (cont.)

Existence & uniqueness for fixed point equation:

This proof follows standard arguments from variational theory, i.e., we prove the existence of a unique minimizer for $\mathcal{J}$ (by showing coercivity, lower-semicontinuity, strict convexity of the functional $\mathcal{J}$).

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Existence of a fixed point:

We are going to apply:

Theorem (Schauder's fixed point theorem)

Suppose that $K \subset X$ is compact and convex, and assume also $A : K \to K$ is continuous. Then A has a fixed point.

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We start with proving that

$$\|\mathcal{A}(v)\|_2^2 = \|u\|_2^2 \leq \beta \|v\|_2^2 + \alpha$$

for constants $\beta < 1$ and $\alpha > 0 \to K = B(0, M) = \{u \in L^2(\Omega) : \|u\|_2 \leq M\}$ into itself for an appropriate constant $M > 0$. We conclude with showing the compactness of K and the continuity of the fixed point operator $\mathcal{A}$.

The $\Gamma-$ Limit $\epsilon \to 0$

The sequence of Cahn-Hilliard functionals

$$CH(u) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u) \right) \, dx$$

Γ-converges in the topology $L^1(\Omega)$ to

$$TV(u) = \begin{cases} C_0 \, |Du| \, (\Omega) & \text{if } u = \chi_E \text{ for a Borel measurable subset } E \text{ of } \Omega \\ +\infty & \text{otherwise} \end{cases}$$

as $\epsilon \to 0$, where $C_0 = 2 \int_{-1}^{1} \sqrt{F(s)} \, ds$.[8]

[8] Modica & Mortola 1977

The $\Gamma-$ Limit $\epsilon \to 0$ (cont.)

Definition

Let $X = (X, d)$ be a metric space and (F_h), $h \in \mathbb{N}$ be family of functions $F_h : X \to [0, +\infty]$. We say that (F_h) Γ-converges to a function $F : X \to [0, +\infty]$ on X as $h \to \infty$ if $\forall x \in X$ we have

(i) for every sequence x_h with $d(x_h, x) \to 0$ we have

$$F(x) \leq \liminf_h F_h(x_h);$$

(ii) there exists a sequence $\bar{x_h}$ such that $d(\bar{x_h}, x) \to 0$ and

$$F(x) = \lim_h F_h(\bar{x_h})$$

(or, equivalently, $F(x) \geq \limsup_h F_h(\bar{x_h})$).

Then F is the Γ-limit of (F_h) in X and we write: $F(x) = \Gamma - \lim_h F_h(x)$, $x \in X$.

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The formulation of the Γ-limit for $\epsilon \to 0$ is analogous by defining a sequence ϵ_h with $\epsilon_h \to 0$ as $h \to \infty$.

The $\Gamma-$ Limit $\epsilon \rightarrow 0$ (cont.)

The important property of Γ-convergent sequences of functions F_h: their minima converge to minima of the Γ-limit F. In fact we have the following theorem

Theorem

Let (F_h) be like in the previous definition and additionally equicoercive, that is there exists a compact set $K \subset X$ (independent of h) such that

$$\inf_{x \in X} \{F_h(x)\} = \inf_{x \in K} \{F_h(x)\}.$$

If F_h Γ-converges on X to a function F we have

$$\min_{x \in X} \{F(x)\} = \lim_h \inf_{x \in X} \{F_h(x)\}.$$

Outline

TV-H^{-1} inpainting[9]

Motivated by this Γ-convergence result we propose the following higher-order inpainting method: the inpainted image u of $g \in L^2(\Omega)$, shall evolve via

$$u_t = \Delta p + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u), \quad p \in \partial TV(u),$$

where $\partial TV(u)$ denotes the subdifferential of

$$TV(u) = \begin{cases} |Du|(\Omega) & \text{if } |u(x)| \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

[9] $TV - H^{-1}$ denoising: S. Osher, A. Sole and L. Vese (2002); L. Lieu and L. Vese (2005)

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... and this is now a fourth-order inpainting model for grayvalue images!

