EVEN MORE INFINITE BALL PACKINGS FROM LORENTZIAN COXETER SYSTEMS

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Abstract. Boyd (1974) proposed a class of infinite ball packings that are generated by inversions. Later, Maxwell (1983) interpreted Boyd's construction in terms of Lorentzian Coxeter systems. In Maxwell's work, the simple roots form a basis of the representations space of the Coxeter group. With a new definition for the notion of "level", we extend Boyd–Maxwell's results to the case where the simple roots are positively independent. Our main theorem is that, under the new definition, the space-like weights of a Lorentzian Coxeter system correspond to a ball packing if and only if the Coxeter system is of level 2. We also present a partial classification of level-2 Coxeter d-polytopes with $d+2$ facets.

1. Introduction

The title refers to a paper of Boyd titled "A new class of infinite sphere packings" [Boy74], in which he described a class of infinite ball packings that are generated by inversions, generalising the famous Apollonian disk packing. Later, Maxwell [Max82] generalized Boyd's construction by interpreting the ball packing as the space-like weights associated to an infinite root system in Lorentz space. In particular, Maxwell proved that the space-like weights correspond to a ball packing if and only if the associated Coxeter system is of "level 2". In this paper, we propose a new definition for the notion of "level" and extend Maxwell's results to the new definition.

Inspired by recent works [DHR13, HLR14] on limit roots (i.e. accumulation points of roots), Labbé and the author [CL14] revisited Maxwell's work. For Lorentzian Coxeter systems of level 2, we proved that the accumulation points of the roots and of the weights coincide on the light cone in the projective space, and that the set of limit roots is the residue set of the ball packing described by Boyd and Maxwell. Furthermore, we gave a geometric interpretation for Maxwell's notion of level, described the tangency graph of the Boyd–Maxwell ball packing in terms of the Coxeter complex, and completed the enumeration of 326 Coxeter graphs that is of level 2 in Maxwell's sense.

Our results in [CL14] establish a connection between [Max82] and [DHR13, HLR14], but the latter are more general in several ways: First, the Coxeter systems considered in [DHR13, HLR14] are not necessarily of level 2. Lorentzian Coxeter systems of level $\neq 2$ were also investigated in [CL14]. It turns out that no ball appears for Coxeter systems of level 1, and balls may intersect for Coxeter systems of level > 2. In either case, it remains true that the set of limit roots is the residue set of the balls corresponding to the space-like weights. Second, the Coxeter systems considered in [DHR13, HLR14] are not necessarily Lorentzian. For non-Lorentzian Coxeter systems, we conjectured in [CL14] that accumulation points of roots still coincide with accumulation points of the weights.

The current paper deals with a third gap between [DHR13, HLR14] and [Max82]. Maxwell only considered the case where the simple roots form a basis of the representation space of the Coxeter group. In this case, one can define the fundamental weights as the dual basis. However, in [HLR14], the simple roots are only required to be positively independent, but not necessarily linearly independent. We propose in Definition 2.2 a new definition for the fundamental weights, which makes sense even if the simple roots are not linearly independent. Our definition uses the

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notion of facial subset adapted from [DHR13]; see Definition 2.1. Consequently, based on the geometric interpretation of level in [CL14], we propose in Definition 2.3 a geometric definition for the notion of "level", to replace Maxwell's graph theoretical definition. Many Coxeter systems of level > 2 in Maxwell's sense becomes of level 2 under our definition.

Our definitions are more suitable for geometric studies. In Section 3, we extend one by one the results in [Max82] and [CL14] to the new definition. It turns out that

Main Result. *All the results in* [Max82] *and in* [CL14] *remain valid under our new definitions.*

However, many proofs need to be revised. Whiles Maxwell's proofs rely on decomposition of vectors into basis vectors (i.e. the simple roots), our proofs make heavy use of projective geometry.

In particular, the new definition includes more Lorentzian Coxeter systems of level 2, corresponding to many new infinite ball packings generated by inversions. In Section 4, we provide a partial classification of Coxeter systems of level 2 under our definition. More specifically, we try to classify the level-2 Coxeter d-polytopes with $d+2$ facets. For this, we follow the approach of [Kap74, Ess96, Tum04] for enumerating hyperbolic Coxeter d-polytopes with $d+2$ facets, and take advantage of previous enumerations of Coxeter systems, such as [Lan50, Che69, CL14]. Our enumeration makes contribution to the study of infinite-covolume hyperbolic reflection groups.

2. Geometric Coxeter systems and levels

2.1. **Lorentz space.** A *pseudo-Euclidean space* is a pair $(V,\mathcal{B})$ where V is a real vector space and $\mathcal{B}$ is a symmetric bilinear form on V.

Two vectors $\mathbf{x},\mathbf{y}\in V$ are said to be *orthogonal* if $\mathcal{B}(\mathbf{x},\mathbf{y})=0$. For a subspace $U\subseteq V$, its *orthogonal companion* is the set

$$U^\perp=\{\mathbf{x}\in V\mid \mathcal{B}(\mathbf{x},\mathbf{y})=0 \text{ for all } \mathbf{y}\in U\}$$

The orthogonal companion $V^\perp$ of the whole space V is called the *radical.* We say that $(V,\mathcal{B})$ is *degenerate* if the radical $V^\perp$ contains non-zero vectors. In this case, the matrix of $B=(\mathcal{B}(\mathbf{e}_i,\mathbf{e}_j))$ is singular for any basis $\{\mathbf{e}_i\}$ of V.

We say that $(V,\mathcal{B})$ is a *Euclidean space* if the matrix B is positive definite, or a *Lorentz space* if the matrix B is nonsingular and has exactly one negative eigenvalue. The group of linear transformations of V that preserve the bilinear form $\mathcal{B}$ is called a *pseudo-orthogonal group*, and is denoted by $O_\mathcal{B}(V)$. The pseudo-orthogonal group of an Euclidean space is called an *orthogonal group*, and that of a Lorentz space is called a *Lorentz group.*

The set

$$Q=\{\mathbf{x}\in V\mid \mathcal{B}(\mathbf{x},\mathbf{x})=0\}$$

is called the *isotropic cone*, and vectors in Q are said to be *isotropic.* In Lorentz space, the isotropic cone is called the *light cone*, and isotropic vectors are said to be *light-like.* Two light-like vectors are orthogonal if and only if one is the scalar multiple of the other. A non-isotropic vector $\mathbf{x}\in V$ is said to be *space-like* (resp. *time-like*) if $\mathcal{B}(\mathbf{x},\mathbf{x})>0$ (resp. <0). A subspace $U\subseteq V$ is said to be *space-like* if its non-zero vectors are all space-like, *light-like* if it contains some non-zero light-like vector but no time-like vector, or *time-like* if it contains time-like vectors.

