STACKING DISORDER IN PERIODIC MINIMAL SURFACES

HAO CHEN AND MARTIN TRAIZET

Abstract. We construct 1-parameter families of non-periodic embedded minimal surfaces of infinite genus in $T \times \mathbb{R}$, where T denotes a flat 2-tori. Each of our families converges to a foliation of $T \times \mathbb{R}$ by T. These surfaces then lift to minimal surfaces in $\mathbb{R}^3$ that are periodic in horizontal directions but not periodic in the vertical direction. In the language of crystallography, our construction can be interpreted as disordered stacking of layers of periodically arranged catenoid necks. Limit positions of the necks are governed by equations that appear, surprisingly, in recent studies on the Mean Field Equation and the Painlevé VI Equation. This helps us to obtain a rich variety of disordered minimal surfaces. Our work is motivated by experimental observations of twinning defects in periodic minimal surfaces, which we reproduce as special cases of stacking disorder.

1. Introduction

1.1. **Background.** Triply periodic minimal surfaces (TPMSs) is a topic of trans-disciplinary interest. On the one hand, the mathematical notion has been employed to model many structures in nature (e.g. biological membrane) and in laboratory (e.g. lyotropic liquid crystals); we refer the readers to the book [HBL$^+$96] for more information. On the other hand, natural scientists have been contributing with important mathematical discoveries, many long precede the rigorous mathematical treatment. Examples include the famous gyroid discovered in [Sch70] and proved in [GBW96], as well as its deformations discovered in [FHL93, FH99] and recently proved in [Che19a].

The current paper is another example in which mathematics is inspired by natural sciences. In [HXBC11], mesoporous crystals exhibiting the structure of Schwarz' D surface are synthesized. Remarkably, a twinning structure, which looks like two copies of Schwarz' D surface glued along a reflection plane, is observed. In other word, the periodicity is broken in the direction orthogonal to the reflection plane. Thereafter, many other crystal defects are experimentally observed in more TPMS structures, leading to a growing demand of mathematical understanding.

Recently, the first named author [Che19b] responded to this demand with numerical experiments in Surface Evolver [Bra92]. More specifically, periodic twinning defects are numerically introduced into rPD surfaces (see Figure 1) and the gyroid. Success of these experiments provide strong evidences for the existence of single twinning defects.

Moreover, he also became aware of the node-opening techniques developed by the second named author [Tra02]. The idea is to glue catenoid necks among horizontal planes. When

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the planes are infinitesimally close, the necks degenerate to singular points termed nodes. It is proved that, if the limit positions of the nodes satisfy a balancing condition and a non-degeneracy condition, then it is possible to push the planes a little bit away from each other, giving a 1-parameter family of minimal surfaces along the way. The technique has been used to construct TPMSs [Tra08] by gluing necks among finitely many flat tori, and non-periodic minimal surfaces with infinitely many planar ends [MT12] by gluing necks among infinitely many Riemann spheres.

In this paper, we combine the techniques in [Tra08] and [MT12] to glue necks among infinitely many flat tori. Then each balanced and non-degenerate arrangement of nodes gives rise to a 1-parameter families of minimal surfaces. Seen in $T \times \mathbb{R}$, each of these family converges to a foliation of $T \times \mathbb{R}$ by T. Seen in $\mathbb{R}^3$, the minimal surfaces are periodic in two independent horizontal directions but not periodic in any other independent direction.

Our motivation is to rigorously construct twinning defects, but the examples produced by our construction is far richer. In the language of crystallography, our construction can be seen as stacking layers of periodically arranged catenoid necks. In the case that T is the 60-degree torus, for example, we will see that any bi-infinite sequence of 5 stacking patterns gives arise to a 1-parameter family of minimal surfaces. These are then uncountably many families. In particular, a twinning defect arises from a stacking fault, which is not periodic but still quite ordered from a physics point of view. But most of our examples does not exhibit any order, hence should be considered as stacking disorders.

Back to the twinning, experiments and simulations have shown that TPMSs with twinning defects decay exponentially to the standard TPMSs. We will provide mathematical proof to this physics phenomenon, hence finally justify the term "TPMS twinning". More specifically, we will prove that if a configuration is eventually periodic, then the corresponding minimal surface is asymptotic to a TPMS. The proof uses weighted Banach space as in [Tra13].

Our construction uses Implicit Function Theorem, hence only works near the degenerate limit of foliations, which is not physically plausible. However, physicists have proposed formation mechanisms for TPMSs in nature and in laboratory (e.g. [CF97, MBF94, CCM+06, TBC+15]) that are very similar to node-opening, some even with experiment evidences. Hence we may hope that some of the minimal surfaces constructed in this paper, including those with stacking disorders, would be one day observed in laboratory.

1.2. **Mathematical setting.** A doubly periodic minimal surface (DPMS) M is invariant by two independent translations, which we may assume to be horizontal. Let Γ be the two-dimensional lattice generated by these translations, then M projects to a minimal surface M/Γ in $\mathbb{R}^3/\Gamma = T \times \mathbb{R}$, where $T = \mathbb{R}^2/\Gamma$ denotes a flat 2-torus. Immediate examples of infinite genus are given by triply periodic minimal surfaces (TPMSs), if one ignores one of their three periods. Motivated by experimental observations mentioned above, we are particularly interested in non-periodic DPMS with infinite genus.

A flat torus in $T \times \mathbb{R}$ is *horizontal* if it has the form $T \times \{h\}$ for some $h \in \mathbb{R}$; then h is called the *height* of the torus. Informally speaking, we construct minimal surfaces that look like infinitely many horizontal flat tori in $T \times \mathbb{R}$, ordered by increasing height, with one catenoid neck between each adjacent pair. The tori are then labeled by $k \in \mathbb{Z}$ in the order of height. The catenoid necks are also labeled by $k \in \mathbb{Z}$, such that the k-th neck is between the k-th and the $(k+1)$-th tori.

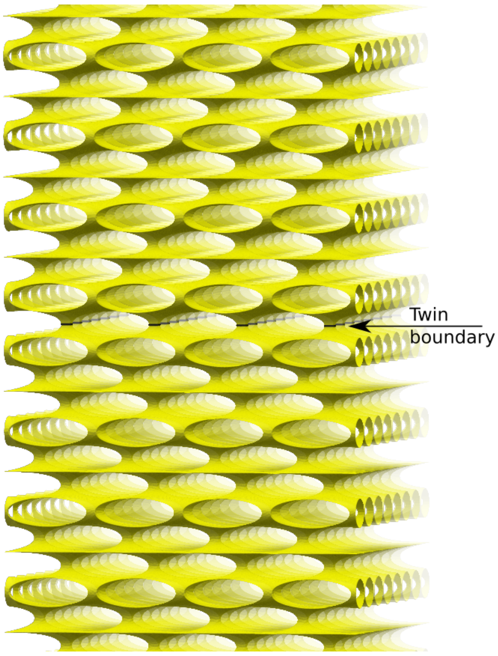

FIGURE 1. Twinning defects in an rPD surface near the catenoid limit, as described in Example 5. This is actually an approximation by a TPMS with large vertical period. The surface has a horizontal symmetry plane in the middle. The image was computed in Surface Evolver [Bra92] using the procedure in [Che19b].

Remark 1.1. Our construction can, in principle, handle finitely many necks between each adjacent pair of tori. But in view of the immediate interest from material sciences, we will only glue one catenoid neck between each adjacent tori. This also eases the notations and facilitates the proofs, but still produces a rich variety of examples.

More formally, we say that a minimal surfaces $M \in T \times \mathbb{R}$ is *stacked* if there is an increasing sequence of real numbers $(h_k)_{k\in\mathbb{Z}}$ satisfying

- $M \cap (T \times \{h_k\})$ has a single connected component that projects to a null-homotopic smooth simple closed curve in T;
- $M \cap (T \times (h_k, h_{k+1}))$ is homeomorphic to T with two disks removed.

Then the k-th neck can be interpreted as an annular neighborhood of $M \cap (T \times \{h_k\})$.

Remark 1.2. The term "stacked" is borrowed from crystallography. Closed-packed structures are often described as a result of stacking layers of periodically arranged atoms, one on top of another. Analogously, a stacked minimal surface can be seen as obtained by stacking layers of periodically arranged catenoid necks.

We intend to construct 1-parameter families M_t, $t > 0$, of stacked minimal surfaces such that, in the limit $t \to 0$, every neck converges to a catenoid after suitable rescaling. If we rescale to keep the size of the torus, then M_t will converge to a foliation of $T \times \mathbb{R}$ by T, and necks converge to singular points, which we call *nodes*.

1.3. **Definitions and main result.**

Definition 1.1. A *node configuration* is a sequence $\boldsymbol{p} = (p_k)_{k\in\mathbb{Z}}$ such that $p_k \in T$ for all $k \in \mathbb{Z}$.

We use p_k to prescribe the limit position of the k-th node, and assume that

Hypothesis 1.2 (Uniform separation)**.** There exists a constant $\varrho > 0$ such that p_k and p_ℓ are at least at distance ϱ apart whenever $|k - \ell| = 1$.

Assume that $2\omega_1 = 1$ and $2\omega_2 = \tau$ generate the lattice Γ, so $T = T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Without loss of generality, we also assume that $\operatorname{Im}\tau > 0$. Then for $p \in T$, we use $x(p;\tau)$ and $y(p;\tau)$ to denote its coordinates in the basis 1 and τ, and define the function

$$\xi(p;\tau) = x(p;\tau)\eta_1(\tau) + y(p;\tau)\eta_2(\tau),$$

where $\eta_i(\tau) = \zeta(z + 2\omega_i;\tau) - \zeta(z;\tau) = 2\zeta(\omega_i;\tau)$ for $i = 1, 2$, and

$$\zeta(z;\tau) = \frac{1}{z} \sum_{0\neq u\in\mathbb{Z}+\tau\mathbb{Z}} \Big(\frac{1}{z-u} + \frac{1}{u} + \frac{z}{u^2}\Big)$$

is the Weierstrass zeta function associated to Γ.

The following definitions are borrowed from [Tra08] and [MT12]. Given a node configuration $\boldsymbol{p}$, the *force* F_k exerted on the node p_k by other nodes is

$$F_k := \zeta(p_{k+1} - p_k;\tau) + \zeta(p_{k-1} - p_k;\tau) + 2\xi(p_k;\tau) - \xi(p_{k+1};\tau) - \xi(p_{k-1};\tau) \tag{1}$$

Definition 1.3. A node configuration is said to be *balanced* if $F_k = 0$ for all $k \in \mathbb{Z}$.

Since our construction uses the Implicit Function Theorem, we need the differential of the force to be invertible in some sense. As explained in [MT12], (p_k) is not the right parameter to formulate non-degeneracy and one needs to introduce the sequence $\boldsymbol{q} = (q_k)$ defined by

$$q_k = p_k - p_{k-1}.$$

Note that the uniform separation hypothesis can be reformulated as (q_k) being bounded away from 0. From now on, we use the term "configuration" for the infinite sequence (q_k).

Under the new variables, (1) becomes

$$F_k := \zeta(q_{k+1};\tau) - \zeta(q_k;\tau) + \xi(q_k;\tau) - \xi(q_{k+1};\tau) = G_{k+1} - G_k.$$

where

$$G_k = G(q_k;\tau) = \zeta(q_k;\tau) - \xi(q_k;\tau).$$

A configuration is then balanced if (G_k) is a constant sequence, i.e. $G_k = G_0$ for all $k \in \mathbb{Z}$.

Definition 1.4. A configuration is said to be *non-degenerate* if the differential of $(G_k)_{k\in\mathbb{Z}}$ with respect to $(q_k)_{k\in\mathbb{Z}}$, as a map from ℓ^∞ to itself, is an isomorphism.

Note that $G(q;\tau)$ is periodic in q but not meromorphic. The function $G(q;\tau)$ is called the *Hecke form* in [Lan95]. It was proved by Hecke [Hec27] that, if $q = (k_1 + k_2\tau)/N$ with $\gcd(k_1, k_2, N) = 1$, then $G(q;\tau)$ is a modular form of weight 1 with respect to the congruence group $\Gamma(N)$. Recently, the Hecke form gained popularity for its importance in the study of PDEs, including the Mean Field Equation, the Painlevé VI Equation, and the (generalized) Lamé Equation; see [Lin16] for a survey. We will exploit some of the recent results [LW10, CKLW18, BE16] in our construction.

Now we are ready to state our main theorems

Theorem 1.5. *If a configuration $\boldsymbol{q}$ is balanced, non-degenerate, and satisfies the uniform separation hypothesis, then there exists in $T \times \mathbb{R}$ a 1-parameter family $(M_t)_{0<t<\epsilon}$ of embedded stacked minimal surfaces which, in the limit $t \to 0$, converges to a foliation of $T \times \mathbb{R}$ by T. Moreover, the necks have asymptotically catenoidal shape and their limiting positions in T are prescribed by $\boldsymbol{p}$.*

Theorem 1.6. *Let $\boldsymbol{q}$ and $\boldsymbol{q}'$ be two balanced and non-degenerate configurations that satisfy the uniform separation hypothesis. Assume that (q_k) is periodic (in the sense $q_{k+N} = q_k$) and $q'_k = q_k$ for all $k \geq 0$. Let (M_t) and (M'_t) denote the corresponding 1-parameter families of minimal surfaces. Then M_t is a TPMS and M'_t is asymptotic to a translation of M_t as $x_3 \to \infty$.*

The paper is organized by increasing technicality. After reviewing some examples in Section 2, we prove our main theorems in Section 3 and Section 4, respectively. Technical ingredients of the proofs are delayed to later sections. In Section 5 we prove the existence and smooth dependence on parameters of a holomorphic 1-form ω that we used in the Weierstrass data in Section 3. In Section 6, we study the asymptotic behavior of ω and other parameters, which is crucial for proving the TPMS asymptotic behavior in Section 4.

2. Examples

Given an infinite sequence of planes, if only one node is opened between each adjacent pair, the result is necessarily a Riemann minimal example [MT12]. We now show that opening nodes among flat tori is a sharp contrast. Although we only open one node between each adjacent pair of tori, we still obtain a rich variety of balanced configurations. We produce configurations using the following

Proposition 2.1. *Let $\widetilde{q}_0, \widetilde{q}_1, \cdots, \widetilde{q}_{n-1}$ be n solutions of the equation*

$$G(q;\tau) = C \tag{2}$$

for the same complex constant C. Assume that the differential of G with respect to q at $\widetilde{q}_k$, as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, is non-singular for each k. Then any bi-infinite sequence $(q_k)_{k\in\mathbb{Z}}$ of elements in the set $\{\widetilde{q}_0, \widetilde{q}_1, \cdots, \widetilde{q}_{n-1}\}$ is a balanced, non-degenerate configuration satisfying the uniform separation hypothesis.

Proof. The configuration is balanced by definition. Since there is only a finite number of points $\widetilde{q}_k$, the uniform separation hypothesis is satisfied and the differentials of G with respect to q at $\widetilde{q}_k$, as well as their inverses, are uniformly bounded. Then the differential of (G_k) with respect to (q_k) is clearly an automorphism of ℓ^∞. □

We can produce a rich variety of examples thanks to the fact that (2) often has several solutions, which we can combine in any arbitrary way to form stacking disorders. We first discuss the number of solutions of Equation (2).

2.1. Solutions with $C = 0$.

The solutions to $G(q;\tau) = 0$ are critical points of the Green function on a flat torus. The number of critical points and their non-degeneracy has been investigated in [LW10, CKLW18, BE16]. Recall that the function G is odd and Γ-periodic in the variable q. Hence for any τ, the 2-division points $1/2$, $\tau/2$ and $(1+\tau)/2$ are trivial solutions to $G(q;\tau) = 0$. Moreover, it is recently proved [CKLW18] (see also [BMMS17]) that all three trivial solutions are non-degenerate for a generic τ, and at least two of them are non-degenerate for any τ. Using Proposition 2.1, any flat 2-torus T admits uncountably many balanced and non-degenerate configurations, giving rise to uncountably many 1-parameter families of non-periodic minimal surfaces in $T \times \mathbb{R}$.

Since $G(q;\tau)$ is odd in q, non-trivial solutions of $G(q;\tau) = 0$ must appear in pairs. Using a deep connection with the mean field equation, Lin and Wang proved that $G(q;\tau) = 0$ has at most one non-trivial solution pair for a fixed τ [LW10, Theorem 1.2]. In other words, $G(q;\tau) = 0$ has either three or five solutions. A direct and simpler proof was later provided by Bergweiler and Eremenko [BE16], who also give an explicit criterion distinguishing τ's with three and five solutions. Moreover, in the case of five solutions, all solutions are non-degenerate [LW17]. For an explicit example, with $\tau = \exp(\mathrm{i}\pi/3)$, the non-trivial pair of solutions are

$$q = \pm(1 + \tau)/3. \tag{3}$$

2.2. **TPMS examples.** For a fixed τ, any bi-infinite sequence of the solutions of $G(q;\tau) = 0$ is a balanced configuration, hence gives rise to a family of minimal surfaces. From a crystallographic point of view, most of these surfaces would be considered as disordered.