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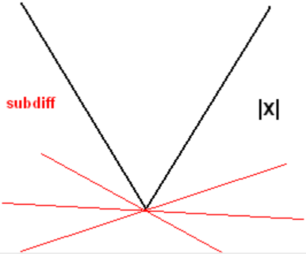
Subdifferential

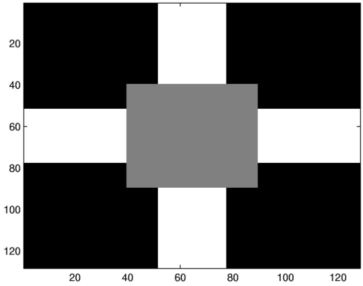
Definition

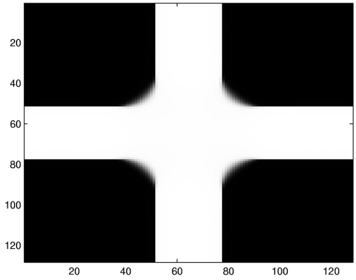
Let X be a locally convex space, X' its dual, $\langle .,. \rangle$ the bilinear pairing over $X \times X'$ and $\mathcal{J}$ a mapping of X into $\mathbb{R}$. The subdifferential of $\mathcal{J}$ at $u \in X$ is defined as

$$\partial\mathcal{J}(u) = \left\{p \in X' | \, \langle v - u, p\rangle \leq \mathcal{J}(v) - \mathcal{J}(u), \forall v \in X\right\}.$$



Inpainting example - connects edges across large gaps





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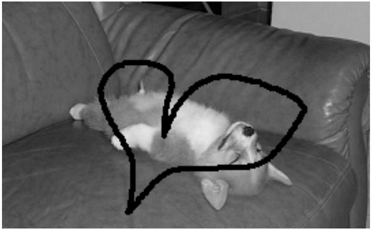
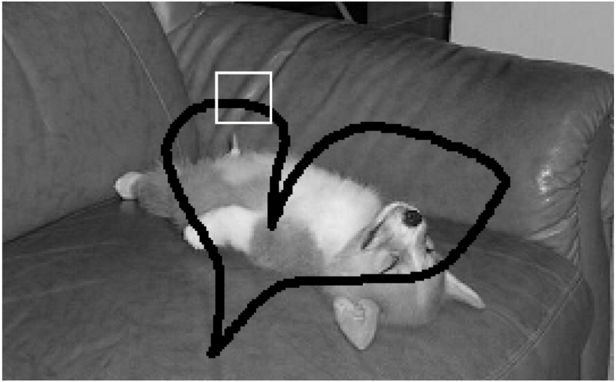
[10] u(10000) with $\lambda = 2 \cdot 10^{-5}$.

Inpainting example

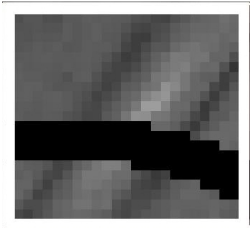
  [11]

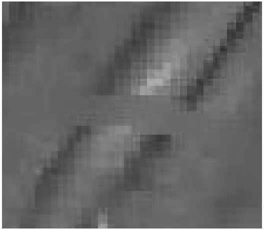
[11] u(1000) with $\lambda = 10^{-3}$.

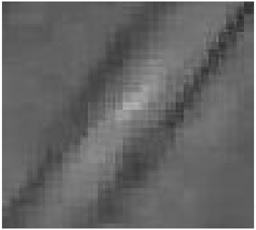
Inpainting example - lets take a closer look



4th order versus 2nd order method




[12]


[13]

[12] TV-L^2 u(5000) with $\lambda = 10^{-3}$.
[13] TV-H^{-1} u(1000) with $\lambda = 10^{-3}$.

TV-H^{-1} inpainting (cont.)