For a non-isotropic vector $\alpha\in V$, the *reflection* in α is defined as the map

$$s_\alpha(\mathbf{x})=\mathbf{x}-2\frac{\mathcal{B}(\alpha,\mathbf{x})}{\mathcal{B}(\alpha,\alpha)}\alpha,\quad \text{for all } \mathbf{x}\in V.$$

The orthogonal companion of $\alpha\mathbb{R}$, denoted by

$$H_\alpha=\{\mathbf{x}\in V\mid \mathcal{B}(\mathbf{x},\alpha)=0\},$$

is fixed by the reflection s_α, and is called the *reflecting hyperplane* of α. One verifies that α is space-like (resp. light-like, time-like) if and only if H_α is time-like (resp. light-like, space-like).

For a subset $X\in V$, its *dual* X^* is the set

$$X^*=\{\mathbf{x}\in V\mid \mathcal{B}(\mathbf{x},\mathbf{y})>0 \text{ for all } \mathbf{y}\in X\}$$

2.2. **Representation of Coxeter system.** An abstract *Coxeter system* is a pair (W,S), where S is a finite set of generators and the *Coxeter group* W is generated by S with the relations $(st)^{m_{st}}=e$ where $s,t\in S$, $m_{ss}=1$ and $m_{st}=m_{ts}\geq 2$ or $=\infty$ if $s\neq t$. The cardinality $n=|S|$ is the *rank* of the Coxeter system (W,S). For an element $w\in W$, the *length* of w, denoted by $\operatorname{len}(w)$, is the smallest natural number k such that $w=s_1s_2\dots s_k$ for $s_i\in S$. The readers are invited to consult [Bou68, Hum92] for more details.

Let $(V,\mathcal{B})$ be a pseudo-Euclidean space. A *simple system* Δ in $(V,\mathcal{B})$ is a set of vectors in V such that

(1) $\mathcal{B}(\alpha,\alpha)=1$ for all $\alpha\in\Delta$;
(2) $\mathcal{B}(\alpha,\beta)\in(-\infty,-1]\cup\{-\cos(\pi/k),\ k\in\mathbb{Z}_{\geq 2}\}$ for all $\alpha\neq\beta\in\Delta$;
(3) Δ is *positively independent*. That is, a linear combination of Δ with non-negative coefficient vanishes only if all the coefficients vanishes.

Let $S=\{s_\alpha\mid\alpha\in\Delta\}$ be the set of reflections in vectors of Δ, and W be the reflection subgroup generated by S. Then (W,S) is a *Coxeter system*, where the order of $s_\alpha s_\beta$ is k if $\mathcal{B}(\alpha,\beta)=-\cos(\pi/k)$, or ∞ if $\mathcal{B}(\alpha,\beta)\leq -1$. Let Φ be the orbit of Δ under the action of W, then the pair (Δ,Φ) is called a *root system*. Vectors in Δ are called *simple roots*, and vectors in Φ are called *roots*. The *rank* of a root system is the cardinality of Δ. The roots Φ are partitioned into *positive roots* $\Phi^+=\operatorname{Cone}(\Delta)\cap\Phi$ and *negative roots* $\Phi^-=-\Phi^+$.

Conversely, a Coxeter system may be represented with different root systems. Let (W,S) be a Coxeter system. We associate a matrix B to (W,S) such that

$$B_{st}=\begin{cases}-\cos(\pi/m_{st}) & \text{if} \quad m_{st}<\infty,\\ -c_{st} & \text{if} \quad m_{st}=\infty,\end{cases}$$

for $s,t\in S$, where c_{st} are chosen arbitrarily with $c_{st}=c_{ts}\geq 1$. We now describe a representation of (W,S) depending on the associated matrix B.

Let V be a real vector space of dimension n, equipped with a basis $\{\mathbf{e}_s\}_{s\in S}$ indexed by the elements in S. The matrix B defines a bilinear form $\mathcal{B}$ on V by $\mathcal{B}(\mathbf{e}_s,\mathbf{e}_t)=\mathbf{e}_s^\intercal B\mathbf{e}_t$ for $s,t\in S$. Then $\{\mathbf{e}_s\}_{s\in S}$ is a simple system in $(V,\mathcal{B})$. The homomorphism that maps $s\in S$ to the reflection in $\mathbf{e}_s$ is a faithful *geometric representation* of the Coxeter group W as a discrete reflection subgroup of the pseudo-orthogonal group $O_\mathcal{B}(V)$. In the literature, this linear independent simple system $\{\mathbf{e}_s\}_{s\in S}$ is said to be "free" [HRT97], "classical" [Kra09] or "canonical" in [HLR14]. In the present paper, we use $\{\mathbf{e}_s\}_{s\in S}$ to represent (W,S) only when the associated matrix B is non-degenerate or positive semidefinite.

If B is degenerate, the dimension of the *radical* is $n-d$, where d is the rank of matrix B. If B is not positive semidefinite, let U be the quotient space $V/V^\perp$, then the bilinear form $\mathcal{B}$ restricted to U is non-degenerate. Let α_s be the projection of $\mathbf{e}_s$ onto V for all $s\in S$. Then the vectors $\Delta=\{\alpha_s\mid s\in S\}$ are positively independent [Kra09, Proposition 6.1.2] and form a simple system in $(U,\mathcal{B})$ such that $\mathcal{B}(\alpha_s,\alpha_t)=B_{st}$. The homomorphism that maps $s\in S$ to the reflection in α_s is a faithful *geometric representation* of the Coxeter group W as a discrete reflection subgroup of the pseudo-orthogonal group $O_\mathcal{B}(U)$. In the present paper, we use Δ to represent (W,S) if the associated matrix B is degenerate and not positive semidefinite.