TPMSs with perfect periodic patterns, arising from periodic configurations, are certainly the most interesting cases for crystallographers. In the following, we list some TPMSs of genus three that arise from configurations with period 2 (namely $q_{2k} = q_0$ and $q_{2k+1} = q_1$ for all $k \in \mathbb{Z}$). This completes the discussion in Section 4.3.3 of [Tra08] which was incomplete.

Example 1. If $q_1 = q_0$ (so $\boldsymbol{q}$ is constant) the configuration is trivially balanced. These configurations give rise to TPMSs in Meeks' 5-parameter family. Some famous examples are:

- $\operatorname{Re}\tau = 0$, $q_0 = (1 + \tau)/2$, gives an orthorhombic deformation family of Schwarz' P surface (named oPa in [FH92]), which reduces to Schwarz' tP family when $\tau = \mathrm{i}$.
- $|\tau| = 1$, $q_0 = (1 + \tau)/2$, gives another orthorhombic deformation family of Schwarz' P surface (named oPb in [FH92]), which reduces to Schwarz' tP family when $\tau = \mathrm{i}$.
- $\operatorname{Re}\tau = 0$, $q_0 = 1/2$, gives an orthorhombic deformation family of Schwarz' CLP surface (named oCLP' in [FH92]).
- $\tau = \exp(\mathrm{i}\pi/3)$, $q_0 = (1 + \tau)/3$, gives a rhombohedral deformation family of Schwarz' D and P surfaces (known as rPD).

Note that the first three examples are obtained from trivial solutions of $G(q;\tau) = 0$, and the fourth one is obtained from the non-trivial solutions (3).

Example 2. When $\operatorname{Re}\tau = 0$, $q_0 = 1/2$ and $q_1 = \tau/2$, gives the newly discovered $o\Delta$ surfaces [CW18a]. Note that this example is obtained by alternating two trivial solutions of the equation $G(q;\tau) = 0$.

Example 3. Examples with $q_1 = -q_0$ were studied in [CW18b]. In particular

- $\tau = \exp(\mathrm{i}\pi/3)$, $q_0 = (1 + \tau)/3$, gives hexagonal Schwarz' H family. Note that this example is obtained by alternating the two non-trivial solutions (3).
- There exists a real number $\pi/2 > \theta^* > \pi/3$ such that whenever $\tau = \exp(i\theta)$ with $\theta < \theta^*$, the configuration is balanced with $q_0 = c(1 + \tau)$ for a unique $c < 1/2$. This leads to the orthorhombic deformations of Schwarz' H surfaces in [CW18b]. Existence and uniqueness of θ^* was essentially proved [Web02, WHW09], and independently

in [LW10]. Its value was computed in [CW18b] explicitly as

$$\theta^* = 2 \arctan \frac{K'(m)}{K(m)} \approx 1.23409, \tag{4}$$

where m is the unique solution of $2E(m) = K(m)$, and $K(m)$, $K'(m)$ and $E(m)$ are elliptic integrals of the first kind, associated first kind, and second kind, respectively.

- For general τ, non-trivial q_0's that give balanced configurations are studied in [Web02, LW10, CKLW18] and numerically in [CW18b].

Remark 2.1. For crystallographers, the rPD and the H surfaces are analogous to, respectively, the cubic and hexagonal close-packing.

Remark 2.2. Interestingly, configurations in Examples 1 and 3 give rise to TPMSs no matter their degeneracy. Those in Example 1 form a 4-parameter family, and they are limits of Meeks 5-parameter family. Those in Example 3 are degenerate only if q_0 is a 2-division point, hence reduces to Example 1. In particular, the degenerate configuration with $\tau = \exp(i\theta^*)$ and $q_0 = q_1 = (1+\tau)/2$ is considered in [CW18b]. It is the limit of a 1-parameter family of TPMSs that are degenerate in the sense that the same deformation of the lattices may lead to different deformations of the TPMSs. It is not clear to what extent does this phenomenon generalize.

2.3. **Examples of TPMSs with defects.** From the TPMS examples above, we obtain the following examples with asymptotic TPMS behavior by Theorem 1.6. From a crystallographic point of view, they are TPMSs with planar defects.

Example 4. We may combine oPa, oCLP' and oΔ surfaces using the trivial solutions of $G(q;\tau) = 0$ at the 2-division points. For example:

- The configuration $(q_k)_{k\in\mathbb{Z}}$ defined by

$$\operatorname{Re}\tau = 0, \qquad q_k = \begin{cases} 1/2 & \text{if } k < 0, \\ (1+\tau)/2 & \text{if } k \geq 0, \end{cases}$$

gives rise to non-periodic minimal surfaces in $T \times \mathbb{R}$ which are asymptotic to oPa surfaces as $x_3 \to +\infty$ and oCLP' surfaces as $x_3 \to -\infty$.

- The configuration $(q_k)_{k\in\mathbb{Z}}$ defined by

$$\operatorname{Re}\tau = 0, \qquad q_k = \begin{cases} 1/2 & \text{if } k < 0, \\ \tau/2 & \text{if } k \geq 0, \end{cases}$$

gives rise to non-periodic minimal surfaces in $T\times\mathbb{R}$ which are asymptotic, as $x_3 \to +\infty$ and $x_3 \to -\infty$, to two different oCLP' surfaces. When $\tau = \mathrm{i}$, the two oCLP' surfaces differ only by a 180-degree rotation with horizontal axis, hence can be seen as a rotation twin.

- The configuration $(q_k)_{k\in\mathbb{Z}}$ defined by

$$\operatorname{Re}\tau = 0, \qquad q_k = \begin{cases} 1/2 & \text{if } k < 0 \text{ and odd}, \\ \tau/2 & \text{if } k < 0 \text{ and even}, \\ (1+\tau)/2 & \text{if } k \geq 0, \end{cases}$$

gives rise to non-periodic minimal surfaces in $T \times \mathbb{R}$ which are asymptotic to oPa surfaces as $x_3 \to +\infty$ and the newly discovered oΔ surfaces [CW18a] as $x_3 \to -\infty$.

We certainly did not list all possible combinations. Note that these examples generalize to other τ's in an obvious way, giving rise to non-periodic minimal surfaces that are asymptotic to unnamed TPMSs.

Example 5. We may combine H and rPD surfaces using the pair of non-trivial solutions (3). For example:

- The configuration defined by
$$\tau = \exp(\mathrm{i}\pi/3), \qquad q_k = \begin{cases} (1+\tau)/3 & \text{if } k < 0, \\ -(1+\tau)/3 & \text{if } k \geq 0, \end{cases}$$
gives rise to non-periodic minimal surfaces in $T \times \mathbb{R}$ which are symptotic, as $x_3 \to +\infty$ and $x_3 \to -\infty$, to two Schwarz rPD surfaces that differ only by a reflection, hence are twins of Schwarz rPD-surfaces (see Figure 1). Such a D-twin has been observed experimentally.
- The configuration defined by
$$\tau = \exp(\mathrm{i}\pi/3), \qquad q_k = \begin{cases} (1+\tau)/3 & \text{if } k < 0 \text{ and even}, \\ -(1+\tau)/3 & \text{otherwise}, \end{cases}$$
gives rise to non-periodic minimal surfaces in $T \times \mathbb{R}$ which are asymptotic to Schwarz rPD surfaces as $x_3 \to +\infty$ and Schwarz' H surfaces as $x_3 \to -\infty$.
- The configuration defined by
$$\tau = \exp(\mathrm{i}\pi/3), \qquad q_k = \begin{cases} (1+\tau)/3 & \text{if } k < 0 \text{ and even or if } k > 0 \text{ and odd}, \\ -(1+\tau)/3 & \text{otherwise}, \end{cases}$$
gives rise to non-periodic minimal surfaces in $T \times \mathbb{R}$ which are asymptotic, as $x_3 \to +\infty$ and $x_3 \to -\infty$, to two different Schwarz' H surfaces that differ only by a horizontal translation.

We certainly did not list all possible combinations. These examples generalize, in an obvious way, to any other τ's such that $G(q;\tau) = 0$ has a pair of non-trivial solutions, giving rise to non-periodic minimal surfaces that are asymptotic to unnamed TPMSs.

2.4. **Historical remarks.** The Hecke form $G(q;\tau)$ has been studied independently by the PDE and minimal surface communities. Hence we would like to point out some connections between their approaches.

Solutions to $G(q;\tau) = 0$ are particularly interesting as they are the critical points of the Green function on a flat torus T_τ [LW10, CKLW18]. This is no surprise in the context of node-opening construction of TPMSs. In [Tra08], the forces between nodes are compared to electrostatic forces between electric charges. The Green function is nothing but the potential function of the electric field generated by periodically arranged charges. The balancing condition asks that all charges are in equilibrium, hence at a critical point of the potential.

A 2-division point ω is degenerate if
$$\frac{\tau\wp(\omega;\tau) + \eta_2(\tau)}{\wp(\omega;\tau) + \eta_1(\tau)}$$
is real. This is the quotient of periods of the elliptic function $\wp(z;\tau) - \wp(\omega;\tau)$. So if ω is degenerate, the torus T_τ admits a meromorphic 1-form with a double pole, a double zero, and only real periods. Such tori are no stranger to the minimal surface theory. In particular,

the unique rhombic torus with period quotient -1 was used to construct helicoids with handles [Web02, WHW09], and its angle has an explicit expression as given in (4) (see [CW18b]). On the PDE side, existence of this torus was independently proved in [LW10].

Following [WHW09], we propose a simple construction for the torus and the 1-form: Slit the complex plane along the real segment $[-1,1]$. Identify the top edge of $[-1,-x]$ (resp. $[-x,1]$) with the bottom edge of $[x,1]$ (resp. $[-1,x]$), where $x \in [0,1)$ and $(x+1)/(x-1)$ is the quotient of periods. The result is a torus carrying a cone metric with two cone singularities, one of cone angle 6π at the point identified with ± 1 and $\pm x$, the other of cone angle -2π at ∞. Its periods are obviously real. The same torus with flat metric is T_τ.

It follows easily from [Web02] that there exists a unique torus for each real period quotient. This essentially proves Theorem 6.1(1) in [CKLW18].

2.5. **Solutions with $C \neq 0$.** As said before, [BE16] proved again that the equation $G(z;\tau) = 0$ has either three or five solutions. Their elegant argument can be adapted to the case $C \neq 0$ and yields the following result:

Theorem 2.2. *For given τ and $C \in \mathbb{C}$, the equation $G(q;\tau) = C$ has at least 1 and at most 5 solutions.*

Proof. We adapt the argument of [BE16] to the case $C \neq 0$. First of all, following [BE16], we write

$$G(z;\tau) = \zeta(z;\tau) + az + b\bar{z} \quad \text{with } a = \frac{\pi}{\operatorname{Im}(\tau)} - \eta_1 \quad \text{and} \quad b = -\frac{\pi}{\operatorname{Im}(\tau)},$$

and define the anti-meromorphic function g by

$$g(z) = -\frac{1}{b}\left(\overline{\zeta(z)} + a\bar{z} - \overline{C}\right) = z - \frac{1}{b}\left(\overline{G(z;\tau)} - \overline{C}\right).$$

The only difference with [BE16] is that g is not odd anymore if $C \neq 0$.

It is proved in Lemma 4 of [BE16] using complex dynamics that g has at most two attracting fixed points modulo Γ. The proof carries over to the case $C \neq 0$ with no change. The function g satisfies $g(z+\omega) = g(z) + \omega$ for all $\omega \in \Gamma$. Hence we may define a map $\phi : \mathbb{C}/\Gamma \to \mathbb{C} \cup \{\infty\}$ by $\phi(z) = z - g(z)$. The equation $G(z;\tau) = C$ is equivalent to $\phi(z) = 0$. Since ϕ has a single simple pole, where the differential reverses orientation, its degree (as a map between compact manifolds of the same dimension) is -1. Hence the equation $\phi(z) = 0$ always has at least one solution. In fact, $\phi(z) = 0$ has exactly one solution if $|C|$ is sufficiently large.

We have $\det(d\phi) = 1 - |\bar{\partial} g|^2$. If 0 is a regular value of ϕ, then writing N^+ and N^- for the number of zeros of ϕ with respectively positive and negative determinant of $d\phi$, we see that N^+ is the number of attracting fixed points of g, so $N^+ \leq 2$. Then since $\deg(\phi) = -1$, we have $N^- = N^+ + 1$ so the total number of zeros of ϕ is ≤ 5. If 0 is a critical value of ϕ, the number of zeros of ϕ is still less than 5 by the same argument as in [BE16], Lemma 5.

Observe that if $C = 0$, then ϕ is odd so has the three half-lattice points as trivial zeros: this is the only place in this part of the argument of [BE16] where the parity of ϕ is really used. □

Note that to apply Theorem 2.2 to minimal surfaces, we still need to study the non-degeneracy of the solutions.

3. Construction

3.1. Parameters. The parameters of the construction are a real number t in a neighborhood of 0 and four sequences of complex numbers

$$\boldsymbol{a} = (a_k)_{k\in\mathbb{Z}}, \quad \boldsymbol{b} = (b_k)_{k\in\mathbb{Z}}, \quad \boldsymbol{v} = (v_k)_{k\in\mathbb{Z}}, \quad \text{and} \quad \boldsymbol{\tau} = (\tau_k)_{k\in\mathbb{Z}}$$

in ℓ^∞. Each parameter is in a small ℓ^∞-neighborhood of a central value denoted with an underscore. We will calculate that the central value of the parameters are:

$$\underline{a}_k = -\frac{1}{2}, \quad \underline{b}_k = \frac{1}{2}\xi(\underline{v}_k; \underline{\tau}_k), \quad \underline{v}_k = (-\operatorname{conj})^k q_k, \quad \underline{\tau}_k = (-\operatorname{conj})^k \tau, \tag{5}$$

where conj denotes conjugation, τ and (q_k) prescribe the flat 2-torus and the configuration as in the introduction. We use $\boldsymbol{x} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}, \boldsymbol{\tau})$ to denote the vector of all parameters but t. When required, the dependence of objects on parameters will be denoted with a bracket as in $\Sigma[t, \boldsymbol{x}]$, but will be omitted most of the time.

3.2. Opening nodes and the Gauss map. We denote by $T_k = T_k[\boldsymbol{x}]$ the torus $\mathbb{C}/(\mathbb{Z}+\tau_k\mathbb{Z})$. The point $z = 0$ in T_k is denoted by 0_k. We define the elliptic function $g_k = g_k[\boldsymbol{x}]$ on T_k by

$$g_k(z) = a_k\big(\zeta(z; \tau_k) - \zeta(z - v_k; \tau_k)\big) + b_k.$$

It has two simple poles at 0_k and v_k, with residues a_k and $-a_k$, respectively. Observe that $1/g_k$ is a local complex coordinate in a neighborhood of 0_k and v_k. Hence for a sufficiently small $\varepsilon > 0$, $1/g_k$ gives a diffeomorphism z_k^+ from a neighborhood of v_k in T_k to the disk $D(0, 2\varepsilon) \subset \mathbb{C}$, and a diffeomorphism z_k^- from a neighborhood of 0_k in T_k to the disk $D(0, 2\varepsilon)$. Provided that $\boldsymbol{x}$ is sufficiently close to $\underline{\boldsymbol{x}}$ and by Hypothesis 1.2, ε can be chosen independent of k and $\boldsymbol{x}$. Let $D_k^\pm$ be the disk $|z_k^\pm| < \varepsilon$ in T_k.

Consider the disjoint union of all T_k for $k \in \mathbb{Z}$. If $t = 0$, identify $v_k \in T_k$ and $0_{k+1} \in T_{k+1}$ to create a node. The resulting Riemann surface with nodes is denoted $\Sigma[0, \boldsymbol{x}]$. If $t \neq 0$ and $|t| < \varepsilon$, then for each $k \in \mathbb{Z}$, remove the disks $|z_k^\pm| \le t^2/\varepsilon$ from T_k, and let $A_k^\pm$ be the annuli $t^2/\varepsilon < |z_k^\pm| < \varepsilon$. Identify A_k^+ and A_{k+1}^- by $z_k^+ z_{k+1}^- = t^2$. This opens nodes and creates a neck between T_k and T_{k+1}. The resulting Riemann surface is denoted $\Sigma = \Sigma[t, \boldsymbol{x}]$.