Applying the same strategy that we used for the existence of a stationary solution in the Cahn-Hilliard case, we can prove:

Theorem

Let $g \in L^2(\Omega)$. The stationary equation

$$\Delta p + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u) = 0, \quad p \in \partial TV(u)$$

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Reference: M. Burger, L. He, C.-B. Schönlieb, *Cahn-Hilliard inpainting and a generalization for grayvalue images*, SIAM J. Imaging Sci. Volume 2, Nr. 4, pp. 1129-1167 (2009)

Error Estimation With the Bregman Distance

Let f_{true} be the original image and $\hat{u}$ a stationary solution of of the TV-H^{-1} inpainting equation. In our considerations we use the symmetric Bregman distance defined as

$$D_{TV}^{symm}(\hat{u}, f_{true}) = \langle \hat{u} - f_{true}, \hat{p} - \xi \rangle_2 , \quad \hat{p} \in \partial TV(\hat{u}),\ \xi \in \partial TV(f_{true}).$$

We prove the following result

Theorem

Let f_{true} fulfill the so-called source condition, namely that

$$\textit{there exists } \xi \in \partial TV(f_{true}) \textit{ such that } \xi = \Delta^{-1} q \textit{ for a source element } q \in H^{-1}(\Omega),$$

and $\hat{u} \in BV(\Omega)$ be a stationary solution, given by $\hat{u} = u^s + u^d$, where u^s is a smooth function and u^d is a piecewise constant function.

Error Estimation With the Bregman Distance (cont.)

Theorem

Then the inpainting error reads

$$D_{TV}^{symm}(\hat{u}, f_{true}) + \frac{\lambda_0}{2}\|\hat{u} - f_{true}\|_{-1}^2 \leq \frac{1}{\lambda_0}\|\xi\|_1^2 + C\lambda_0 |D|^{(r-2)/r} \mathrm{err}_{\mathrm{inpaint}},$$

with $2 < r < \infty$*, constant* $C > 0$ *and*

$$\mathrm{err}_{\mathrm{inpaint}} := K_1 + K_2 \left(|D| \, C\left(M(u^s), \beta\right) + 2\left|R(u^d)\right|\right),$$

where K_1 *and* K_2 *are appropriate positive constants, and* C *is a constant depending on the smoothness bound* $M(u^s)$ *for* u^s *and* β *that is determined from the shape of* D*. The error region* $R(u^d)$ *is determined by the level lines of* u^d*.*

Error Estimation With the Bregman Distance (cont.)

Key ingredients of the proof:

- Source condition for f_{true} is equivalent to say that f_{true} solves the TV-H^{-1} inpainting equation for some rhs.
- Source condition for f_{true} has a consequence on the Bregman distance:

$$\begin{aligned} D_R^{\xi}(u, f_{true}) &= R(u) - R(f_{true}) - \langle u - f_{true}, \xi \rangle_2 \\ &= R(u) - R(f_{true}) - \langle q, Au - Af_{true} \rangle_2 \,, \end{aligned}$$

  meaning that the Bregman distance can be related to both the error in the regularization functional $(R(u) - R(f_{true}))$ and the output error $(Au - Af_{true})$.
- Chan and Kang estimates for the output/inpainting error.

Error Estimation With the Bregman Distance (cont.)

Key observations

- Inpainting error consists of the smoothing error due to the regularizer (on the whole image) and the error due to the construction of the inpainted image (inside of the inpainting domain).
- Scaling which minimizes the inpainting error:

$$\lambda_0^2 \, |D|^{\frac{r-2}{r}} \sim 1,$$

i.e., in two space dimensions r can be chosen arbitrarily big and hence

$$\lambda_0 \sim 1/\sqrt{|D|}.$$

Open Analytical Problems

- Asymtotic behaviour of fourth-order inpainting approaches; convergence of the evolution equation to a stationary state.

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- Geometric interpretation of interpolation process.

Next time . . .

Numerical methods for higher-order inpainting:

- Convexity splitting:

$$\begin{cases} u_t = -\nabla \mathcal{J}(u) & \text{in } \Omega, \\ u(.,t=0) = u_0 & \text{in } \Omega, \end{cases}$$

  Split the functional $\mathcal{J}$ into a convex and a concave part $\Rightarrow$ semi implicit scheme where the convex part is treated implicitly and the concave one explicitly in time.
  (Application to Cahn-Hilliard/TV-H^{-1}/LCIS-inpainting)
- A dual approach for TV-H^{-1} inpainting (Chambolle)

End of Part II

Thank you for your kind attention!

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