Since the geometric representation we use depends on the associated matrix B, we say that the Coxeter system (W,S) associated with B is a *geometric Coxeter system*, and denote it by $(W,S)_B$. For a fixed geometric Coxeter system, the representation described above is uniquely determined, and is referred to as the *canonical geometric representation*. Correspondingly, the associated simple system and root system are also said to be *canonical*. The advantage of the canonical geometric representation is that the representation space is non-degenerate in most cases. It uses a positively independent simple system, which has been the framework of several recent studies of infinite Coxeter systems, including [HRT97, Kra09, DHR13, HLR14] etc. and traces back to Vinberg [Vin71]. In the following, we write $w(\mathbf{x})$ for the action of $w\in W$ in the canonical representation.

If W is a finite group, then B is positive definite, and we say that $(W,S)_B$ is of *spherical type* since W can be represented as a reflection group in spherical space. If B is positive semidefinite, we say that $(W,S)_B$ is of *Euclidean type* since W can be represented as a reflection group in Euclidean space. We say that $(W,S)_B$ is of *Lorentzian type* if B has exactly one negative eigenvalue. In this case, W is canonically represented as a reflection group in Lorentz space.

2.3. **Weights and level.** We pass to the projective representation space $\mathbb{P}V$, i.e. the space of 1-subspaces of V. For a non-zero vector $\mathbf{x} \in V$, we denote by $\widehat{\mathbf{x}} \in \mathbb{P}V$ the 1-subspace spanned by $\mathbf{x}$. The geometric representation then induces a *projective representation*

$$w \cdot \widehat{\mathbf{x}} = \widehat{w(\mathbf{x})}, \quad w \in W, \quad \mathbf{x} \in V.$$

For a set $X \subset V$, we have the corresponding projective set

$$\widehat{X} := \{\widehat{\mathbf{x}} \in \mathbb{P}V \mid \mathbf{x} \in X\}$$

In this sense, we have the projective simple roots $\widehat{\Delta}$, projective roots $\widehat{\Phi}$ and the projective isotropic cone $\widehat{Q}$. We use $\operatorname{Conv}(\widehat{X})$ and $\operatorname{Aff}(\widehat{X})$ to denote $\widehat{\operatorname{Cone}(X)}$ and $\widehat{\operatorname{Span}(X)}$ respectively.

The projective space $\mathbb{P}V$ can be identified with an affine subspace plus a *hyperplane at infinity*. We usually fix a vector $\mathbf{t}$ and take the affine subspace

$$H^1_{\mathbf{t}} = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{t},\mathbf{x}) = 1\},$$

Then for a vector $\mathbf{x} \in V$, we represent $\widehat{\mathbf{x}} \in \mathbb{P}V$ by the vector $\mathbf{x}/\mathcal{B}(\mathbf{t},\mathbf{x}) \in H^1_{\mathbf{t}}$ if $\mathcal{B}(\mathbf{t},\mathbf{x}) \neq 0$, or some point at infinity if $\mathcal{B}(\mathbf{t},\mathbf{x}) = 0$. If the representation space is a Lorentz space, the affine picture of the projective light cone is projectively equivalent to a sphere. In this case, it is often convenient to choose $\mathbf{t}$ as a time-like vector, so that $\widehat{Q}$ is closed in the affine picture. Then the subspace $H_{\mathbf{t}} = \mathbf{t}^\perp$ is space-like and divides the space into two parts. Vectors on the same side as $\mathbf{t}$ are said to be *past-directed*, and those on the other side are said to be *future directed*. It then makes sense to call $\mathbf{t}$ the *direction of past*.

Let $(W,S)_B$ be a geometric Coxeter system and Δ be its canonical simple system. The projective simple roots $\widehat{\Delta}$ are in convex position. Indeed, for a simple root $\alpha \in \Delta$, the orthogonal hyperplane H_α separates α from other simple roots. If the direction of past $\mathbf{t}$ is a negative combination of the simple roots, $\mathcal{P} = \operatorname{Conv}(\widehat{\Delta}) \subset \mathbb{P}V$ appears as a convex polytope in the affine picture. We call $\mathcal{P}$ the *positive polytope*, since $\operatorname{Cone}(\Delta)$ is called the *positive cone* [Hum92].

Definition 2.1 (Facial subset [DHR13, § 4])**.** Let $(W,S)_B$ be a geometric Coxeter system and $\Delta \subset (V,\mathcal{B})$ be its canonical simple system. For a subset $I \subset S$, let $\Delta_I \subset \Delta$ be the corresponding subset of simple root. We say that I is a *k-facial subset* of S if $\operatorname{Conv}(\widehat{\Delta}_I)$ is a face of codimension k of the positive polytope $\mathcal{P}$. In this case, let W_I be the group generated by I, we say that $(W_I,I)_B$ is a *k-facial subsystem.* 1-facial subsets of S are simply said to be *facial.*

Note that the bilinear form $\mathcal{B}$ may be degenerate on the subspace $\operatorname{Span}(\Delta_I)$, in which case Δ_I is not the canonical simple system for $(W_I,I)_B$. However, the notion of "facial" is always defined with respect to the canonical representation.

Definition 2.2 (Fundamental weights)**.** Let $(W,S)_B$ be a geometric Coxeter system and $\Delta \subset (V,\mathcal{B})$ be its canonical simple system. For a facial subset I of S, there is a unique vector ω_I such that

$$\mathcal{B}(\alpha,\omega_I) = \begin{cases} = 0, & \alpha \in \Delta_I \\ > 0, & \alpha \notin \Delta_I \end{cases}$$

and

$$\min_{\alpha \notin \Delta_I} \mathcal{B}(\alpha,\omega_I) = 1.$$

We say that ω_I is a *fundamental weight* of the geometric Coxeter system.

This definition extends the notion of fundamental weights to all simple systems, not necessarily a basis of V. The 1 in the second condition is there for the convenience of the proofs.