If $t \neq 0$, we define the Gauss map $g = g[t, \boldsymbol{x}]$ explicitly on $\Sigma[t, \boldsymbol{x}]$ by

$$g(z) = (t g_k(z))^{(-1)^{k+1}} \quad \text{in } T_k.$$

Then g takes the same value at the points that are identified when defining Σ. So g is a well-defined meromorphic function on Σ.

3.3. Height differential. We define

$$\Omega_k := T_k \setminus \left(\overline{D_k^+} \cup \overline{D_k^-}\right) \quad \text{and} \quad \Omega := \bigsqcup_{k\in\mathbb{Z}} \Omega_k \subset \Sigma.$$

All circles ∂D_k^- are homologous in Σ. This homology class is denoted γ. We denote by α_k and β_k the standard generators of the homology of T_k, namely the homology classes of $[0, 1]$ and $[0, \tau_k]$ modulo $\mathbb{Z} + \tau_k\mathbb{Z}$. We choose representatives of α_k and β_k within Ω_k, so they can be seen as curves on Σ.

By Proposition 5.1 in Section 5, for t small enough, there exists a holomorphic 1-form $\omega = \omega[t, \boldsymbol{x}]$ on $\Sigma[t, \boldsymbol{x}]$ with imaginary periods on α_k, β_k for all $k \in \mathbb{Z}$ and $\int_\gamma \omega = 2\pi\mathrm{i}$. We define the height differential dh by

$$dh = t\omega.$$

If $t = 0$, ω is allowed to have simple poles at the nodes (a so-called *regular 1-form* on a Riemann surface with nodes). So ω has simple poles at 0_k and v_k, with residues 1 and -1 respectively, and imaginary periods on α_k and β_k. By Proposition 5.1, we have explicitly

$$\omega[0, \boldsymbol{x}] = (\zeta(z; \tau_k) - \zeta(z - v_k; \tau_k) - \xi(v_k; \tau_k))\, dz \quad \text{in } T_k. \tag{6}$$

Finally, $\omega[t, \boldsymbol{x}]$ restricted to Ω depends smoothly on $(t, \boldsymbol{x})$ in a sense which we now explain. Some care is required because the domain Ω depends on the parameters. To formulate the smooth dependence, we pullback ω to a fixed domain as follows. Let $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + \mathrm{i}\mathbb{Z})$ be the standard square torus. Let $\psi_k[\boldsymbol{x}]$ be the diffeomorphism defined by

$$\psi_k[\boldsymbol{x}] : \mathbb{T} \to T_k[\boldsymbol{x}], \quad \psi_k(x + \mathrm{i}y) = x + \tau_k y.$$

Let $\widetilde{v}_k = \psi_k^{-1}(v_k)$. Fix a small $\varepsilon' > 0$ and define

$$\widetilde{\Omega}_k = \mathbb{T} \setminus \left(\overline{D(0, \varepsilon')} \cup \overline{D(\underline{\widetilde{v}}_k, \varepsilon')}\right) \quad \text{and} \quad \widetilde{\Omega} = \bigsqcup_{k \in \mathbb{Z}} \widetilde{\Omega}_k.$$

If ε' is small enough and $\boldsymbol{x}$ is close enough to $\underline{\boldsymbol{x}}$, we have $\Omega_k \subset \psi_k(\widetilde{\Omega}_k)$. Moreover, if t is small enough, the disks $|z_k^{\pm}| < t^2/\varepsilon$ which were removed when opening nodes are outside $\psi_k(\widetilde{\Omega}_k)$, so we can see $\psi_k(\widetilde{\Omega}_k)$ as a domain in Σ. We define $\psi : \widetilde{\Omega} \to \Sigma$ by $\psi = \psi_k$ on $\widetilde{\Omega}_k$. Then $\psi^*\omega$ is a smooth 1-form on $\widetilde{\Omega}$. (Note that $\psi^*\omega$ is not holomorphic, because ψ_k is not conformal.)

We define the pointwise norm of a (not necessarily holomorphic) 1-form η on a domain U in $\mathbb{C}$ or a torus $\mathbb{C}/\Gamma$ by $|\eta(z)| = \sup_{X \in \mathbb{C}^*} \frac{|\eta(z)X|}{|X|}$. We denote $C^0(U)$ the Banach space of 1-forms $\eta = f_1\, dx + f_2\, dy$ with f_1, f_2 bounded continuous functions on U, with the sup norm. We can now state:

Proposition 3.1. *The map $(t, \boldsymbol{x}) \mapsto \psi^*\omega[t, \boldsymbol{x}]$ is smooth from a neighborhood of $(0, \underline{\boldsymbol{x}})$ to $C^0(\widetilde{\Omega})$.*

This is the content of Proposition 5.1(c).

Remark 3.1. The point here is that $\widetilde{\Omega}$ is a fixed domain, independent of the parameters. The domain $\psi(\widetilde{\Omega}) \subset \Sigma$ depends on (τ_k) but not on t nor the other parameters. It contains the domain Ω which fully depends on $\boldsymbol{x}$. Since τ_k is in a neighborhood of $\underline{\tau}_k$, we have $c^{-1} \leq |\psi_k^* dz| \leq c$ for some uniform constant c. Hence for any holomorphic 1-form η on Σ,

$$c^{-1} \|\psi_k^*\eta\|_{C^0(\widetilde{\Omega})} \leq \|\eta\|_{C^0(\psi(\widetilde{\Omega}))} \leq c\|\psi_k^*\eta\|_{C^0(\widetilde{\Omega})}.$$

3.4. **Zeros of the height differential.** We use the Weierstrass parametrization

$$\Sigma[t, \boldsymbol{x}] \ni z \mapsto \operatorname{Re} \int_{z_0}^{z} (\Phi_1, \Phi_2, \Phi_3) \in \mathbb{R}^3,$$

where

$$(\Phi_1, \Phi_2, \Phi_3) := \Big(\frac{1}{2}(g^{-1} - g), \frac{\mathrm{i}}{2}(g^{-1} + g), 1\Big) dh$$

are holomorphic differentials, so that the Weierstrass parametrization is an immersion. So we need to solve the Regularity Problem, which asks that dh has a zero at each zero or pole of g, with the same multiplicity, and no other zeros.

The elliptic function $g_k[\boldsymbol{x}]$ has degree 2 so it has two zeros in T_k which we denote $Z_{k,1}[\boldsymbol{x}]$ and $Z_{k,2}[\boldsymbol{x}]$. Since $g_k[\boldsymbol{x}]$ has poles at 0_k and v_k, its zeros are in Ω_k provided that ε is small enough. Note that we cannot rule out the possibility of a double zero $Z_{k,1} = Z_{k,2}$ at $\underline{\boldsymbol{x}}$, in which case $Z_{k,1}$ and $Z_{k,2}$ are not smooth functions of $\boldsymbol{x}$. Nevertheless, by Weierstrass Preparation Theorem, in a neighborhood of $\underline{\boldsymbol{x}}$, $Z_{k,1}+Z_{k,2}$ and $Z_{k,1}Z_{k,2}$ are smooth functions of $\boldsymbol{x}$. The gauss map g, by definition, has zeros (resp. poles) at $Z_{k,1}$ and $Z_{k,2}$ for k odd (resp. k even).

Proposition 3.2. *For $(t,\boldsymbol{x})$ close to $(0,\underline{\boldsymbol{x}})$, ω has two zeros in Ω_k for $k\in\mathbb{Z}$ (counting multiplicity) and no other zeros. The Regularity Problem is equivalent to*

$$\frac{\omega}{dz}(Z_{k,1})+\frac{\omega}{dz}(Z_{k,2})=0 \tag{7}$$

and

$$\int_{\partial\Omega_k} g_k^{-1}\omega=0 \tag{8}$$

for $k\in\mathbb{Z}$.

Proof. By (6), $\omega[0,\boldsymbol{x}]/dz$ is an elliptic function of degree 2 in T_k, with simple poles at 0_k and v_k. So it has two zeros in Ω_k as long as ε is small enough. Hence $\psi^*\omega[0,\boldsymbol{x}]$ has two zeros in $\widetilde{\Omega}_k$, counting multiplicity. By Proposition 3.1 and the Argument Principle, for t small enough, $\psi^*\omega[t,\boldsymbol{x}]$ has two zeros in $\widetilde{\Omega}_k$, so $\omega[t,\boldsymbol{x}]$ has two zeros in $\psi(\widetilde{\Omega}_k)$. By the same proof of Corollary 2 in [Tra13], ω has no zero in the annuli $A_k^\pm$, so it has two zeros in Ω_k for each $k\in\mathbb{Z}$, and no further zero.

If $Z_{k,1}\neq Z_{k,2}$, then dz/g_k is a meromorphic 1-form on T_k with two simple poles at $Z_{k,1}$ and $Z_{k,2}$. Therefore, by the Residue Theorem,

$$\frac{1}{g_k'(Z_{k,1})}+\frac{1}{g_k'(Z_{k,2})}=0,$$

$$\begin{aligned}\int_{\partial\Omega_k} g_k^{-1}\omega&=2\pi\mathrm{i}\left(\frac{\omega}{g_k'\,dz}(Z_{k,1})+\frac{\omega}{g_k'\,dz}(Z_{k,2})\right)\\&=\frac{2\pi\mathrm{i}}{g_k'(Z_{k,1})}\left(\frac{\omega}{dz}(Z_{k,1})-\frac{\omega}{dz}(Z_{k,2})\right).\end{aligned}$$

Hence (7) and (8) are equivalent to $\frac{\omega}{dz}(Z_{k,1})=\frac{\omega}{dz}(Z_{k,2})=0$.

If $Z_{k,1}=Z_{k,2}$, g_k has a double zero at $Z_{k,1}$. It is then easy to see, using the Residue Theorem, that (7) and (8) are equivalent to ω having a double zero at $Z_{k,1}$. ☐

Proposition 3.3. *For t in a neighborhood of 0, there exists a unique value of $(b_k)\in\ell^\infty$, depending smoothly on t and the other parameters, such that (7) is solved for all $k\in\mathbb{Z}$. Moreover, at $t=0$,*

$$b_k=-a_k\xi(v_k;\tau_k)$$

so

$$g_k\,dz=a_k\omega\quad\text{in } T_k. \tag{9}$$

Proof. Define

$$\mathcal{E}_k(t,\boldsymbol{x})=\frac{\omega}{dz}(Z_{k,1})+\frac{\omega}{dz}(Z_{k,2}).$$

Then by the Residue Theorem,

$$\mathcal{E}_k(t,\boldsymbol{x})=\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega_k}\frac{\omega}{z-Z_{k,1}}+\frac{\omega}{z-Z_{k,2}}=\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega_k}\frac{(2z-(Z_{k,1}+Z_{k,2}))\omega}{z^2-(Z_{k,1}+Z_{k,2})z+Z_{k,1}Z_{k,2}}.$$

Using Proposition 3.1, $(\mathcal{E}_k)_{k\in\mathbb{Z}}$ is a smooth map with value in ℓ^∞. At $t=0$, we have by (6)

$$\frac{\omega}{dz}=\frac{1}{a_k}(g_k(z)-b_k)-\xi(v_k;\tau_k)$$

Since $g_k(Z_{k,i})=0$,

$$\mathcal{E}_k(0,\boldsymbol{x})=-2\left(\frac{b_k}{a_k}+\xi(v_k;\tau_k)\right).$$

Since $\underline{a}_k=-1/2$, the partial differential of $(\mathcal{E}_k)$ with respect to (b_k) is an automorphism of ℓ^∞. Proposition 3.3 then follows from the Implicit Function Theorem. □

If b_k is given by Proposition 3.3, then at $t=0$, ω and g_k are proportional in T_k hence have the same zeros. Then (8) is satisfied at $t=0$ disregard of the value of the other parameters. So we cannot easily solve (8) for $t\neq 0$ using the Implicit Function Theorem. We will solve (8) in Section 3.6.

3.5. **The Period Problem.** From now on, we assume that (b_k) is given by Proposition 3.3 and $\boldsymbol{x}=(a_k,v_k,\tau_k)_{k\in\mathbb{Z}}$ denotes the remaining parameters. The height differential has imaginary periods by definition. It remains to solve the following Period Problems for all $k\in\mathbb{Z}$:

$$\operatorname{Re}\int_{\alpha_k}\Phi_1=(-1)^k,\quad \operatorname{Re}\int_{\alpha_k}\Phi_2=0, \tag{10}$$

$$\operatorname{Re}\int_{\beta_k}\Phi_1=\operatorname{Re}\tau,\qquad \operatorname{Re}\int_{\beta_k}\Phi_2=\operatorname{Im}\tau, \tag{11}$$

$$\operatorname{Re}\int_{\gamma}\Phi_1=0,\qquad \operatorname{Re}\int_{\gamma}\Phi_2=0. \tag{12}$$

Proposition 3.4. *For t small enough, there exists unique values for the parameters (a_k) and (τ_k) in ℓ^∞, depending smoothly on t and (v_k), such that* (10) *and* (11) *are satisfied for all $k\in\mathbb{Z}$. Moreover, at $t=0$, $a_k=-1/2$ and $\tau_k=(-\operatorname{conj})^k(\tau)$, disregard of the value of (v_k).*

Proof. we define

$$\mathcal{P}_{k,1}(t,\boldsymbol{x})=\operatorname{conj}\Big(\int_{\alpha_k}g^{-1}\,dh\Big)-\int_{\alpha_k}g\,dh$$

$$\mathcal{P}_{k,2}(t,\boldsymbol{x})=\operatorname{conj}\Big(\int_{\beta_k}g^{-1}\,dh\Big)-\int_{\beta_k}g\,dh$$

Equations (10) and (11) are equivalent to

$$\begin{cases}\mathcal{P}_{k,1}(t,\boldsymbol{x})=2(-1)^k\\ \mathcal{P}_{k,2}(t,\boldsymbol{x})=2\tau.\end{cases} \tag{13}$$

We have in T_k:

$$g\,dh=\begin{cases}g_k^{-1}\omega & \text{if } k \text{ even}\\ t^2g_k\omega & \text{if } k \text{ odd}\end{cases}\quad\text{and}\quad g^{-1}dh=\begin{cases}t^2g_k\omega & \text{if } k \text{ even}\\ g_k^{-1}\omega & \text{if } k \text{ odd}\end{cases}$$

We can take $\alpha_k=\psi_k(\widetilde{\alpha}_k)$ and $\beta_k=\psi_k(\widetilde{\beta}_k)$ where $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ are fixed curves in $\widetilde{\Omega}_k$. By Proposition 3.1, $(\mathcal{P}_{k,1})_{k\in\mathbb{Z}}$ and $(\mathcal{P}_{k,2})_{k\in\mathbb{Z}}$ are smooth maps with value in ℓ^∞. At $t=0$, we have by (9) that: For k even,

$$\mathcal{P}_{k,1}(0,\boldsymbol{x})=-\int_{\alpha_k}a_k^{-1}dz=\frac{-1}{a_k}\quad\text{and}\quad\mathcal{P}_{k,2}(0,\boldsymbol{x})=-\int_{\beta_k}a_k^{-1}dz=\frac{-\tau_k}{a_k}.$$

The solution to (13) is then $a_k = -1/2$ and $\tau_k = \tau$. For k odd,

$$\mathcal{P}_{k,1}(0,\boldsymbol{x}) = \operatorname{conj} \int_{\alpha_k} a_k^{-1} dz = \operatorname{conj} \frac{1}{a_k} \quad \text{and} \quad \mathcal{P}_{k,2}(0,\boldsymbol{x}) = \operatorname{conj} \int_{\beta_k} a_k^{-1} dz = \operatorname{conj} \frac{\tau_k}{a_k}.$$

The solution to (13) is then $a_k = -1/2$ and $\tau_k = -\operatorname{conj}\tau$. The partial differential of $((\mathcal{P}_{k,1}),(\mathcal{P}_{k,2}))$ with respect to $((a_k),(\tau_k))$ is clearly an automorphism of $\ell^\infty \times \ell^\infty$. Proposition 3.4 then follows from the Implicit Function Theorem. □

3.6. **Balancing.** From now on, we assume that the parameters (a_k) and (τ_k) are given by Proposition 3.4. So the only remaining parameters are t and $\boldsymbol{v} = (v_k)$. It remains to solve (8) and (12). We define

$$\mathcal{G}_k(t,\boldsymbol{v}) = \operatorname{conj}^k \int_{\partial D_k^-} g_k \omega.$$

Proposition 3.5. *For $t \neq 0$, (8) and (12) are equivalent to $\mathcal{G}_k(t,\boldsymbol{v}) = \mathcal{G}_0(t,\boldsymbol{v})$ for all $k \in \mathbb{Z}$.*

Proof. We have for $k \in \mathbb{Z}$:

$$\begin{aligned}
\int_{\partial\Omega_k} g_k^{-1}\omega &= -\int_{\partial D_k^-} z_k^- \omega - \int_{\partial D_k^+} z_k^+ \omega \\
&= \int_{\partial D_{k-1}^+} \frac{t^2}{z_{k-1}^+}\omega + \int_{\partial D_{k+1}^-} \frac{t^2}{z_{k+1}^-}\omega \\
&= t^2 \int_{\partial D_{k-1}^+} g_{k-1}\omega + t^2 \int_{\partial D_{k+1}^-} g_{k+1}\omega \\
&= -t^2 \int_{\partial D_{k-1}^-} g_{k-1}\omega + t^2 \int_{\partial D_{k+1}^-} g_{k+1}\omega \quad (\text{because } g_{k-1}\omega \text{ is holomorphic in } \Omega_{k-1}) \\
&= t^2 \operatorname{conj}^{k+1}(\mathcal{G}_{k+1} - \mathcal{G}_{k-1}).
\end{aligned}$$

Hence (8) is equivalent to $\mathcal{G}_{k+1} = \mathcal{G}_{k-1}$ for $k \in \mathbb{Z}$.

We chose ∂D_1^- as a representative of γ. Equation (12) is equivalent to

$$\int_{\partial D_1^-} g^{-1}\, dh - \operatorname{conj}\Big(\int_{\partial D_1^-} g\, dh\Big) = 0.$$

We have

$$\int_{\partial D_1^-} \overline{g\, dh} = t^2 \int_{\partial D_1^-} \overline{g_1 \omega} = t^2 \mathcal{G}_1.$$

$$\int_{\partial D_1^-} g^{-1}\, dh = \int_{\partial D_1^-} g_1^{-1}\omega = -t^2 \int_{\partial D_0^+} g_0 \omega = t^2 \int_{\partial D_0^-} g_0\omega = t^2 \mathcal{G}_0.$$

Hence (12) is equivalent to $\mathcal{G}_1 = \mathcal{G}_0$. □

Proposition 3.6. *Assume that the configuration $\boldsymbol{q} = (q_k)$ is balanced and non-degenerate. For t in a neighborhood of 0, there exists a unique $\boldsymbol{v}(t)$ in ℓ^∞, depending smoothly on t, such that $v_k(0) = (-\operatorname{conj})^k q_k$ and*

$$\mathcal{G}_k(t,\boldsymbol{v}(t)) = -2\pi \mathrm{i} G(q_0;\tau) \tag{14}$$

for all $k \in \mathbb{Z}$, so (8) and (12) are solved.

Proof. First of all, $(\mathcal{G}_k(t,\boldsymbol{v}))$ is a smooth function of $(t,\boldsymbol{v})$ with value in ℓ^∞ by Proposition 3.1. At $t=0$, we have in T_k:

$$g_k\frac{\omega}{dz}=a_k\left(\frac{\omega}{dz}\right)^2=\frac{-1}{2}\left(\zeta(z;\tau_k)-\zeta(z-v_k;\tau_k)-\xi(v_k;\tau_k)\right)^2$$

$$\begin{aligned}\operatorname{Res}_{0_k}(g_k\omega)&=\frac{-1}{2}\operatorname{Res}_{0_k}\left[\zeta(z;\tau_k)^2-2\zeta(z;\tau_k)\big(\zeta(z-v_k;\tau_k)+\xi(v_k;\tau_k)\big)\right]\\&=-\zeta(v_k;\tau_k)+\xi(v_k;\tau_k)=-G(v_k;\tau_k).\end{aligned}$$

Here we used the fact that ζ is odd, hence ζ^2 is even and has no residue at 0. Then by the Residue Theorem,

$$\mathcal{G}_k(0,\boldsymbol{v})=\operatorname{conj}^k\big(-2\pi\mathrm{i}\,G(v_k;\tau_k)\big)=-2\pi\mathrm{i}(-\operatorname{conj})^kG(v_k;\tau_k).$$

Recall that $\tau_k=(-\operatorname{conj})^k(\tau)$ at $t=0$. Now change to the variable $u_k=(-\operatorname{conj})^kv_k$ with central value $\underline{u}_k=(-\operatorname{conj})^k\underline{v}_k=q_k$. Using the definition of ζ, ξ and G, one easily checks that

$$G(-\overline{z};-\overline{\tau})=-\operatorname{conj}G(z;\tau).$$

Hence

$$\mathcal{G}_k(0,\boldsymbol{v})=-2\pi\mathrm{i}\,G(u_k;\tau).$$

Then Proposition 3.6 follows from the balance and non-degeneracy of (q_k) and the Implicit Function Theorem. $\square$

Remark 3.2. The horizontal component of the flux of γ, identified with a complex number, is given by

$$-\mathrm{i}\int_\gamma g\,dh=-\mathrm{i}\int_{\partial D_1^-}t^2g_1\omega=-\mathrm{i}t^2\overline{\mathcal{G}_1}.$$

So (14) for $k=1$ normalizes the horizontal part of the flux of γ (which can also be taken as a free parameter).

3.7. **Embeddedness.** We denote by $\boldsymbol{x}(t)$ the value of the parameters given by Proposition 3.3, 3.4 and 3.6, (Σ_t,g_t,dh_t) the corresponding Weierstrass data, $f_t:\Sigma_t\to\mathbb{R}^3/\Gamma$ the immersion given by Weierstrass Representation and $M_t=f_t(\Sigma_t)$. Recall that Γ is the 2-dimensional lattice generated by the horizontal vectors $(1,0,0)$ and $(\operatorname{Re}\tau,\operatorname{Im}\tau,0)$. The goal of this section is to prove that M_t is embedded and has the geometry described in Section 1.2. The argument is very similar to Section 4.10 of [MT12], so we will only sketch it.

We write $f_t=(X_t,h_t)$ with

$$X_t(z)=\frac{1}{2}\operatorname{conj}\Big(\int_{z_0}^z g_t^{-1}dh_t\Big)-\frac{1}{2}\int_{z_0}^z g_t\,dh_t\quad\text{and}\quad h_t(z)=\operatorname{Re}\int_{z_0}^z dh_t.$$

We fix a base point $\widetilde{O}_k$ in $\widetilde{\Omega}_k$, away from the zeros of $g_k\circ\psi_k$, and let $O_k=\psi_k(\widetilde{O}_k)\in\Omega_k$. Let $w_k(t)\in\Sigma_t$ be the point $z_k^+=t$ which is identified with the point $z_{k+1}^-=t$ (the "middle" of the k-th neck). For $r>0$, we denote $\Omega_{k,r}$ the torus T_k minus the two disks $|z_k^\pm|\le r$. By the computations in Section 3.5, we have in $\Omega_{k,r}$

$$\lim_{t\to0}dX_t=(-\operatorname{conj})^kdz\tag{15}$$

so the image of $\Omega_{k,r}$ is a graph for t small enough and

$$\lim_{t\to 0} X_t(w_k(t)) - X_t(w_{k-1}(t)) = (-\operatorname{conj})^k \int_{0_k}^{\underline{v}_k} dz = (-\operatorname{conj})^k(\underline{v}_k) = q_k.$$

Remark 3.3. This computation is not rigorous because the limit (15) only holds in $\Omega_{k,r}$. It can be made rigorous by expanding ω_t in Laurent series in the annuli $A_k^\pm$, see details in Appendix A of [MT12].

Recall that $q_k = p_k - p_{k-1}$. We may translate f_t horizontally so that $X_t(w_0(t)) = p_0$. Then

$$\lim_{t\to 0} X_t(w_k(t)) = p_k.$$

In other words, p_k is the limit position of the k-th neck. Note however that the convergence is not uniform with respect to k. By (6), we have for $z \in \Omega_{k,r}$

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}(h_t(z) - h_t(O_k)) &= \operatorname{Re} \lim_{t\to 0} \int_{O_k}^z \omega_t \\ &= \operatorname{Re} \int_{O_k}^z (\zeta(z;\underline{\tau}_k) - \zeta(z-\underline{v}_k;\underline{\tau}_k) - \xi(\underline{v}_k;\underline{\tau}_k))\, dz =: \Upsilon_k(z).\end{aligned}$$

The function Υ_k is bounded in $\Omega_{k,r}$ by a uniform constant $C(r)$ depending only on r. By Lemma A.2 in [MT12],

$$\int_{O_{k-1}}^{O_k} \omega_t \simeq -2\log t \operatorname{Res}_{0_k}(\omega_0) = -2\log t \quad \text{as } t \to 0.$$

Hence

$$h_t(O_k) - h_t(O_{k-1}) \sim -2t\log t.$$

This ensures that for t small enough, $f_t(\Omega_{k,r})$ lies strictly above $f_t(\Omega_{k-1,r})$ so the images $f_t(\Omega_{k,r})$ for $k \in \mathbb{Z}$ are disjoint.

Since the function Υ_k has two logarithmic singularities at 0_k and $\underline{v}_k$, for c large enough, the level lines $\Upsilon_k = \pm c$ are convex curves, which are included in $\Omega_{k,r}$ provided r is small enough. Define

$$h_k^\pm = h_t(O_k) \pm tc.$$

Then for t small enough, M_t intersects the planes $x_3 = h_k^+$ and $x_3 = h_k^-$ in two convex curves denoted γ_k^+ and γ_k^-, which are included in $f_t(\Omega_{k,r})$. Then $M_t \cap \{h_k^+ < x_3 < h_{k+1}^-\}$ is a minimal annulus bounded by two convex curves in parallel planes. Such an annulus is foliated by convex curves by a theorem of Shiffman [Shi56]. This proves that M_t is embedded.

4. Convergence to TPMSs

In this section, we study the asymptotic behavior of the minimal surfaces that we just constructed. Readers who are not interested in the asymptotic behavior may skip this part and jump directly to the next section.

Assume that the configuration (q_k) is periodic. Then the corresponding minimal surface M_t is a TPMS (as an easy consequence of uniqueness in the Implicit Function Theorem). Let (q'_k) be another balanced, non-degenerate configuration with the same horizontal lattice Γ, and assume that $q'_k = q_k$ for all $k \geq 0$. Let M'_t be the family of minimal surfaces corresponding to the configuration (q'_k). In this section, we prove that M'_t is asymptotic to a translation of M_t as the vertical coordinate $x_3 \to +\infty$. This finally justify the term “TPMS twinning”.

We use primes for all objects associated to the configuration (q'_k). So $\Sigma' = \Sigma[t, \boldsymbol{x}'(t)]$ and $f' : \Sigma' \to \mathbb{R}^3/\Gamma$ is the immersion given by the Weierstrass data $g' = g[t, \boldsymbol{x}'(t)]$ and $dh' = t\omega[t, \boldsymbol{x}'(t)]$. We will omit the dependence on t, hence will write, for instance, $\omega = \omega[t, \boldsymbol{x}(t)]$ and $\omega' = \omega[t, \boldsymbol{x}'(t)]$, and in the same way $\psi_k = \psi_k[t, \boldsymbol{x}(t)]$ and $\psi'_k = \psi_k[t, \boldsymbol{x}'(t)]$. Otherwise notations are as in Section 3.7.

We use the same letter C to denote any constant that is independent of $t > 0$ and $k \in \mathbb{Z}$.

Fix an arbitrary $\delta > 1$. We will prove in Proposition 6.5 that, for t sufficiently small, $\boldsymbol{x} - \boldsymbol{x}'$ decays like δ^{-k}. That is

$$\|x_k - x'_k\| \leq \frac{C}{\delta^k}.$$

Then Proposition 6.1 implies that, for t sufficiently small, the difference between $\psi_k^*\omega$ and $(\psi'_k)^*\omega'$ in $\widetilde{\Omega}_k$ also decays like δ^{-k}; see Equation (17) below.

It is convenient to scale the third coordinate of f and f' by t^{-1}. So let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $S(x_1, x_2, x_3) = (x_1, x_2, t^{-1}x_3)$ and define $\widehat{f} = S \circ f$ and $\widehat{f'} = S \circ f'$.

Proposition 4.1. *For $k \in \mathbb{N}$,*

$$\|d(\widehat{f} \circ \psi_k) - d(\widehat{f'} \circ \psi'_k)\|_{C^0(\widetilde{\Omega}_k)} \leq \frac{C}{\delta^k}$$

Proof. We have in Ω_k

$$d\widehat{f} = \begin{cases} (\frac{1}{2}(t^2\overline{g_k\omega} - g_k^{-1}\omega), \operatorname{Re}\omega) & \text{if } k \text{ is even} \\ (\frac{1}{2}(\overline{g_k^{-1}\omega} - t^2 g_k\omega), \operatorname{Re}\omega) & \text{if } k \text{ is odd} \end{cases} \tag{16}$$

and similar formulas for $d\widehat{f'}$. By Propositions 6.1 and 6.5, we have

$$\|\psi_k^*\omega - (\psi'_k)^*\omega'\|_{C^0(\widetilde{\Omega}_k)} \leq \frac{C}{\delta^k}. \tag{17}$$

Using Proposition 6.5 and the definition of g_k, we obtain

$$\|\psi_k^*(g_k\omega) - (\psi'_k)^*(g'_k\omega')\|_{C^0(\widetilde{\Omega}_k)} \leq \frac{C}{\delta^k}. \tag{18}$$

Let $\widetilde{D}_{k,1}$ and $\widetilde{D}_{k,2}$ be two disks containing the two zeros of $g_k \circ \psi_k$, so that $g_k^{-1} \circ \psi_k$ is uniformly bounded outside these disks. Then

$$\|\psi_k^*(g_k^{-1}\omega) - (\psi'_k)^*((g'_k)^{-1}\omega')\|_{C^0(\widetilde{\Omega}_k \setminus \widetilde{D}_{k,1} \cup \widetilde{D}_{k,2})} \leq \frac{C}{\delta^k}. \tag{19}$$

Since the Regularity Problem is solved, $(g'_k)^{-1}\omega'$ extends holomorphically to the zeros of g'_k. Since $\psi_k(x + \mathrm{i}y) - \psi'_k(x + \mathrm{i}y) = (\tau_k - \tau'_k)y$, we have

$$\|\psi_k^*((g'_k)^{-1}\omega') - (\psi'_k)^*((g'_k)^{-1}\omega')\|_{C^0(\widetilde{\Omega}_k)} \leq \frac{C}{\delta^k}. \tag{20}$$

From (19) and (20), we obtain by the Triangular Inequality

$$\|\psi_k^*\left[g_k^{-1}\omega - (g'_k)^{-1}\omega'\right]\|_{C^0(\partial\widetilde{D}_{k,i})} \leq \frac{C}{\delta^k}.$$

Since $g_k^{-1}\omega - (g'_k)^{-1}\omega'$ is holomorphic, we obtain by the Maximum Principle (for the holomorphic structure on $\mathbb{T}$ induced by ψ_k)

$$\|\psi_k^*\left[g_k^{-1}\omega - (g'_k)^{-1}\omega'\right]\|_{C^0(\widetilde{D}_{k,i})} \leq \frac{C}{\delta^k}.$$

Using (20),

$$\|\psi_k^*(g_k^{-1}\omega) - (\psi_k')^*((g_k')^{-1}\omega')\|_{C^0(\widetilde{D}_{k,i})} \le \frac{C}{\delta^k}. \tag{21}$$

Proposition 4.1 follows from (17), (18), (19) and (21). □

Recall that $O_k = \psi_k(\widetilde{O}_k)$ and $O_k' = \psi_k'(\widetilde{O}_k)$.

Proposition 4.2. *For $k \in \mathbb{N}$,*

$$\|\widehat{f}(O_{k+1}) - \widehat{f}(O_k) - \widehat{f}'(O_{k+1}') + \widehat{f}'(O_k')\| \le \frac{C}{\delta^k}.$$

Hence the sequence $(\widehat{f}(O_k) - \widehat{f}'(O_k'))_{k\in\mathbb{N}}$ is Cauchy. By translation, we may assume that its limit is zero, so

$$\|\widehat{f}(O_k) - \widehat{f}'(O_k')\| \le \frac{C'}{\delta^k} \quad \text{with } C' = \frac{C\delta}{\delta - 1}.$$

Proof. Recall from Section 3.7 that $w_k \in \Sigma$ denotes the point $z_k^+ = t$, identified with $z_{k+1}^- = t$. Similarly, we introduce $w_k' \in \Sigma'$ to denote the point $z_k'^+ = t$, identified with $z_{k+1}'^- = t$. Moreover, let $m_k^\pm \in \Sigma$ denotes the point $z_k^\pm = \varepsilon$, and $m_k'^\pm \in \Sigma'$ denotes the point $z_k'^\pm = \varepsilon$.