Let $\Delta^* = \{\omega_I \mid I \text{ is facial}\}$ be the set of fundamental weights. Vectors in the orbit $\Omega = W(\Delta^*)$ are called *weights*. The cone $\operatorname{Cone}(\Delta^*)$ spanned by the fundamental weights is called the *fundamental cone*. It is dual to the positive cone $\operatorname{Cone}(\Delta)$. In other words,

$$\operatorname{Cone}(\Delta^*) = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x},\alpha) > 0 \text{ for all } \alpha \in \Delta\}$$

Therefore, $\mathrm{Cone}(\Delta^*)$ is a polyhedral cone supported by the reflecting hyperplanes of the simple roots. The Coxeter group W is isomorphic to the group generated by reflections in the facets of $\mathrm{Cone}(\Delta^*)$, and the stabilizer of a face of $\mathrm{Cone}(\Delta^*)$ is generated by reflections in the facets that contains this face. The cone $T = \mathrm{Cone}(\Omega)$ over all the weights is called the Tits cone. For Coxeter systems of spherical (resp. Euclidean) type, the Tits cone is the entire representation space V (resp. a closed half-space) [AB08, § 2.6.3].

If the direction of past $\mathbf{t}$ is a positive combination of the fundamental weights, the polytope $\mathcal{C} = \mathrm{Conv}(\widehat{\Delta}^*) \subset \mathbb{P}V$ is a convex polytope in the affine picture, and we call it the *Coxeter polytope*. The Coxeter polytope $\mathcal{C}$ is the dual polytope of the positive polytope $\mathcal{P}$. We then have another definition for k-facial subsets: A subset of simple roots $I \subseteq S$ is k-facial if $\cap_{\alpha\in\Delta_I} \widehat{H}_\alpha$ is a $(k-1)$-face of the Coxeter polytope $\mathcal{C}$.

We now define the central notion of this paper.

Definition 2.3 (Level)**.** A geometric Coxeter system $(W, S)_B$ is of *level* 0 if it is of spherical or Euclidean type. Otherwise, the *level* of $(W, S)_B$ is the biggest integer ℓ such that $(W, S)_B$ has an $(\ell - 1)$-facial subsystem that is *not* of level 0.

This definition agrees with Maxwell's definition if B is non-degenerate or positive semi-definite. Otherwise, the level in Maxwell's sense is in general different from the level under Definition 2.3. The purpose of this paper is to demonstrate that Definition 2.3 is more suitable for geometric studies.

We can reformulate Definition 2.3 in a way similar to Maxwell's original formulation: $(W, S)_B$ is of *level* $\leq \ell$ if the ℓ-facial subsystems of $(W, S)_B$ are all of level 0. It is of *level* ℓ if it is of level $\leq \ell$ but not of level $\leq \ell - 1$. A Coxeter system of level ℓ is *strict* if its ℓ-facial subsystems are all of spherical type. In this case, we also say that the Coxeter system is *strictly* of level ℓ. In the canonical geometric representation of a level-ℓ Coxeter system, the bilinear form $\mathcal{B}$ is positive semidefinite on $\mathrm{Span}(\Delta_I)$ for every I that is ℓ-facial, and indefinite for some I that is $(\ell - 1)$-facial.

If the Coxeter system $(W, S)_B$ is Lorentzian, we can reformulate Definition 2.3 in terms of its positive polytope $\mathcal{P}$ and Coxeter polytope $\mathcal{C}$: The level of $(W, S)_B$ is 1+the maximum codimension of the time-like faces of $\mathcal{P}$, and 2+the maximum dimension of the space-like faces of $\mathcal{C}$. Here, we use the conventions that a face of codimension 0 is the polytope itself, and a face of dimension -1 is empty and is considered to be space-like.

In the following, unless otherwise stated, the term level takes the meaning as in Definnition 2.3.

3. Extending Maxwell's results

In this part, we extend one by one the major results of [Max82] to positively independent simple systems. For the statement of the results, we mimic intentionally the formulations in [Max82]. However, a same word may carry different meanings. It is easy to find Coxeter systems that are level 2 in our sense but not in Maxwell's sense; see Figure 2 for instance. Detailed proofs are given only if there is a significant difference from Maxwell's proof.

First of all, the following two results are proved in [Max82] for linearly independent simple roots, and are extended to positively independent simple roots in [HRT97]. See also [Dye13, §9.4], where the dual of the Tits cone is called the *imaginary cone*.

Proposition 3.1 ([HRT97, Proposition 3.4], extending [Max82, Proposition 1.2])**.** *For every vector* $\mathbf{x}$ *in the dual of the Tits cone,* $\mathcal{B}(\mathbf{x}, \mathbf{x}) \leq 0$.

Proposition 3.2 ([HRT97, Proposition 3.7], extending [Max82, Corollary 1.3])**.** *The Tits cone of a Lorentzian Coxeter system contains one component of the light cone* $Q \setminus \{0\}$.

3.1. **Lorentzian Coxeter systems of level 2.** Two vectors $\mathbf{x}, \mathbf{y}$ in $(V, \mathcal{B})$ are said to be *disjoint* if $\mathcal{B}(\mathbf{x}, \mathbf{y}) \leq 0$ and $\mathcal{B}$ is *not* positive definite on the subspace $\mathrm{Span}(\{\mathbf{x}, \mathbf{y}\})$.

The proofs of the following results are apparently very different from Maxwell's argument [Max82]. Indeed, while Maxwell's proofs make heavy use of basis, our proofs rely primarily on projective geometry. However, the basic idea is the same. In the case where the simple roots form a basis of the representation space, our proofs are just geometric interpretations of Maxwell's proofs.

Proposition 3.3 (extending [Max82, Proposition 1.4])**.** *Coxeter systems of level* 1 *are Lorentzian. All fundamental weights are pairwise disjoint and none are space-like.*

Proof. Let $(W,S)_B$ be a level-1 Coxeter system and (Δ,Φ) be its canonical root system in $(V,\mathcal{B})$.