Proposition 4.2 follows from the following three estimates: For $k \in \mathbb{N}$:

$$\|\widehat{f}(O_k) - \widehat{f}(m_k^\pm) - \widehat{f}'(O_k') + \widehat{f}'(m_k'^\pm)\| \le \frac{C}{\delta^k}, \tag{22}$$

$$\|\widehat{f}(m_k^+) - \widehat{f}(w_k) - \widehat{f}'(m_k'^+) + \widehat{f}'(w_k')\| \le \frac{C}{\delta^k}, \tag{23}$$

$$\|\widehat{f}(m_{k+1}^-) - \widehat{f}(w_k) - \widehat{f}'(m_{k+1}'^-) + \widehat{f}'(w_k')\| \le \frac{C}{\delta^k}. \tag{24}$$

Inequality (22) follows from Proposition 4.1,

$$\psi_k^{-1}(O_k) = (\psi_k')^{-1}(O_k') = \widetilde{O}_k \quad \text{and} \quad |\psi_k^{-1}(m_k^\pm) - (\psi_k')^{-1}(m_k'^\pm)| \le \frac{C}{\delta_k}.$$

To prove (23), we follow the proof of Lemma A.1 in [MT12]. We write the Laurent series of ω in the annulus A_k^+ in term of the complex coordinate $z = z_k^+$ as

$$\omega = \frac{-dz}{z} + \sum_{n\ge1} c_{k,n}^+ z^{n-1}dz + \sum_{n\ge1} t^{2n} c_{k,n}^- \frac{dz}{z^{n+1}}$$

where

$$c_{k,n}^+ = \frac{1}{2\pi\mathrm{i}}\int_{|z|=\varepsilon} \frac{\omega}{z^n} = \frac{1}{2\pi\mathrm{i}}\int_{\partial D_k^+} \frac{\omega}{(z_k^+)^n}$$

$$c_{k,n}^- = \frac{t^{-2n}}{2\pi\mathrm{i}}\int_{|z|=\varepsilon} z^n\omega = \frac{1}{2\pi\mathrm{i}}\int_{\partial D_k^+} (t^{-2}z_k^+)^n\omega = \frac{-1}{2\pi\mathrm{i}}\int_{\partial D_{k+1}^-} \frac{\omega}{(z_{k+1}^-)^n}.$$

We expand ω' in the annulus $A_k'^+$ in the same way, with coefficients $c_{k,n}'^\pm$ given by similar formulas. Using estimates (17) and (18), we obtain the following estimate

$$|c_{k,n}^\pm - c_{k,n}'^\pm| \le \frac{Cn}{\delta^k(2\varepsilon)^n}.$$

By integration, we obtain the following estimates (see details in Appendix A of [MT12]):

$$\left|\int_{z=\varepsilon}^{t}\omega-\omega'\right|\leq\frac{C}{\delta^k}$$

$$\left|\int_{z=\varepsilon}^{t}z(\omega-\omega')\right|\leq\frac{C}{\delta^k}$$

$$\left|\int_{z=\varepsilon}^{t}t^2z^{-1}(\omega-\omega')\right|\leq\frac{C}{\delta^k}$$

Then (23) follows from (16). (24) is proved in the same way, using $z=z_{k+1}^-$ as a local coordinate. $\square$

Assume that the configuration (q_k) is periodic with even period N, i.e. $q_{k+N}=q_k$. By uniqueness in the Implicit Function Theorem, the resulting immersion f is periodic. More precisely, if we define $\sigma:\Sigma\to\Sigma$ by $z\in T_k\mapsto z\in T_{k+N}$ then $f\circ\sigma=f+\mathcal{T}$ (where the period $\mathcal{T}\in\mathbb{R}^3$ depends on t).

Proposition 4.3. *Let $M=f(\Sigma)$ and $M'=f'(\Sigma')$. Define $M'_\ell=M'-\ell\mathcal{T}$ for $\ell\in\mathbb{N}$. Then*

$$\lim_{\ell\to\infty}M'_\ell=M.$$

Here the limit is for the smooth convergence on compact subsets of $\mathbb{R}^3/\Gamma$.

Proof. By periodicity we have

$$\widetilde{\Omega}_{k+N}=\widetilde{\Omega}_k\quad\text{and}\quad f\circ\psi_{k+N}=f\circ\psi_k+\mathcal{T}\quad\text{in }\widetilde{\Omega}_k.$$

By Propositions 4.1 and 4.2, we have for $\ell\in\mathbb{N}$

$$\|f\circ\psi_{k+\ell N}-f'\circ\psi'_{k+\ell N}\|_{C^0(\widetilde{\Omega}_k)}\leq\frac{C}{\delta^{k+\ell N}}.$$

Define $f'_\ell=f-\ell\mathcal{T}$. Then

$$\|f\circ\psi_k-f'_\ell\circ\psi'_{k+\ell N}\|_{C^0(\widetilde{\Omega}_k)}\leq\frac{C}{\delta^{k+\ell N}}.$$

Hence

$$\lim_{\ell\to\infty}f'_\ell\circ\psi'_{k+\ell N}(\widetilde{\Omega}_k)=f\circ\psi_k(\widetilde{\Omega}_k).\tag{25}$$

Recall from Section 3.7 that we have defined heights $h_k^\pm$ such that $M\cap\{h_k^-<x_3<h_k^+\}$ is included in $f\circ\psi_k(\widetilde{\Omega}_k)$ and is bounded by two convex curves denoted γ_k^+ and γ_k^-. Then for ℓ large enough, $M'_\ell\cap\{h_k^-<x_3<h_k^+\}$ is included in $f'_\ell\circ\psi'_{k+\ell N}(\widetilde{\Omega}_k)$ and is bounded by two convex curves denoted $\gamma'^+_{k,\ell}$ and $\gamma'^-_{k,\ell}$. By (25) we have

$$\lim_{\ell\to\infty}M'_\ell\cap\{h_k^-<x_3<h_k^+\}=M\cap\{h_k^-<x_3<h_k^+\}.$$

Let $A'_{k,\ell}=M'_\ell\cap\{h_k^+<x_3<h_{k+1}^-\}$. Then $A'_{k,\ell}$ is an unstable minimal annulus bounded by $\gamma'^+_{k,\ell}$ and $\gamma'^-_{k+1,\ell}$. By Theorem 2(a) in [Tra10], $A'_{k,\ell}$ converges subsequentially as $\ell\to\infty$ to an unstable annulus bounded by γ_k^+ and γ_{k+1}^-. By [MW91], this annulus is unique so the whole sequence converges and

$$\lim_{\ell\to\infty}A'_{k,\ell}=M\cap\{h_k^+<x_3<h_{k+1}^-\}.$$

Hence for any integers $k_1 < k_2$, we have

$$\lim_{\ell\to\infty} M'_\ell \cap \{h^+_{k_1} < x_3 < h^+_{k_2}\} = M \cap \{h^+_{k_1} < x_3 < h^+_{k_2}\}$$

which proves Proposition 4.3 since $\lim_{k\to\pm\infty} h^+_k = \pm\infty$. □

5. The holomorphic 1-forms ω

The goal of this section is to prove:

Proposition 5.1.

(a) *For $(t, \boldsymbol{x})$ in a neighborhood of $(0, \underline{\boldsymbol{x}})$, there exists a holomorphic (regular if $t = 0$) 1-form $\omega[t, \boldsymbol{x}]$ on $\Sigma[t, \boldsymbol{x}]$ with imaginary periods on α_k and β_k for all $k \in \mathbb{Z}$ and $\int_\gamma \omega = 2\pi\mathrm{i}$.*

(b) *At $t = 0$, we have for all $k \in \mathbb{Z}$:*

$$\omega[0, \boldsymbol{x}] = (\zeta(z; \tau_k) - \zeta(z - v_k; \tau_k) - \xi(v_k; \tau_k))\, dz \quad \text{in } T_k.$$

(c) *The pullback $\psi^*\omega$ is in $C^0(\widetilde{\Omega})$ and depends smoothly on $(t, \boldsymbol{x})$ in a neighborhood of $(0, \underline{\boldsymbol{x}})$.*

Remark 5.1. If the configuration is periodic with period $2N$ (namely, $q_{k+2N} = q_k$), the quotient of Σ by its period is a compact Riemann surface, obtained by opening $2N$ nodes between $2N$ tori. In this case, the existence of ω follows from the standard theory of Opening Nodes [Fay73]. To prove the existence of ω in the non-periodic case, we adapt the argument in [Tra13] which allows for infinitely many nodes. The difference is that Riemann spheres are replaced by tori.

In the following, we use the same letter C to designate all uniform constants.

5.1. Preliminaries.

Definition 5.2.

(a) For $p, q \in T_k$, $p \neq q$, we denote by $\omega_{k,p,q}$ the unique meromorphic 1-form on T_k with simple poles at p, q with residues 1 and -1, and imaginary periods on α_k and β_k. So $\omega_{k,p,q}$ is an abelian differential of the third kind with real normalisation.

(b) For $n \geq 2$, we denote $\omega^+_{k,n}$ (resp. $\omega^-_{k,n}$) the unique meromorphic 1-form on T_k with a pole of multiplicity n at v_k (resp. 0_k) with principal part

$$\frac{dz^\pm_k}{(z^\pm_k)^n}$$

and imaginary periods on α_k and β_k. So $\omega^\pm_{k,n}$ are abelian differentials of the second kind with real normalisation. Recall that it depends on the choice of the local coordinate $z^\pm_k$ used to define the principal part.

Lemma 5.3. *The abelian differential $\omega_{k,p,q}$ is explicitely given by*

$$\omega_{k,p,q} = (\zeta(z - p; \tau_k) - \zeta(z - q; \tau_k) - \xi(q - p; \tau_k))\, dz. \tag{26}$$

Proof. Let $\omega'_{k,p,q}$ be the right-hand side of (26). Then $\omega'_{k,p,q}$ has simple poles at p, q with residues 1 and -1. Using the quasi-periodicity of ζ, $\omega'_{k,p,q}$ is independent of the choice of the representatives of p and q modulo $\mathbb{Z} + \tau_k\mathbb{Z}$. We take these representatives in the fundamental parallelogram spanned by 1 and τ_k. We may represent α_k and β_k by curves which do not intersect the segment $[p,q]$. Using the quasi-periodicity of ζ (and omitting the argument τ_k everywhere)

$$\frac{\partial}{\partial q}\int_{\alpha_k}(\zeta(z-p)-\zeta(z-q))dz = \int_{\alpha_k}\frac{d\zeta}{dz}(z-q)dz = [\zeta(z-q)]_{\alpha_k(0)}^{\alpha_k(1)} = \eta_1.$$

Hence

$$\int_{\alpha_k}(\zeta(z-p)-\zeta(z-q))dz = (q-p)\eta_1.$$

Write $q-p = x+y\tau_k$ and recall the definition of ξ and Legendre Relation $\eta_1\tau_k - \eta_2 = 2\pi\mathrm{i}$, where we assumed that $\operatorname{Im}\tau > 0$. Then

$$\int_{\alpha_k}\omega'_{k,p,q} = (x+y\tau_k)\eta_1 - (x\eta_1+y\eta_2) = y(\eta_1\tau_k-\eta_2) = 2\pi\mathrm{i}y.$$

In the same way,

$$\int_{\beta_k}\omega'_{k,p,q} = (x+y\tau_k)\eta_2 - (x\eta_1+y\eta_2)\tau_k = x(\eta_2-\eta_1\tau_k) = -2\pi\mathrm{i}x.$$

Hence $\omega'_{k,p,q}$ has imaginary periods so is equal to $\omega_{k,p,q}$ by uniqueness. □

Lemma 5.4. *There exists a uniform constant C such that for $(t,\boldsymbol{x})$ in a neighborhood of $(0,\underline{\boldsymbol{x}})$ and $k\in\mathbb{Z}$:*

$$\|\omega^{\pm}_{k,n}\|_{C^0(\Omega_k)} \leq C\left(\frac{2}{\varepsilon}\right)^{n-1}.$$

Proof. We only prove the + case. The − case follows similarly.

Let $\eta^+_{k,n}$ be the unique meromorphic 1-form on T_k with a pole at v_k with the same principal part as $\omega^+_{k,n}$ and normalized by $\int_{\alpha_k}\eta^+_{k,n} = 0$. These two differentials are related by

$$\omega^+_{k,n} = \eta^+_{k,n} + \frac{\mathrm{i}}{\operatorname{Im}(\tau_k)}\left(\operatorname{Re}\int_{\beta_k}\eta^+_{k,n}\right)dz. \tag{27}$$

Indeed, the right-hand side has imaginary periods.

Write $\eta^+_{k,n} = f^+_{k,n}dz$. For p,q in $T_k\setminus D^+_k$, we have by the Residue Theorem in $T_k\setminus D^+_k$:

$$\int_{\partial D^+_k} f^+_{k,n}\omega_{k,p,q} = -2\pi\mathrm{i}\left(f^+_{k,n}(p) - f^+_{k,n}(q)\right).$$

On the other hand, since $\omega_{k,p,q}$ is holomorphic in D^+_k and by definition of the principal part of $f^+_{k,n}$ at v_k:

$$\int_{\partial D^+_k} f^+_{k,n}\omega_{k,p,q} = \int_{\partial D^+_k}\frac{dz^+_k}{dz}(z^+_k)^{-n}\omega_{k,p,q}.$$

Hence

$$f^+_{k,n}(p) - f^+_{k,n}(q) = -\chi^+_{k,n}(p,q) \quad \text{with} \quad \chi^+_{k,n}(p,q) = \frac{1}{2\pi\mathrm{i}}\int_{\partial D^+_k}\frac{dz^+_k}{dz}(z^+_k)^{-n}\omega_{k,p,q}.$$

Integrating with respect to q on α_k, we obtain the following integral representation of $f^+_{k,n}(p)$ for $p\in T_k\setminus D^+_k$:

$$f^+_{k,n}(p) = \int_{\alpha_k}(f^+_{k,n}(p) - f^+_{k,n}(q))dq = -\int_{\alpha_k}\chi^+_{k,n}(p,q)dq. \tag{28}$$

By Cauchy Theorem, we may replace the circle ∂D_k^+ by the circle $|z_k^+| = \frac{\varepsilon}{2}$ in the definition of $\chi_{k,n}^+$. Using Lemma 5.3, there exists a uniform constant C (independent of $k \in \mathbb{Z}$ and $\boldsymbol{x}$ in a neighborhood of $\underline{\boldsymbol{x}}$) such that for all $p, q \in \Omega_k$ and z on the circle $|z_k^+| = \frac{\varepsilon}{2}$:

$$|\omega_{k,p,q}(z)| \leq C.$$

Then for $p, q \in \Omega_k$:

$$|\chi_{k,n}^+(p,q)| \leq \frac{C}{2\pi} \int_{|z_k^+|=\varepsilon/2} \frac{|dz_k^+|}{|z_k^+|^n} = \frac{C}{2\pi} \left(\frac{2}{\varepsilon}\right)^n 2\pi \frac{\varepsilon}{2} = C \left(\frac{2}{\varepsilon}\right)^{n-1}.$$

By (28),

$$\|\eta_{k,n}^+\|_{C^0(\Omega_k)} \leq C \left(\frac{2}{\varepsilon}\right)^{n-1}. \tag{29}$$

Lemma 5.4 then follows from (27). □

5.2. **Existence.** Let Λ denotes the set $\mathbb{Z} \times \{n \in \mathbb{N} : n \geq 2\} \times \{+, -\}$. We look for ω in the form $\omega[t, \boldsymbol{x}] = \widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x}))$ where

$$\widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda}) = \omega_{k,0_k,v_k} + \sum_{n=2}^{\infty} \rho^{n-1} \left(\lambda_{k,n}^+ \omega_{k,n}^+ + \lambda_{k,n}^- \omega_{k,n}^-\right) \quad \text{in } T_k, \tag{30}$$

where $\rho \leq \frac{\varepsilon}{4}$ is a fixed positive number, and $\boldsymbol{\lambda} = (\lambda_{k,n}^s)_{(k,n,s)\in\Lambda} \in \ell^\infty$ is a sequence of complex numbers to be determined as a function of $(t, \boldsymbol{x})$. Observe that, formally, $\widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda})$ has the desired periods. Regarding convergence, by Lemma 5.4, we have in Ω_k:

$$\sum_{n=2}^{\infty} \rho^{n-1} \left|(\lambda_{k,n}^+ \omega_{k,n}^+ + \lambda_{k,n}^- \omega_{k,n}^-)\right| \leq 2C\|\boldsymbol{\lambda}\|_\infty \sum_{n=2}^{\infty} \left(\frac{2\rho}{\varepsilon}\right)^{n-1} \leq 2C\|\boldsymbol{\lambda}\|_\infty. \tag{31}$$

so the series (30) converges absolutely in Ω_k and by Lemma 5.3,

$$\|\widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda})\|_{C^0(\Omega_k)} \leq C(1 + \|\boldsymbol{\lambda}\|_\infty). \tag{32}$$

We assume for now the convergence outside Ω_k.

Lemma 5.5. *Assume that the series* (30) *converges in the annulus $A_k^\pm$ for all $k \in \mathbb{Z}$. For $t \neq 0$, $\widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda})$ is a well-defined 1-form on $\Sigma[t, \boldsymbol{x}]$ if and only if for all $k \in \mathbb{Z}$ and $n \geq 2$,*

$$\lambda_{k,n}^+ = \frac{-1}{2\pi \mathrm{i}} \int_{\partial D_{k+1}^-} \left(\frac{t^2}{\rho\, z_{k+1}^-}\right)^{n-1} \widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda}) \quad \textit{and}$$

$$\lambda_{k,n}^- = \frac{-1}{2\pi \mathrm{i}} \int_{\partial D_{k-1}^+} \left(\frac{t^2}{\rho\, z_{k-1}^+}\right)^{n-1} \widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda}).$$

Proof. Fix $t \neq 0$ and define a diffeomorphism

$$\varphi_k = (z_{k+1}^-)^{-1} \circ \frac{t^2}{z_k^+} : A_k^+ \to A_{k+1}^-$$

so $z \in A_k^+$ is identified with $\varphi_k(z) \in A_{k+1}^-$ when opening nodes. Then $\widetilde{\omega}$ is well-defined on Σ if and only if $\varphi_k^* \widetilde{\omega} = \widetilde{\omega}$ in the annulus A_k^+ for all $k \in \mathbb{Z}$. Using the theorem on Laurent series, this is equivalent to

$$\int_{\partial D_k^+} (z_k^+)^n (\varphi_k^* \widetilde{\omega} - \widetilde{\omega}) = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } n \in \mathbb{Z}. \tag{33}$$

We have

$$\begin{aligned}
\int_{\partial D_k^+} (z_k^+)^n \varphi_k^* \widetilde{\omega} &= \int_{\partial D_k^+} \varphi_k^* \left[\left(\frac{t^2}{z_{k+1}^-} \right)^n \widetilde{\omega} \right] && \text{by definition of } \varphi_k \\
&= \int_{\varphi_k(\partial D_k^+)} \left(\frac{t^2}{z_{k+1}^-} \right)^n \widetilde{\omega} && \text{by a change of variable} \\
&= -\int_{\partial D_{k+1}^-} \left(\frac{t^2}{z_{k+1}^-} \right)^n \widetilde{\omega} && \text{because } \widetilde{\omega} \text{ is holomorphic in } A_{k+1}^-.
\end{aligned}$$

For $n = 0$, (33) is always satisfied, because

$$-\int_{\partial D_{k+1}^-} \widetilde{\omega} - \int_{\partial D_k^+} \widetilde{\omega} = -2\pi \mathrm{i}\ \mathrm{Res}_{0_{k+1}}(\omega_{k+1,0_{k+1},v_{k+1}}) - 2\pi \mathrm{i}\ \mathrm{Res}_{v_k}(\omega_{k,0_k,v_k}) = 0.$$

For $n \geq 1$, we have by the Residue Theorem and definition of $\widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda})$

$$\int_{\partial D_k^+} (z_k^+)^n \widetilde{\omega} = 2\pi \mathrm{i}\, \rho^n \lambda_{k,n+1}^+,$$

so (33) is equivalent to

$$2\pi \mathrm{i}\, \rho^n \lambda_{k,n+1}^+ = -\int_{\partial D_{k+1}^-} \left(\frac{t^2}{z_{k+1}^-} \right)^n \widetilde{\omega}.$$

For $n \leq -1$, we have by the Residue Theorem

$$\int_{\partial D_{k+1}^-} \left(\frac{t^2}{z_{k+1}^-} \right)^n \widetilde{\omega} = 2\pi \mathrm{i}\, t^{2n} \rho^{-n} \lambda_{k+1,-n+1}^-$$

so (33) is equivalent to

$$2\pi \mathrm{i}\, t^{2n} \rho^{-n} \lambda_{k+1,-n+1}^- = -\int_{\partial D_k^+} (z_k^+)^n \widetilde{\omega}$$

which, after replacing n by $-n$ and k by $k-1$, becomes

$$2\pi \mathrm{i}\, \rho^n \lambda_{k,n+1}^- = -\int_{\partial D_{k-1}^+} \left(\frac{t^2}{z_{k-1}^+} \right)^n \widetilde{\omega}$$

for all $n \geq 1$. Collecting all results gives Lemma 5.5. □

In view of Lemma 5.5, we define

$$L_{k,n}^{\pm}(t, \boldsymbol{x}, \boldsymbol{\lambda}) = \frac{-1}{2\pi \mathrm{i}} \int_{\partial D_{k\pm1}^{\mp}} \left(\frac{t^2}{\rho\, z_{k\pm1}^{\mp}} \right)^{n-1} \widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda})$$

and $\boldsymbol{L} = (L_{k,n}^s)_{(k,n,s)\in\Lambda}$, so $\widetilde{\omega}(\boldsymbol{x}, \boldsymbol{\lambda})$ is well-defined on $\Sigma[t, \boldsymbol{x}]$ if and only if $\boldsymbol{\lambda} = \boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda})$. Observe that $\boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda})$ is defined for all $\boldsymbol{\lambda} \in \ell^\infty$. In particular, we do not need the convergence in $A_k^{\pm}$ to define $\boldsymbol{L}$. Also, $\widetilde{\omega}$ and $\boldsymbol{L}$ are affine with respect to $\boldsymbol{\lambda}$.

Lemma 5.6. *For $(t, \boldsymbol{x})$ in a neighborhood of $(0, \underline{\boldsymbol{x}})$, $\boldsymbol{\lambda} \mapsto \boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda})$ is contracting from ℓ^∞ to itself, hence has a fixed point $\boldsymbol{\lambda}(t, \boldsymbol{x})$ by the Fixed Point Theorem.*

Proof. We can bound the length of $\partial D_k^\pm$ by a uniform constant ℓ. By Estimate (31):

$$|L_{k,n}^\pm(t, \boldsymbol{x}, \boldsymbol{\lambda}) - L_{k,n}^\pm(t, \boldsymbol{x}, \boldsymbol{0})| \leq \frac{\ell}{2\pi} \left(\frac{t^2}{\rho\varepsilon}\right)^{n-1} 2C\|\boldsymbol{\lambda}\|_\infty.$$

Hence if $t^2 \leq \rho\varepsilon$,

$$\|\boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda}) - \boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{0})\|_\infty \leq \frac{C\ell t^2}{\pi\rho\varepsilon}\|\boldsymbol{\lambda}\|_\infty.$$

so $\boldsymbol{L}$ is contracting for t sufficiently small. □

Now we verify the convergence of (30) outside Ω_k.

Lemma 5.7. *If $\boldsymbol{\lambda} = \boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda})$ and $t \neq 0$ is sufficiently small, the series* (30) *converges absolutely in the annulus $A_k^\pm$ for all $k \in \mathbb{Z}$.*

Proof. We only deal with the convergence of $\sum_{n\geq 2} \rho^{n-1}\lambda_{k,n}^+\omega_{k,n}^+$. Its convergence in A_k^- is straightforward because $\omega_{k,n}^+$ is holomorphic in D_k^- and we already know the convergence on ∂D_k^-. It remains to prove the convergence in A_k^+.

By Cauchy Theorem, we can replace the circle ∂D_{k+1}^- by the circle $|z_{k+1}^-| = 2\varepsilon$ in the definition of $L_{k,n}^+(t, \boldsymbol{x}, \boldsymbol{\lambda})$. Using Estimate (32), this gives

$$|\lambda_{k,n}^+| = |L_{k,n}^+(t, \boldsymbol{x}, \boldsymbol{\lambda})| \leq C(1 + \|\boldsymbol{\lambda}\|_\infty) \left(\frac{t^2}{2\rho\varepsilon}\right)^{n-1}. \tag{34}$$

By definition, the function

$$\frac{\omega_{k,n}^+}{dz} - \frac{(z_k^+)'}{(z_k^+)^n}$$

extends holomorphically to the disk D_k^+. By the maximum principle and Lemma 5.4

$$\sup_{D_k^+}\left|\frac{\omega_{k,n}^+}{dz} - \frac{(z_k^+)'}{(z_k^+)^n}\right| = \max_{\partial D_k^+}\left|\frac{\omega_{k,n}^+}{dz} - \frac{(z_k^+)'}{(z_k^+)^n}\right| \leq C\left(\frac{2}{\varepsilon}\right)^{n-1} + \frac{C}{\varepsilon^n} \leq C\left(\frac{2}{\varepsilon}\right)^n.$$

Hence recalling the definition of A_k^+, and provided $2t^2 \leq \varepsilon^2$:

$$\sup_{A_k^+}|\omega_{k,n}^+| \leq C\left(\frac{2}{\varepsilon}\right)^n + C\left(\frac{\varepsilon}{t^2}\right)^n \leq C\left(\frac{\varepsilon}{t^2}\right)^n.$$

Using Estimate (34), we obtain

$$\sup_{A_k^+}\rho^{n-1}|\lambda_{k,n}^+\omega_{k,n}^+| \leq \frac{C\varepsilon}{t^2 2^n}(1 + \|\boldsymbol{\lambda}\|_\infty).$$

Hence the series $\sum_{n\geq 2}\rho^{n-1}\lambda_{k,n}^+\omega_{k,n}^+$ converges absolutely in A_k^+.

Convergence of $\sum_{n\geq 2}\rho^{n-1}\lambda_{k,n}^-\omega_{k,n}^-$ follows similarly. □

We define $\omega[t,\boldsymbol{x}] = \widetilde{\omega}(t,\boldsymbol{\lambda}(t,\boldsymbol{x}))$. By Lemmas 5.5 and 5.7, $\omega[t,\boldsymbol{x}]$ is a well-defined holomorphic 1-form on $\Sigma[t,\boldsymbol{x}]$ and has the desired periods by definition. This proves Proposition 5.1(a). At $t=0$, $\boldsymbol{L}=\boldsymbol{0}$ so $\boldsymbol{\lambda}=\boldsymbol{0}$ and $\omega=\omega_{k,0_k,v_k}$ in T_k. Proposition 5.1(b) follows from Lemma 5.3.

5.3. **Smooth dependence on parameters.** We denote $\widetilde{\gamma}_k^-$ the circle $|z|=2\varepsilon'$ in $\mathbb{T}=\mathbb{C}/(\mathbb{Z}+\mathrm{i}\mathbb{Z})$ and $\widetilde{\gamma}_k^+$ the circle $|z-\widetilde{v}_k|=2\varepsilon'$. These two circles are fixed and included in the domain $\widetilde{\Omega}_k$. Define for $\eta\in C^0(\widetilde{\Omega})$

$$\widetilde{L}^{\pm}_{k,n}(t,\boldsymbol{x},\eta)=\frac{-1}{2\pi\mathrm{i}}\int_{\widetilde{\gamma}^{\mp}_{k\pm1}}\left(\frac{t^2}{\rho\, z^{\mp}_{k\pm1}\circ\psi_{k\pm1}}\right)^{n-1}\eta$$

and let $\widetilde{\boldsymbol{L}}=(\widetilde{L}^s_{k,n})_{(k,n,s)\in\Lambda}$. Using a change of variable

$$\widetilde{L}^{\pm}_{k,n}(t,\boldsymbol{x},(\psi^*\widetilde{\omega})(\boldsymbol{x},\boldsymbol{\lambda}))=\frac{-1}{2\pi\mathrm{i}}\int_{\psi_{k\pm1}(\widetilde{\gamma}^{\mp}_{\pm1})}\left(\frac{t^2}{\rho\, z^{\mp}_{k\pm1}}\right)^{n-1}\widetilde{\omega}(\boldsymbol{x},\boldsymbol{\lambda})=L^{\pm}_{k,n}(t,\boldsymbol{x},\boldsymbol{\lambda}).$$

Hence

$$\boldsymbol{L}(t,\boldsymbol{x},\boldsymbol{\lambda})=\widetilde{\boldsymbol{L}}(t,\boldsymbol{x},(\psi^*\widetilde{\omega})(\boldsymbol{x},\boldsymbol{\lambda})). \tag{35}$$

By Lemma 5.8 below and composition, $\boldsymbol{L}$ is smooth so its fixed point $\boldsymbol{\lambda}(t,\boldsymbol{x})$ depends smoothly on $(t,\boldsymbol{x})$. By the first point of Lemma 5.8, $(\psi^*\omega)[t,\boldsymbol{x}]=(\psi^*\widetilde{\omega})(\boldsymbol{x},\boldsymbol{\lambda}(t,\boldsymbol{x}))$ depends smoothly on $(t,\boldsymbol{x})$. This proves Proposition 5.1(c).

Lemma 5.8.

(a) *$\psi^*\widetilde{\omega}$ is a smooth function of $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$ and $\boldsymbol{\lambda}\in\ell^\infty$, with value in $C^0(\widetilde{\Omega})$.*

(b) *$\widetilde{\boldsymbol{L}}$ is a smooth function of t in a neighborhood of 0, $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$ and $\eta\in C^0(\widetilde{\Omega})$, with value in ℓ^∞.*

For any infinite set K, if $(V_k)_{k\in K}$ is a sequence of normed spaces, we denote

$$\Big(\bigoplus_{k\in K}V_k\Big)_\infty=\{x\in\prod_{k\in K}V_k : \|x\|_\infty=\sup_{k\in K}\|x_k\|<\infty\}.$$

To prove Lemma 5.8, we use the following elementary fact.

Proposition 5.9. *For $k\in K$, let $f_k : B(0,r)\subset U_k\to V_k$ be a smooth function between normed spaces. Assume that there exists uniform constants $C(m)$ such that*

$$\forall m\in\mathbb{N},\quad \forall k\in K,\quad \forall x_k\in B(0,r),\quad \|d^m f_k(x_k)\|\le C(m)$$

where $d^m f_k$ denotes the m-th order differential of f_k. Let $U^\infty=(\bigoplus_{k\in K}U_k)_\infty$ and $V^\infty=(\bigoplus_{k\in K}V_k)_\infty$. Define $\boldsymbol{f} : B(0,r)\subset U^\infty\to V^\infty$ by $\boldsymbol{f}(\boldsymbol{x})=(f_k(x_k))_{k\in K}$. Then $\boldsymbol{f}$ is smooth and $d\boldsymbol{f}(\boldsymbol{x})\boldsymbol{h}=(df_k(x_k)h_k)_{k\in K}$.

We summarize the hypothesis of Proposition 5.9 by saying that the functions f_k have *uniformly bounded derivatives.*

Proof. It is straightforward to prove that f is differentiable (with the indicated differential) using Taylor Formula with integral remainder. Smoothness follows by induction. □

Recall that $\Omega_{k,r}$ denotes the torus T_k minus the disks $|z_k^\pm| \le r$. We fix a uniform positive $\varepsilon'' < \varepsilon$ so that for all $\boldsymbol{x}$ in a neighborhood of $\underline{\boldsymbol{x}}$ and $k \in \mathbb{Z}$, $\psi_k(\widetilde{\Omega}_k) \subset \Omega_{k,\varepsilon''}$. We choose ρ such that $\rho \le \varepsilon''/4$.

Claim 5.10.

(a) *For $k \in \mathbb{Z}$, $\psi_k^* \omega_{k,0_k,v_k}$ is a smooth function of (τ_k, v_k) in a neighborhood of $(\underline{\tau}_k, \underline{v}_k)$, with value in $C^0(\widetilde{\Omega}_k)$ and has uniformly bounded derivatives.*

(b) *For $k \in \mathbb{Z}$ and $n \ge 2$, $(\frac{\varepsilon''}{2})^{n-1} \psi_k^* \omega_{k,n}^\pm$ is a smooth function of $x_k = (a_k, b_k, v_k, \tau_k)$ in a neighborhood of $\underline{x}_k$, with value in $C^0(\widetilde{\Omega}_k)$, and has uniformly (with respect to k and n) bounded derivatives.*

(c) *If $t^2 < \rho\varepsilon''$, then for $k \in \mathbb{Z}$ and $n \ge 2$, $\left(\frac{t^2}{\rho z_k^\pm \circ \psi_k}\right)^{n-1}$ restricted to $\widetilde{\gamma}_k^\pm$ is a smooth function of t in a neighborhood of 0 and x_k in a neighborhood of $\underline{x}_k$, with value in $C^0(\widetilde{\gamma}_k^\pm)$, and has uniformly bounded derivatives.*

Proof.

(a) follows from the explicit formula in Lemma 5.3 and Hypothesis 1.2.