For a vector $\mathbf{x}$ such that $\mathcal{B}(\mathbf{x},\mathbf{x})\leq 0$, we claim that $\widehat{\mathbf{x}}\in\mathcal{P}$. If it is not the case, we can find in $\mathbb{P}V$ a line passing through $\widehat{\mathbf{x}}$ and intersect the boundary of $\mathcal{P}$ at two points, say $\widehat{\mathbf{x}}_+$ and $\widehat{\mathbf{x}}_-$, such that $\widehat{\mathbf{x}}_+\in\mathrm{Conv}(\widehat{\Delta}_I)$ and $\widehat{\mathbf{x}}_-\in\mathrm{Conv}(\widehat{\Delta}_J)$ for two disjoint facial subset $I,J\subset S$. View in the representation space V, this means that there are two vectors $\mathbf{x}_+\in\mathrm{Cone}(\Delta_I)$ and $\mathbf{x}_-\in\mathrm{Cone}(\Delta_J)$ such that $\mathbf{x}=\mathbf{x}_+-\mathbf{x}_-$. Since $\mathcal{B}(\mathbf{x},\mathbf{x})=\mathcal{B}(\mathbf{x}_+,\mathbf{x}_+)+\mathcal{B}(\mathbf{x}_-,\mathbf{x}_-)-2\mathcal{B}(\mathbf{x}_+,\mathbf{x}_-)\leq 0$ and $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_-)<0$ (because I and J are disjoint), we have either $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_+)<0$ or $\mathcal{B}(\mathbf{x}_-,\mathbf{x}_-)<0$, both contradict the fact that $(W,S)_B$ is of level 1. Our claim is then proved. If $\mathcal{B}(\mathbf{x},\mathbf{x})<0$, since $\mathcal{B}$ is positive semidefinite on the facets, $\widehat{\mathbf{x}}$ must be in the interior of $\mathcal{P}$. If $\mathcal{B}(\mathbf{x},\mathbf{x})=0$, it is possible that $\widehat{\mathbf{x}}$ is in the interior of a facet of $\mathcal{P}$.

Now assume that the representation space V is not a Lorentz space. Then it contains a pair of orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathcal{B}(\mathbf{u},\mathbf{u})<0$ and $\mathcal{B}(\mathbf{v},\mathbf{v})=0$. For any linear combination $\mathbf{x}=\lambda\mathbf{u}+\mu\mathbf{v}$ we have $\mathcal{B}(\mathbf{x},\mathbf{x})<0$ as long as $\lambda\neq 0$. In the projective space, $\mathrm{Aff}(\{\widehat{\mathbf{u}},\widehat{\mathbf{v}}\})$ is a line passing through $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$. This line intersects with $\mathrm{Aff}(\widehat{\Delta}_I)$ for every facial subset $I\subset S$. For every intersection point $\mathbf{x}$, we have $\mathcal{B}(\mathbf{x},\mathbf{x})<0$, with at most one exception (namely $\widehat{\mathbf{v}}$). This contradicts the fact that $(W,S)_B$ is of level 1. We then proved that $(W,S)_B$ is Lorentzian.

Let I be any facial subset of S. Since (W,S) is of level 1, the subspace $\mathrm{Span}(\Delta_I)$ is not time-like, so its orthogonal companion $\omega_I\mathbb{R}$ is not space-like. This proves that no fundamental weight is space-like.

It is clear that any two fundamental weights span a Lorentz space. For the disjointness, we only needs to prove that $\mathcal{B}(\omega_I,\omega_J)\leq 0$ for any two facial subset $I\neq J\subset S$. Since ω_I is not space-like, we have seen that $\widehat{\omega}_I\in\mathcal{P}$, so $\mathcal{B}(\omega_I,\omega_J)$ has the same sign (possibly 0) for all facial $J\subset S$, which is ≤ 0 if ω_I is time-like. If ω_I is light-like, notice that ω_I can be written as a linear combination of the simple roots in Δ_I with coefficients of the same sign (possibly 0). For any $s\notin I$, we have $\mathcal{B}(\omega_I,\alpha_s)>0$ and $\mathcal{B}(\alpha_s,\alpha_t)\leq 0$ for any $t\in\Delta$, so ω_I must be a negative combination of the simple roots. We then conclude that $\mathcal{B}(\omega_I,\omega_J)\leq 0$. □

As a consequence, the Tits cone of a level-1 Coxeter systems equals the set of non-space-like vectors.

Proposition 3.4 (extending [Max82, Proposition 1.6])**.** *Coxeter systems of level* 2 *are Lorentzian. The fundamental weights are pairwise disjoint. A fundamental weight ω_I is space-like if and only if the facial subsystem $(W_I,I)_B$ is of level* 1*, in which case we have $\mathcal{B}(\omega_I,\omega_I)\leq 1$.*

Proof. Let $(W,S)_B$ be a level-2 Coxeter system and (Δ,Φ) be its canonical root system in $(V,\mathcal{B})$. If V is of dimension 3, it is immediate that $(W,S)_B$ is Lorentzian. So we assume that the dimension of V is ≥ 4, and the positive polytope $\mathcal{P}=\mathrm{Conv}(\widehat{\Delta})$ is of dimension ≥ 3.

For a vector $\mathbf{x}$ such that $\mathcal{B}(\mathbf{x},\mathbf{x})\leq 0$, assume that $\widehat{\mathbf{x}}\notin\mathcal{P}$. As in the previous proof, we can write $\mathbf{x}=\mathbf{x}_+-\mathbf{x}_-$ where $\mathbf{x}_+\in\mathrm{Cone}(\Delta_I)$ and $\mathbf{x}_-\in\mathrm{Cone}(\Delta_J)$ and I and J are two disjoint facial subset of S. Again, since $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_-)<0$, we have either $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_+)$ or $\mathcal{B}(\mathbf{x}_-,\mathbf{x}_-)<0$. Since $(W,S)_B$ is of level 2, we must have either $\mathbf{x}_+$ or $\mathbf{x}_-$ in the interior of a facet of $\mathcal{P}$. Therefore, for *any* line in $\mathbb{P}V$ that passes through $\widehat{\mathbf{x}}$ and intersects two disjoint faces of $\mathcal{P}$, one intersection point must be in the interior of a facet. We then conclude that there is only one facial subset $K\subset S$ such that $\mathrm{Aff}(\widehat{\Delta}_K)$ separates $\widehat{\mathbf{x}}$ from the interior of $\mathcal{P}$. Otherwise, as $\mathcal{P}$ is of dimension ≥ 3, it would be possible that $\mathbf{x}_+$ and $\mathbf{x}_-$ are both on faces of codimension ≥ 2, so neither $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_+)$ nor $\mathcal{B}(\mathbf{x}_-,\mathbf{x}_-)$ is negative.