(b) By (29), we have, for some uniform constant C

$$\|\eta_{k,n}^\pm\|_{C^0(\Omega_{k,\varepsilon''})} \le C \left(\frac{2}{\varepsilon''}\right)^{n-1}.$$

Hence

$$\|\psi_k^* \eta_{k,n}^\pm\|_{C^0(\widetilde{\Omega}_k)} \le 2C \left(\frac{2}{\varepsilon''}\right)^{n-1}.$$

Observe that ψ_k depends holomorphically on τ_k. Also, $\eta_{k,n}^\pm$ depends holomorphically on x_k ($\omega_{k,n}^\pm$ does not). Hence for $z \in \widetilde{\Omega}_k$, $\psi_k^* \eta_{k,n}^\pm(z)$ depends holomorphically on x_k. By Cauchy Estimate, restricting the parameter $\boldsymbol{x}$ to a smaller neighborhood of $\underline{\boldsymbol{x}}$,we have

$$\|d^m \psi_k^* \eta_{k,n}^\pm\|_{C^0(\widetilde{\Omega}_k)} \le C(m) \left(\frac{2}{\varepsilon''}\right)^{n-1}$$

for some uniform constants $C(m)$, where d^m denotes the m-th order differential with respect to x_k. (b) follows from (27).

(c) Since $\psi_k(\widetilde{\gamma}_k^\pm) \subset \psi_k(\widetilde{\Omega}_k) \subset \Omega_{k,\varepsilon''}$, we have $|z_k^\pm \circ \psi_k| \ge \varepsilon''$ on $\widetilde{\gamma}_k^\pm$. Hence

$$\left\| \left(\frac{t^2}{\rho z_k^\pm \circ \psi_k}\right)^{n-1} \right\|_{C^0(\widetilde{\gamma}_k^\pm)} \le 1.$$

Since $z_k^\pm \circ \psi_k(z)$ depends holomorphically on x_k, we obtain uniform estimates of the derivatives by Cauchy Estimate.

∎

Proof of Lemma 5.8. We write $\widetilde{\omega}_1$ for the first term in the definition of $\widetilde{\omega}$ and $\widetilde{\omega}_2$ for the second term (the sum for $n \ge 2$).

(a) $(\psi^*\widetilde{\omega}_1)(\boldsymbol{x}) \in C^0(\widetilde{\Omega})$ and is a smooth function of $\boldsymbol{x}$. This follows from Claim 5.10(a) and Proposition 5.9. $(\psi^*\widetilde{\omega}_2)(\boldsymbol{x},\boldsymbol{\lambda}) \in C^0(\widetilde{\Omega})$ and is a smooth function of $\boldsymbol{x}$ and $\boldsymbol{\lambda}$. This follows from Claim 5.10(b), Proposition 5.9 and the fact that the bilinear operator

$$(\eta,\boldsymbol{\lambda}) \mapsto \left(\sum_{n=2}^{\infty}\left(\frac{2\rho}{\varepsilon''}\right)^{n-1}(\lambda_{k,n}^+\eta_{k,n}^+ + \lambda_{k,n}^-\eta_{k,n}^-)\right)_{k\in\mathbb{Z}} \tag{36}$$

is bounded from $\big(\bigoplus_{(k,n,s)\in\Lambda} C^0(\widetilde{\Omega}_k)\big)_\infty \times \ell^\infty$ to $C^0(\widetilde{\Omega})$. (Since $\rho \le \varepsilon''/4$, it has norm at most 2). This proves Lemma 5.8(a).

(b) Lemma 5.8(b) follows from Claim 5.10(c), Proposition 5.9 and the fact that the bilinear operator

$$(\boldsymbol{f},\eta) \mapsto \left(\frac{-1}{2\pi\mathrm{i}}\int_{\widetilde{\gamma}_{k+s}^{-s}} f_{k+s,n}^{-s}\eta\right)_{(k,n,s)\in\Lambda} \tag{37}$$

is bounded from $\big(\bigoplus_{(k,n,s)\in\Lambda} C^0(\widetilde{\gamma}_k^s)\big)_\infty \times C^0(\widetilde{\Omega})$ to ℓ^∞.

$\square$

6. Asymptotic behavior

Assume that we are given two configurations (q_k) and (q'_k) such that $q_k = q'_k$ for all $k \ge 0$. In this section we prove that the holomorphic 1-form ω' and the parameters $\boldsymbol{x}'$ are asymptotic to ω and $\boldsymbol{x}$. These results were used in Section 4 to prove the asymptotic behaviors of the minimal surfaces.

6.1. **Asymptotic behavior of ω.** For any infinite set K equipped with a weight function $\sigma : K \to [1,\infty)$, if $(V_k)_{k\in K}$ is a sequence of normed spaces, we define

$$\big(\bigoplus_{k\in K} V_k\big)_{\infty,\sigma} = \{\boldsymbol{x} \in \prod_{k\in K} V_k : \|\boldsymbol{x}\|_{\infty,\sigma} = \sup_{k\in K}\sigma(k)\|x_k\| < \infty\}.$$

If $V_k = \mathbb{C}^n$ for all $k \in K$, then we simply use the notation $\ell^{\infty,\sigma}(K)$. The argument K will be omitted if it is clear in the context.

Let $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{x}}'$ be the central values corresponding to the configurations (q_k) and (q'_k), as given by (5). Our goal is to compare $\omega[t,\boldsymbol{x}]$ to $\omega[t,\boldsymbol{x}']$ for $\boldsymbol{x},\boldsymbol{x}'$ in a neighborhood of $\underline{\boldsymbol{x}},\underline{\boldsymbol{x}}'$. For this purpose, we replace the definition of $\widetilde{\Omega}_k$ and $\widetilde{\Omega}$ in Section 3.3 by

$$\widetilde{\Omega}_k = \mathbb{T}\setminus\big(\overline{D}(0,\varepsilon')\cup\overline{D}(\underline{\widetilde{v}}_k,\varepsilon')\cup\overline{D}(\underline{\widetilde{v}}'_k,\varepsilon')\big) \quad\text{and}\quad \widetilde{\Omega} = \bigsqcup_{k\in\mathbb{Z}}\widetilde{\Omega}_k.$$

Now $(\psi^*\omega)[t,\boldsymbol{x}]$ and $(\psi^*\omega)[t,\boldsymbol{x}']$ are both defined on the same fixed domain $\widetilde{\Omega}$ so we can compare them.

Fix $\delta > 1$ and define the weight $\sigma : \mathbb{Z} \to [1,\infty)$ by $\sigma(k) = 1$ if $k \le 0$ and $\sigma(k) = \delta^k$ if $k \ge 0$. We extend this weight to Λ by $\sigma(k,n,s) = \sigma(k)$ for all $(k,n,s) \in \Lambda$. We have $\underline{x}_k = \underline{x}'_k$ for all $k \ge 0$ so $\underline{\boldsymbol{x}}' - \underline{\boldsymbol{x}} \in \ell^{\infty,\sigma}$. It will be convenient to write

$$\Delta\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}.$$

We define the weighted space $C^{0,\sigma}(\widetilde{\Omega})$ (not to be confused with a Hölder space) by

$$C^{0,\sigma}(\widetilde{\Omega}) = \big(\bigoplus_{k\in\mathbb{Z}} C^0(\widetilde{\Omega}_k)\big)_{\infty,\sigma}$$

So functions in $C^{0,\sigma}(\widetilde{\Omega})$ decay like δ^{-k} in $\widetilde{\Omega}_k$ as $k \to +\infty$.

Proposition 6.1. *For t small enough, $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$ and $\Delta\boldsymbol{x}$ in an $\ell^{\infty,\sigma}$ neighborhood of $\Delta\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}' - \underline{\boldsymbol{x}}$,*

$$(\psi^*\omega)[t, \boldsymbol{x} + \Delta\boldsymbol{x}] - (\psi^*\omega)[t, \boldsymbol{x}] \in C^{0,\sigma}(\widetilde{\Omega})$$

and depends smoothly on t, $\boldsymbol{x} \in \ell^\infty$ and $\Delta\boldsymbol{x} \in \ell^{\infty,\sigma}$.

Proof. Recall that $\boldsymbol{\lambda}(t, \boldsymbol{x})$ is given by Lemma 5.6 and define

$$\boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x}) = \boldsymbol{\lambda}(t, \boldsymbol{x} + \Delta\boldsymbol{x}) - \boldsymbol{\lambda}(t, \boldsymbol{x}) \in \ell^\infty(\Lambda).$$

Our goal is to prove that $\boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x}) \in \ell^{\infty,\sigma}(\Lambda)$.

Let

$$\Delta\boldsymbol{L}(t, \boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}) = \boldsymbol{L}(t, \boldsymbol{x} + \Delta\boldsymbol{x}, \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}) - \boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda}).$$

Then

$$\Delta\boldsymbol{L}(t, \boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x}), \boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x})) = \boldsymbol{L}(t, \boldsymbol{x}+\Delta\boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x}+\Delta\boldsymbol{x})) - \boldsymbol{L}(t, \boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x})) = \boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x}).$$

In other words, $\boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x})$ is a fixed point of $\Delta\boldsymbol{L}$ with respect to the $\Delta\boldsymbol{\lambda}$ variable.

Define

$$\Delta\widetilde{\boldsymbol{L}}(t, \boldsymbol{x}, \Delta\boldsymbol{x}, \eta, \Delta\eta) = \widetilde{\boldsymbol{L}}(t, \boldsymbol{x} + \Delta\boldsymbol{x}, \eta + \Delta\eta) - \widetilde{\boldsymbol{L}}(t, \boldsymbol{x}, \eta)$$

and

$$\boldsymbol{H}(\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}) = (\psi^*\widetilde{\omega})(\boldsymbol{x} + \Delta\boldsymbol{x}, \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}) - (\psi^*\widetilde{\omega})(\boldsymbol{x}, \boldsymbol{\lambda}).$$

Recalling (35), we have

$$\Delta\boldsymbol{L}(t, \boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}) = \Delta\widetilde{\boldsymbol{L}}\big(t, \boldsymbol{x}, \Delta\boldsymbol{x}, (\psi^*\widetilde{\omega})(\boldsymbol{x}, \boldsymbol{\lambda}), \boldsymbol{H}(\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}, \Delta\boldsymbol{\lambda})\big).$$

By Lemma 6.2 below, $\Delta\boldsymbol{L} \in \ell^{\infty,\sigma}$ is smooth with respect to $\boldsymbol{x}, \boldsymbol{\lambda} \in \ell^\infty$ and $\Delta\boldsymbol{x}, \Delta\boldsymbol{\lambda} \in \ell^{\infty,\sigma}$, and is contracting with respect to $\Delta\boldsymbol{\lambda}$, so its unique fixed point $\boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x})$ is in $\ell^{\infty,\sigma}$ and depends smoothly on $t, \boldsymbol{x}$,$\Delta\boldsymbol{x}$ in their respective spaces. Finally, we have by definition

$$\begin{aligned}(\psi^*\omega)[t, \boldsymbol{x} + \Delta\boldsymbol{x}] - (\psi^*\omega)[t, \boldsymbol{x}] &= (\psi^*\widetilde{\omega})(\boldsymbol{x} + \Delta\boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x} + \Delta\boldsymbol{x})) - (\psi^*\widetilde{\omega})(\boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x}))\\ &= \boldsymbol{H}\big(\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}(t, \boldsymbol{x}), \boldsymbol{\mu}(t, \boldsymbol{x}, \Delta\boldsymbol{x})\big)\end{aligned}$$

so Proposition 6.1 follows from Lemma 6.2(a). ∎

Lemma 6.2.

(a) *For $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$, $\Delta\boldsymbol{x}$ in an $\ell^{\infty,\sigma}$-neighborhood of $\Delta\underline{\boldsymbol{x}}$, $\boldsymbol{\lambda} \in \ell^\infty$ and $\Delta\boldsymbol{\lambda} \in \ell^{\infty,\sigma}$,*

$$\boldsymbol{H}(\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}) \in C^{0,\sigma}(\widetilde{\Omega})$$

and depends smoothly on $\boldsymbol{x}, \Delta\boldsymbol{x}, \boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}$.

(b) *For t in a neighborhood of 0, $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$, $\Delta\boldsymbol{x}$ in an $\ell^{\infty,\sigma}$-neighborhood of $\Delta\underline{\boldsymbol{x}}$, $\eta \in C^0(\widetilde{\Omega})$ and $\Delta\eta \in C^{0,\sigma}(\widetilde{\Omega})$,*

$$\Delta\widetilde{\boldsymbol{L}}(t, \boldsymbol{x}, \Delta\boldsymbol{x}, \eta, \Delta\eta) \in \ell^{\infty,\sigma}$$

and depends smoothly on $t, \boldsymbol{x}, \Delta\boldsymbol{x}, \eta, \Delta\eta$.

(c) *For t small enough, $\Delta\boldsymbol{L}$ is contracting with respect to $\Delta\boldsymbol{\lambda}$, as a map from $\ell^{\infty,\sigma}$ to itself.*

We need the following

Proposition 6.3. *Under the same notations and hypothesis as in Proposition 5.9, let* $\sigma : K \to [1,\infty)$ *be an arbitrary weight. Let* $U^{\infty,\sigma} = (\bigoplus_{k\in K} U_k)_{\infty,\sigma}$ *and* $V^{\infty,\sigma} = (\bigoplus_{k\in K} V_k)_{\infty,\sigma}$. *Define for* $\boldsymbol{x} \in B(\mathbf{0}, r/2) \subset U^\infty$ *and* $\Delta\boldsymbol{x} \in B(\mathbf{0}, r/2) \subset U^{\infty,\sigma}$

$$\Delta\boldsymbol{f}(\boldsymbol{x},\Delta\boldsymbol{x}) = \big(f_k(x_k+\Delta x_k) - f_k(x_k)\big)_{k\in K}.$$

Then $\Delta\boldsymbol{f}(\boldsymbol{x},\Delta\boldsymbol{x}) \in V^{\infty,\sigma}$, $\Delta\boldsymbol{f}$ *is smooth and*

$$d(\Delta\boldsymbol{f})(\boldsymbol{x},\Delta\boldsymbol{x})(\boldsymbol{h},\Delta\boldsymbol{h}) = \big(df_k(x_k+\Delta x_k)(h_k+\Delta h_k) - df_k(x_k)h_k\big)_{k\in K}.$$

Proof. By the Mean Value Inequality

$$\sigma(k)\|f_k(x_k+\Delta x_k) - f_k(x_k)\| \le C\sigma(k)\|\Delta x_k\|$$

Hence $\Delta\boldsymbol{f}(\boldsymbol{x},\Delta\boldsymbol{x}) \in V^{\infty,\sigma}$. Define

$$l_k(h_k,\Delta h_k) = df_k(x_k+\Delta x_k)(h_k+\Delta h_k) - df_k(x_k)h_k \quad \text{and} \quad \boldsymbol{l}(\boldsymbol{h},\Delta\boldsymbol{h}) = (l_k(h_k,\Delta h_k))_{k\in K}.$$

Using the Mean Value Inequality, one easily obtains

$$\sigma(k)\|l_k(h_k,\Delta h_k)\| \le C\sigma(k)\big(\|\Delta h_k\| + \|\Delta x_k\|\,\|h_k\|\big).$$

Hence $\boldsymbol{l}$ is a bounded operator from $U^\infty \times U^{\infty,\sigma}$ to $V^{\infty,\sigma}$. Using Taylor Formula with integral remainder, we have

$$\begin{aligned}
&\Delta f_k(x_k+h_k, \Delta x_k+\Delta h_k) - \Delta f_k(x_k,\Delta x_k) - l_k(h_k,\Delta h_k)\\
=&f_k(x_k+\Delta x_k+h_k+\Delta h_k) - f_k(x_k+\Delta x_k)\\
&-df_k(x_k+\Delta x_k)(h_k+\Delta h_k) - [f_k(x_k+h_k) - f_k(x_k) - df_k(x_k)h_k]\\
=&\int_0^1 (1-t)\left[d^2 f_k\big(x_k+\Delta x_k+t(h_k+\Delta h_k)\big)(h_k+\Delta h_k)^2 - d^2 f_k(x_k+th_k)h_k^2\right] dt
\end{aligned}$$

By the Mean Value Inequality

$$\begin{aligned}
&\sigma(k)\|\Delta f_k(x_k+h_k,\Delta x_k+\Delta h_k) - \Delta f_k(x_k,\Delta x_k) - l_k(h_k,\Delta h_k)\|\\
\le&C\sigma(k)\left[2\|h_k\|\|\Delta h_k\| + \|\Delta h_k\|^2 + (\|\Delta x_k\| + \|\Delta h_k\|)\|h_k\|^2\right].
\end{aligned}$$

Hence

$$\|\Delta\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h},\Delta\boldsymbol{x}+\Delta\boldsymbol{h}) - \Delta\boldsymbol{f}(\boldsymbol{x},\Delta\boldsymbol{x}) - \boldsymbol{l}(\boldsymbol{h},\Delta\boldsymbol{h})\|_{\infty,\sigma} = O((\|\boldsymbol{h}\|_\infty + \|\Delta\boldsymbol{h}\|_{\infty,\sigma})^2)$$

so $\Delta\boldsymbol{f}$ is differentiable with $d(\Delta\boldsymbol{f})(\boldsymbol{x},\Delta\boldsymbol{x}) = \boldsymbol{l}$. Smoothness follows by induction. ☐

We shall use the following corollary with $K=\mathbb{Z}$, $K^+ = \mathbb{N}$ and $K^- = \mathbb{Z}\setminus\mathbb{N}$:

Corollary 6.4. *With the same notation as in Proposition 6.3, assume that for* $k \in K$, f_k *is defined in* $B(\underline{x}_k, r) \cup B(\underline{x}'_k, r)$ *in* U_k *and has uniformly bounded derivatives. Assume that* K *admits a partition* (K^+,K^-) *such that for all* $k \in K^+$, $\underline{x}_k = \underline{x}'_k$, *and for all* $k\in K^-$, $\sigma(k) = 1$. *Then for* $\boldsymbol{x} \in B(\underline{\boldsymbol{x}}, r/2) \subset U^\infty$ *and* $\Delta\boldsymbol{x} \in B(\underline{\boldsymbol{x}}' - \underline{\boldsymbol{x}}, r/2) \subset U^{\infty,\sigma}$, $\Delta\boldsymbol{f}(\boldsymbol{x},\Delta\boldsymbol{x}) \in V^{\infty,\sigma}$ *and depends smoothly on* $\boldsymbol{x}$ *and* $\Delta\boldsymbol{x}$.

Proof. We decompose a sequence $\boldsymbol{x} = (x_k)_{k\in K}$ as $\boldsymbol{x} = \boldsymbol{x}^+ + \boldsymbol{x}^-$ with $\boldsymbol{x}^+$ supported on K^+ and $\boldsymbol{x}^-$ supported on K^-. By Proposition 5.9, $\boldsymbol{f}^-(\boldsymbol{x}^-) \in V^\infty$ and $\boldsymbol{f}^-(\boldsymbol{x}^- + \Delta\boldsymbol{x}^-) \in V^\infty$ so $\Delta\boldsymbol{f}^-(\boldsymbol{x}^-,\Delta\boldsymbol{x}^-) \in V^\infty$. Since $\sigma = 1$ on K^-, $\Delta\boldsymbol{f}^-(\boldsymbol{x}^-,\Delta\boldsymbol{x}^-) \in V^{\infty,\sigma}$. Since $\underline{\boldsymbol{x}}^+ = (\underline{\boldsymbol{x}}')^+$, we may use the change of variable $\boldsymbol{x}^+ = \underline{\boldsymbol{x}}^+ + \boldsymbol{y}^+$ and conclude that $\Delta\boldsymbol{f}^+(\boldsymbol{x}^+,\Delta\boldsymbol{x}^+) \in V^{\infty,\sigma}$ by Proposition 6.3. ☐

Proof of Lemma 6.2. Lemma 6.2(a) follows from the following.

- $(\psi^*\widetilde{\omega}_1)(\boldsymbol{x}+\Delta\boldsymbol{x}) - (\psi^*\widetilde{\omega}_1)(\boldsymbol{x}) \in C^{0,\sigma}(\widetilde{\Omega})$ and is a smooth function of $\boldsymbol{x}$, $\Delta\boldsymbol{x}$. This follows from Claim 5.10(a) and Corollary 6.4.
- $(\psi^*\widetilde{\omega}_2)(\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{\lambda}) - (\psi^*\widetilde{\omega}_2)(\boldsymbol{x},\boldsymbol{\lambda}) \in C^{0,\sigma}(\widetilde{\Omega})$ and is a smooth function of $\boldsymbol{x}$, $\Delta\boldsymbol{x}$ and $\boldsymbol{\lambda}$. This follows from Claim 5.10(b), Corollary 6.4 and the fact that the bilinear operator (36) is bounded from $\big(\bigoplus_{(k,n,s)\in\Lambda} C^0(\widetilde{\Omega}_k)\big)_{\infty,\sigma} \times \ell^\infty(\Lambda)$ to $C^{0,\sigma}(\widetilde{\Omega})$.
- $(\psi^*\widetilde{\omega}_2)(\boldsymbol{x}+\Delta\boldsymbol{x},\Delta\boldsymbol{\lambda}) \in C^{0,\sigma}(\widetilde{\Omega})$ and is a smooth function of $\boldsymbol{x}$, $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{\lambda}$. This follows from Claim 5.10(b), Proposition 5.9 and the fact that the bilinear operator (36) is bounded from $\big(\bigoplus_{(k,n,s)\in\Lambda} C^0(\widetilde{\Omega}_k)\big)_{\infty} \times \ell^{\infty,\sigma}(\Lambda)$ to $C^{0,\sigma}(\widetilde{\Omega})$.

Lemma 6.2(b) follows from the following.

- $\widetilde{\boldsymbol{L}}(t,\boldsymbol{x}+\Delta\boldsymbol{x},\eta) - \widetilde{\boldsymbol{L}}(t,\boldsymbol{x},\eta) \in \ell^{\infty,\sigma}(\Lambda)$ and depends smoothly on $t,\boldsymbol{x},\Delta\boldsymbol{x}$ and η. This follows from Claim 5.10(c), Corollary 6.4 and the fact that the bilinear operator (37) is bounded from $\big(\bigoplus_{(k,n,s)\in\Lambda} C^0(\widetilde{\gamma}_k^s)\big)_{\infty,\sigma} \times C^0(\widetilde{\Omega})$ to $\ell^{\infty,\sigma}(\Lambda)$. This uses that $\frac{\sigma(k)}{\sigma(k\pm1)} \le \delta$ and explains our choice of the weight σ.
- $\widetilde{\boldsymbol{L}}(t,\boldsymbol{x}+\Delta\boldsymbol{x},\Delta\eta) \in \ell^{\infty,\sigma}(\Lambda)$ and depends smoothly on $t,\boldsymbol{x},\Delta\boldsymbol{x}$ and $\Delta\eta$. This follows from Claim 5.10(c), Proposition 5.9 and the fact that the bilinear operator (37) is bounded from $\big(\bigoplus_{(k,n,s)\in\Lambda} C^0(\widetilde{\gamma}_k^s)\big)_{\infty} \times C^{0,\sigma}(\widetilde{\Omega})$ to $\ell^{\infty,\sigma}(\Lambda)$.

Finally, we have

$$\Delta\boldsymbol{L}(t,\boldsymbol{x},\Delta\boldsymbol{x},\boldsymbol{\lambda},\Delta\boldsymbol{\lambda}) - \Delta\boldsymbol{L}(t,\boldsymbol{x},\Delta\boldsymbol{x},\boldsymbol{\lambda},0) = \boldsymbol{L}(t,\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{\lambda}+\Delta\boldsymbol{\lambda}) - \boldsymbol{L}(t,\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{\lambda}).$$

Using Estimate (31) as in the proof of Lemma 5.6,

$$\|\boldsymbol{L}(t,\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{\lambda}+\Delta\boldsymbol{\lambda}) - \boldsymbol{L}(t,\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{\lambda})\|_{\infty,\sigma} \le \frac{C\ell\delta t^2}{\pi\rho\varepsilon}\|\Delta\boldsymbol{\lambda}\|_{\infty,\sigma}$$

so $\Delta\boldsymbol{L}$ is contracting with respect to $\Delta\boldsymbol{\lambda}$ for t small enough. □

6.2. **Asymptotic behavior of the parameters.** Let $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$ be the solutions obtained in Section 3 from the configurations (q_k) and (q'_k), respectively.

Proposition 6.5. *For t small enough, $\boldsymbol{x}'(t) - \boldsymbol{x}(t) \in \ell^{\infty,\sigma}$.*

Proof. Recall the definition of $\mathcal{E}_k$ in Section 3.4, $\mathcal{P}_{k,1}$ and $\mathcal{P}_{k,2}$ in Section 3.5 and $\mathcal{G}_k$ in Section 3.6. We have solved equations by three consecutive applications of the Implicit Function Theorem. But we could have solved all of them by one single application. Indeed, consider the change of parameter

$$b_k = -a_k\xi(v_k,\tau_k) + \widehat{b}_k.$$

By the computations in Sections 3.4, 3.5 and 3.6, the jacobian of $(\mathcal{E}_k,\mathcal{P}_{k,1},\mathcal{P}_{k,2},\mathcal{G}_k)$ with respect to $(\widehat{b}_k,a_k,\tau_k,v_k)$ has upper-triangular form with $\mathbb{R}$-linear automorphisms of $\mathbb{C}$ on the diagonal, whose inverses are uniformly bounded with respect to k. Define

$$\mathcal{F}_k(t,\boldsymbol{x}) = (\mathcal{E}_k(t,\boldsymbol{x}),\mathcal{P}_{k,1}(t,\boldsymbol{x}),\mathcal{P}_{k,2}(t,\boldsymbol{x}),\mathcal{G}_k(t,\boldsymbol{x}) + 2\pi\mathrm{i}G(q_0;T)) \quad \text{and} \quad \boldsymbol{\mathcal{F}} = (\mathcal{F}_k)_{k\in\mathbb{Z}}.$$

Then $d_{\boldsymbol{x}}\boldsymbol{\mathcal{F}}(0,\underline{\boldsymbol{x}}')$ is an automorphism of ℓ^∞, and restricts to an automorphism of $\ell^{\infty,\sigma}$. Define, for $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$ and $\Delta\boldsymbol{x}$ in an $\ell^{\infty,\sigma}$-neighborhood of $\Delta\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}' - \underline{\boldsymbol{x}}$

$$\Delta\boldsymbol{\mathcal{F}}(t,\boldsymbol{x},\Delta\boldsymbol{x}) = \boldsymbol{\mathcal{F}}(t,\boldsymbol{x}+\Delta\boldsymbol{x}) - \boldsymbol{\mathcal{F}}(t,\boldsymbol{x}).$$

By Lemma 6.6 below, $\Delta\boldsymbol{\mathcal{F}}(t,\boldsymbol{x},\Delta\boldsymbol{x}) \in \ell^{\infty,\sigma}$. We have

$$\Delta\boldsymbol{\mathcal{F}}(0,\underline{\boldsymbol{x}},\Delta\underline{\boldsymbol{x}}) = \boldsymbol{\mathcal{F}}(t,\underline{\boldsymbol{x}}') - \boldsymbol{\mathcal{F}}(t,\underline{\boldsymbol{x}}) = 0 \quad \text{and} \quad d_{\Delta\boldsymbol{x}}(\Delta\boldsymbol{\mathcal{F}})(0,\underline{\boldsymbol{x}},\Delta\underline{\boldsymbol{x}}) = d_{\boldsymbol{x}}\boldsymbol{\mathcal{F}}(0,\underline{\boldsymbol{x}}').$$

By the Implicit Function Theorem, for t small enough and $\boldsymbol{x}$ in a neighborhood of $\underline{\boldsymbol{x}}$, there exists $\Delta\boldsymbol{x}(t,\boldsymbol{x}) \in \ell^{\infty,\sigma}$ such that $\Delta\boldsymbol{\mathcal{F}}(t,\boldsymbol{x},\Delta\boldsymbol{x}(t,\boldsymbol{x})) = 0$. We substitute $\boldsymbol{x} = \boldsymbol{x}(t)$ and obtain $\boldsymbol{\mathcal{F}}(t,\boldsymbol{x}(t)+\Delta\boldsymbol{x}(t,\boldsymbol{x}(t))) = \boldsymbol{\mathcal{F}}(t,\boldsymbol{x}'(t)) = 0$. By uniqueness, $\boldsymbol{x}'(t) = \boldsymbol{x}(t) + \Delta\boldsymbol{x}(t,\boldsymbol{x}(t))$, which proves Proposition 6.5. □

Lemma 6.6. *For t in a neighborhood of 0, $\boldsymbol{x}$ in an ℓ^∞-neighborhood of $\underline{\boldsymbol{x}}$ and $\Delta\boldsymbol{x}$ in an $\ell^{\infty,\sigma}$-neighborhood of $\Delta\underline{\boldsymbol{x}}$, $\Delta\boldsymbol{\mathcal{F}}(t,\boldsymbol{x},\Delta\boldsymbol{x}) \in \ell^{\infty,\sigma}$ and depends smoothly on t, $\boldsymbol{x}$ and $\Delta\boldsymbol{x}$.*

Proof. Define for $z \in \mathbb{T}$:

$$f_k[\boldsymbol{x}](z) = \frac{\psi_k(z) - (Z_{k,1}+Z_{k,2})}{\psi_k(z)^2 - (Z_{k,1}+Z_{k,2})\psi_k(z) + Z_{k,1}Z_{k,2}} \quad \text{and} \quad \boldsymbol{f}[\boldsymbol{x}] = (f_k[\boldsymbol{x}])_{k\in\mathbb{Z}}.$$

By Cauchy Theorem and a change of variable,

$$\mathcal{E}_k(t,\boldsymbol{x}) = \frac{1}{2\pi\mathrm{i}}\int_{\partial\widetilde{\Omega}_{k,2\varepsilon'}} f_k\psi_k^*\omega[t,\boldsymbol{x}].$$

Hence we can write

$$\boldsymbol{\mathcal{E}}(t,\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{f}[\boldsymbol{x}],(\psi^*\omega)[t,\boldsymbol{x}]) \quad \text{where} \quad \boldsymbol{B}(\boldsymbol{f},\eta) = \left(\frac{1}{2\pi\mathrm{i}}\int_{\partial\widetilde{\Omega}_{k,2\varepsilon'}} f_k\eta_k\right)_{k\in\mathbb{Z}}.$$

Using Weierstrass Preparation Theorem, the symmetric functions of $Z_{k,1}$ and $Z_{k,2}$ are holomorphic functions of x_k. Hence f_k is a smooth function of x_k with value in $C^0(\partial\widetilde{\Omega}_{k,2\varepsilon'})$. Using Corollary 6.4 and that the bilinear operator

$$\boldsymbol{B} : \Big(\bigoplus_{k\in\mathbb{Z}} C^0(\partial\widetilde{\Omega}_{k,2\varepsilon'})\Big)_{\infty,\sigma} \times C^0(\widetilde{\Omega}) \to \ell^{\infty,\sigma}$$

is bounded, we conclude that

$$\boldsymbol{B}(\boldsymbol{f}[\boldsymbol{x}+\Delta\boldsymbol{x}] - \boldsymbol{f}[\boldsymbol{x}],(\psi^*\omega)[t,\boldsymbol{x}]) \in \ell^{\infty,\sigma} \tag{38}$$

and depends smoothly on t, $\boldsymbol{x}$ and $\Delta\boldsymbol{x}$. Using Proposition 6.1 and that the bilinear operator

$$\boldsymbol{B} : \Big(\bigoplus_{k\in\mathbb{Z}} C^0(\partial\widetilde{\Omega}_{k,2\varepsilon'})\Big)_{\infty} \times C^{0,\sigma}(\widetilde{\Omega}) \to \ell^{\infty,\sigma}$$

is bounded, we obtain

$$\boldsymbol{B}(\boldsymbol{f}[\boldsymbol{x}+\Delta\boldsymbol{x}],(\psi^*\omega)[t,\boldsymbol{x}+\Delta\boldsymbol{x}] - (\psi^*\omega)[t,\boldsymbol{x}]) \in \ell^{\infty,\sigma} \tag{39}$$

and depends smoothly on t, $\boldsymbol{x}$ and $\Delta\boldsymbol{x}$. Adding (38) and (39), we conclude that

$$\boldsymbol{\mathcal{E}}(t,\boldsymbol{x}+\Delta\boldsymbol{x}) - \boldsymbol{\mathcal{E}}(t,\boldsymbol{x}) \in \ell^{\infty,\sigma}$$

and depends smoothly on t, $\boldsymbol{x}$ and $\Delta\boldsymbol{x}$.

Finally, $\mathcal{P}_{k,1}$, $\mathcal{P}_{k,2}$ and $\mathcal{G}_k$ are defined as integrals of ω times powers of g_k on certain curves in T_k, so we can deal with them in the same way as $\mathcal{E}_k$. □

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(Chen) Georg-August-Universität Göttingen, Institut für Numerische und Angewandte Mathematik

Email address: h.chen@math.uni-goettingen.de

(Traizet) Institut Denis Poisson, CNRS UMR 7350, Faculté des Sciences et Techniques, Université de Tours

Email address: martin.traizet@lmpt.univ-tours.fr