Take the case where $\widehat{\mathbf{x}}\in\mathcal{P}$ into consideration, we conclude that the sign of $\mathcal{B}(\mathbf{x},\omega_I)$ are the same for all the fundamental weights ω_I except for at most one exception. If $\mathcal{B}(\mathbf{x},\mathbf{x})=0$, it is possible that $\widehat{\mathbf{x}}$ lies on a codimension-2 face of $\mathcal{P}$.

If the representation space V is not a Lorentz space, it contains a pair of orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathcal{B}(\mathbf{u},\mathbf{u})<0$ and $\mathcal{B}(\mathbf{v},\mathbf{v})=0$. Then, for any linear combination $\mathbf{x}=\lambda\mathbf{u}+\mu\mathbf{v}$, we have $\mathcal{B}(\mathbf{x},\mathbf{x})<0$ as long as $\lambda\neq 0$. The subspace $\mathrm{Span}(\mathbf{u},\mathbf{v})$ appears in the projective space as a line L passing through $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{v}}$. The line L intersects with $\mathrm{Aff}(\widehat{\Delta}_I)$ for every facial subset $I\subset S$.

By the conclusion of the previous paragraph, the intersection points must be on the boundary of $\mathcal{P}$, and are in the interior of the facets of $\mathcal{P}$ with at most one exception. Then the only possibility is that $\mathcal{P}$ being a pyramid, in which case L pass through the apex $\widehat{\mathbf{p}}$ and a point in the base. But the apex $\widehat{\mathbf{p}}$ is a projective simple root, so $\mathcal{B}(\mathbf{p},\mathbf{p})>0$ and $\widehat{\mathbf{p}}\notin L$. This contradiction proves that V is a Lorentz space.

We have seen that a time-like vector is separated from $\mathcal{P}$ by at most one defining hyperplane. Consequently, the intersection $\operatorname{Span}(\Delta_I)\cap\operatorname{Span}(\Delta_J)$ is not time-like for any two facial subsets $I\neq J\subset S$. The orthogonal companion of the intersection is the subspace $\operatorname{Span}(\omega_I,\omega_J)$, which is not space-like. For proving the disjointness, one still needs to prove that $\mathcal{B}(\omega_I,\omega_J)\leq 0$.

Assume that ω_I is not space-like. A similar argument as in the proof of Proposition 3.3 shows that $\mathcal{B}(\omega_I,\omega_J)\leq 0$ for all $J\neq I$ with at most one exception. Let $K\neq I$ be this exception, we have seen that $\operatorname{Aff}(\widehat{\Delta}_K)$ is the only defining hyperplane of $\mathcal{P}$ that separates $\widehat{\omega}_I$ from the interior of $\mathcal{P}$. Pick a generator $s\in I\setminus K$, we can write $\omega_I=\lambda\alpha_s-\omega'_I$, where ω'_I is a linear combination of Δ_K with coefficients of same sign, which is also the sign of λ. We have $\mathcal{B}(\alpha_s,\omega_I)=0$ by definition, but this is not the case since $\mathcal{B}(\alpha_s,\alpha_t)\leq 0$ for $t\in K$ while $\mathcal{B}(\alpha_s,\alpha_s)=1$. Therefore, the exception K does not exist.

If ω_I is space-like, then the subspace $\operatorname{Span}(\Delta_I)$ is time-like, so $(\operatorname{Span}(\Delta_I),\mathcal{B})$ is a (non-degenerate) Lorentz space. This proves that $(W_I,I)_B$ is of level 1. Then, for a simple root $\alpha\notin\Delta_I$, let

$$\alpha'=\alpha-\frac{\mathcal{B}(\alpha,\omega_I)}{\mathcal{B}(\omega_I,\omega_I)}\omega_I$$

be the projection of α on $\operatorname{Span}(\Delta_I)$. Since $\mathcal{B}(\alpha',\beta)\leq 0$ for all $\beta\in\Delta_I$, $\widehat{\alpha}'$ is in the Coxeter polytope of Δ_I. Since $(W_I,I)_B$ is of level 1, α' is time-like by Proposition 3.3, i.e.

$$\mathcal{B}(\alpha',\alpha')=\mathcal{B}(\alpha,\alpha)-\frac{\mathcal{B}(\alpha,\omega_I)^2}{\mathcal{B}(\omega_I,\omega_I)}=1-\frac{\mathcal{B}(\alpha,\omega_I)^2}{\mathcal{B}(\omega_I,\omega_I)}\leq 0,$$

which proves that

$$\mathcal{B}(\omega_I,\omega_I)\leq\min_{\alpha\notin\Delta_I}\mathcal{B}(\alpha,\omega_I)^2=1. \tag{1}$$

Let J be a facial subset such that $\alpha\in\Delta_J$, and

$$\omega'_J=\omega_J-\frac{\mathcal{B}(\omega_J,\omega_I)}{\mathcal{B}(\omega_I,\omega_I)}\omega_I$$

be the projection of ω_J on $\operatorname{Span}(\Delta_I)$. Then, since $\alpha'\in\mathcal{P}$,

$$\mathcal{B}(\alpha',\omega'_J)=\mathcal{B}(\alpha,\omega_J)-\frac{\mathcal{B}(\alpha,\omega_I)\mathcal{B}(\omega_J,\omega_I)}{\mathcal{B}(\omega_I,\omega_I)}=-\frac{\mathcal{B}(\alpha,\omega_I)\mathcal{B}(\omega_J,\omega_I)}{\mathcal{B}(\omega_I,\omega_I)}=\mathcal{B}(\alpha',\omega_J)\geq 0,$$

which proves that $\mathcal{B}(\omega_I,\omega_J)\leq 0$. Since J can be chosen as any facial subset $J\neq I\subset S$, this finish the proof of disjointness. □

The following is an interesting corollary.

Corollary 3.5. *Let Δ be a simple system of level 2 and Δ_r^* be the set of space-like fundamental weights, then the set*

$$\Delta\cup\{-\omega/\sqrt{\mathcal{B}(\omega,\omega)}\mid\omega\in\Delta_r^*\}$$

is a simple system of level 1.

The following proposition is proved for linearly independent simple systems in, for instance, [Bou68, Ch. V, § 4.4, Theorem 1; AB08, Lemma 2.58]. An extension for positively independent simple systems can be found in [Kra09, Theorem 1.2.2(b)], who refers to [Vin71] for proof. Note that $\operatorname{Cone}(\Delta^*)$ is closed in the present paper, so the inequalities are not strict.

Proposition 3.6 (extending [Max82, Corollary 1.8])**.** *For $\mathbf{x}\in\operatorname{Cone}(\Delta^*)$, $w\in W$ and $\alpha_s\in\Delta$, either $\mathcal{B}(w(\mathbf{x}),\alpha_s)\geq 0$ and $\operatorname{len}(sw)>\operatorname{len}(w)$, or $\mathcal{B}(w(\mathbf{x}),\alpha_s)\leq 0$ and $\operatorname{len}(sw)<\operatorname{len}(w)$.*

Theorem 3.7 (extending [Max82, Theorem 1.9])**.** *The followings are equivalent:*

(a) $(W,S)_B$ is of level 1 or 2;

(b) $(W,S)_B$ *is Lorentzian and any two weights are disjoint.*

Proof. Maxwell's proof applies with slight modification.

(a)⇒(b): We only need to prove the disjointness. We first prove, for any fundamental weights ω_I and ω_J, that

$$\mathcal{B}(\omega_I, w(\omega_J)) \le 0. \tag{2}$$

by induction on the length of $w \in W$. The case of $w = e$ is already known. One may assume that $\mathrm{len}(tw) > \mathrm{len}(w)$ for all $t \in I$, otherwise one may replace w by tw in (2). So $w = sw'$ for some $s \notin I$ and $\mathrm{len}(w) > \mathrm{len}(w')$. We then have

$$\mathcal{B}(\omega_I, w(\omega_J)) = (s(\omega_I), w'(\omega_J)) = \mathcal{B}(\omega_I, w'(\omega_J)) - 2\mathcal{B}(\alpha_s, \omega_I)\mathcal{B}(\alpha_s, w'(\omega_J)).$$

If $\omega_I \neq w'(\omega_J)$, (2) is proved since $\mathcal{B}(\omega_I, w'(\omega_J)) \le 0$ by inductive hypothesis, $\mathcal{B}(\alpha_s, \omega_I) > 0$ by definition of fundamental weights, and $\mathcal{B}(\alpha_s, w'(\omega_J)) \ge 0$ by Proposition 3.6. Otherwise, if $\omega_I = w'(\omega_J)$, we have

$$\mathcal{B}(\omega_I, w(\omega_J)) = \mathcal{B}(\omega_I, \omega_I) - 2\mathcal{B}(\alpha_s, \omega_I)^2 \le 0$$

by (1).

It remains to prove that $\mathcal{B}$ is not positive definite on the subspace $\mathrm{Span}(\omega_I, w(\omega_J))$. If this is not the case, then $(W_I, I)_B$ is of level 1. Let $\mathbf{v}$ be the projection of $w(\omega_J)$ on $\omega_I^\perp = \mathrm{Span}(\Delta_I)$. The subspace $\mathbf{v}^\perp$ orthogonal to $\mathbf{v}$ in $\omega_I^\perp$ is in the time-like intersection of $\omega_I^\perp$ and $w(\omega_J)^\perp$, so $\mathbf{v}$ must be space-like. On the other hand, for all $t \in I$, $\mathcal{B}(\alpha_t, w(\omega_J)) \ge 0$ because $\mathrm{len}(tw) \ge \mathrm{len}(w)$. We then conclude that $\mathbf{v}$ is in the Coxeter polytope of $(W_I, I)_B$, so $\mathbf{v}$ must be time-like. This contradiction finishes the proof of disjointness.

(b)⇒(a): Since $\mathcal{B}$ is not positive definite on the subspace spanned by any two fundamental weights, the orthogonal companion of these intersections, including the codimension-2 faces of $\mathcal{P}$, are not time-like. So $\mathcal{B}$ is positive semidefinite on all codimension-2 faces, which proves that $(W,S)_B$ is of level 1 or 2 (not level-0 because $(W,S)_B$ is Lorentzian). □

3.2. **Infinite ball packings.** For a *space-like* vector $\mathbf{x}$ in the Lorentz space $(V, \mathcal{B})$, the *normalized vector* $\overline{\mathbf{x}}$ of $\mathbf{x}$ is given by

$$\overline{\mathbf{x}} = \mathbf{x}/\sqrt{\mathcal{B}(\mathbf{x}, \mathbf{x})}.$$

It lies on the one-sheet hyperboloid $\mathcal{H} = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{x}) = 1\}$. Note that $\widehat{\mathbf{x}} = \widehat{-\mathbf{x}}$ is the same point in $\mathbb{P}V$, but $\overline{\mathbf{x}}$ and $\overline{-\mathbf{x}}$ are two different vectors in opposite directions in V. One verifies that two space-like vectors $\mathbf{x}, \mathbf{y}$ are disjoint if and only if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \le -1$.

A correspondence between space-like directions in $(d+2)$-dimensional Lorentz space $(V, \mathcal{B})$ and d-dimensional balls is introduced in [Max82, §2], see also [HJ03, § 1.1; Cec08, § 2.2]. Fix a time-like direction of past $\mathbf{t}$ so that the projective light cone $\widehat{Q}$ appears as a closed sphere on $H^1_{\mathbf{t}}$. Then in the affine picture, given a space-like vector $\mathbf{x} \in V$, the intersection of $\widehat{Q}$ with the half-space $\widehat{H}^-_{\mathbf{x}} = \{\mathbf{x}' \in H^1_{\mathbf{t}} \mid \mathcal{B}(\mathbf{x}, \mathbf{x}') \le 0\}$ is a closed ball (spherical cap) on $\widehat{Q}$. We denote this ball by $\mathsf{Ball}(\mathbf{x})$. After a stereographic projection, $\mathsf{Ball}(\mathbf{x})$ becomes an d-dimensional ball in Euclidean space. Here, we also regard closed half-spaces as closed balls of curvature 0, and complement of open balls as closed balls of negative curvature. For two past-directed space-like vectors $\mathbf{x}$ and $\mathbf{y}$, we have

- $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ are disjoint if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) < -1$;
- $\mathsf{Ball}(\mathbf{x})$ is tangent to $\mathsf{Ball}(\mathbf{y})$ if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = -1$;
- $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ *overlap* (i.e. their interiors intersect) if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) > -1$;
- $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ *heavily overlap* (i.e. their boundary intersect transversally at an obtuse angle, or one is contained in the other) if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) > 0$.

If one of the vectors is future-directed and $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \le -1$, then either the interiors or the exteriors of $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ are disjoint.

A *ball packing* is a collection balls with disjoint interiors. It is then clear that a ball packing correspond to a set of space-like vectors $X \in V$ such that any two vectors are disjoint and at most one vector is future-directed. Conversely, Maxwell proved that every such set of space-like vectors correspond to a ball packing [Max82, Proposition 3.1]. So the following theorem follows directly from Theorem 3.7.

Theorem 3.8 (extending [Max82, Theorem 3.2])**.** *Let Ω_r be the set of space-like weights, then $\{\mathsf{Ball}(\omega) \mid \omega \in \Omega_r\}$ is a ball packing if and only if the associated Coxeter system is of level 2.*

We say that the ball packing in Theorem 3.8 is generated by the corresponding Coxeter system. A ball packing is maximal if one can not add any additional ball into the packing without overlapping other balls.

Theorem 3.9. *The ball packing generated by a level-2 Lorentzian root system is maximal.*

Proof. It follows from an extension of [Max82, Theorem 3.3], whose proof applies directly for positively independent simple systems. To verify the condition of the extended [Max82, Theorem 3.3], one still needs to extend [Max89, Theorem 6.1]: Let Ω_r denotes the set of space-like weights, then we have $\mathrm{Cone}(\Omega_r) = \mathrm{Cone}(\Omega)$ if $(W, S)_B$ is irreducible, Lorentzian and of level ≥ 2.

For this, one may pick a basis $\tilde{\Delta}^*$ of V from the fundamental weights Δ^*, in which some space-like weights $\tilde{\Delta}_r^*$ are included. They are the fundamental weights of a linear independent simple subsystem $\tilde{\Delta} \in \Delta$. Let $\tilde{W}$ be the Coxeter group associated to $\tilde{\Delta}$, $\tilde{\Omega}_r = \tilde{W}(\tilde{\Delta}_r^*)$ and $\tilde{\Omega} = \tilde{W}(\tilde{\Delta}^*)$, then

$$\mathrm{Cone}(\Omega_r) \supset \mathrm{Cone}(\tilde{\Omega}_r) = \mathrm{Cone}(\tilde{\Omega}) \supset Q$$

So every non-space-like weight is included in $\mathrm{Cone}(\Omega_r)$, which finishes the extension of [Max89, Theorem 6.1]. □

We have extended most of the major results of [Max82] to positively independent simple systems. We now continue to extend results from [CL14].

The *limit roots* are the accumulation points of $\widehat{\Phi} \subset \mathbb{P}V$. In [HLR14], it was proved that the limit roots lies on $\widehat{Q}$. A key notion in the studies of limit roots is the height $h(\mathbf{x})$ of a vector $\mathbf{x} \in V$, which is defined as the sum of the coordinates when the simple roots Δ form a basis of V. In the case where Δ is positively independent, let $\mathbf{t}$ be a time-like negative combination of Δ and take it as the direction of past. Then the affine hyperplane $H_{\mathbf{t}}^1$ is transverse to Φ^+, and we can define $h(\mathbf{x}) = \mathcal{B}(\mathbf{t}, \mathbf{x})$, see [HLR14, § 5.2]. Furthermore, we can always find a basis for V such that all the positive roots have only positive coordinates, so $h(\mathbf{x})$ is a L_1-norm on Φ^+.

The *residual set* of a collection of balls is the complement of the interiors of the balls. With the necessary adaptions mentioned above, the proofs in [CL14] applies to the following extensions.

Theorem 3.10 (extending [CL14, Theorem 3.6])**.** *The set of limit roots of a level-2 Lorentzian Coxeter system is equal to the residual set of the ball packing generated by the Coxeter system.*

Theorem 3.11 (extending [CL14, Theorem 1.1; §3.4])**.** *For a Lorentzian Coxeter system of level ≥ 3, $\{\mathsf{Ball}(\omega) \mid \omega \in \Omega_r\}$ is a maximal collection of balls with no heavily overlapping balls. In this case, the set of limit roots is again the residual set of the ball cluster.*

The polytopes in the orbit $W \cdot \mathcal{C}$ of the Coxeter polytope $\mathcal{C}$ are called *chambers*. Analogous to the situation when $\mathcal{C}$ is a simplex [AB08], the chambers form a cell decomposition of the projective Tits cone $\widehat{T}$, whose vertices correspond to projective weights. We call it the *Coxeter complex*, and denote it by $\mathscr{C}$. It is a pure polyhedral cell complex of dimension $d-1$ (dimension of $\mathbb{P}V$). The 1-cells of $\mathcal{C}$ are called *edges*, and $(d-2)$-cells are called *panels*.

The *tangency graph* of a ball packing takes the balls as vertices and the tangent pairs as edges. Since $\mathcal{C}$ is the fundamental domain for the action of W on the projective Tits cone $\widehat{T}$, the orbit of two different fundamental weights are disjoint. So the vertices of the Coxeter complex admits a coloring by Δ^*, i.e. a vertex u is colored by $\omega \in \Delta^*$ if $u \in W \cdot \widehat{\omega}$. Panels are orbits of the facets of Δ^*, therefore they can be colored by the simple roots, i.e. a panel is colored by $\alpha \in \Delta$ if it is the orbit of the facet of Δ^* corresponding to α. Vertices with time- or light-like colors are called *imaginary vertices*; vertices with space-like colors are called *real vertices* because they correspond to balls in the packing, and are therefore vertices in the tangency graph. An edge of the Coxeter complex connecting two real vertices of color ω and ω' is said to be *real* if $\mathcal{B}(\overline{\omega}, \overline{\omega}') = -1$. Real edges correspond to tangent pairs in the packing, and are therefore edges in the tangency graph. For a Lorentzian Coxeter systems of level 2, vertices colored by $\omega \in \Delta^*$ such that $\mathcal{B}(\omega, \omega) = 1$ are said to be *surreal*. Two distinct surreal vertices of the same color ω are said to be *adjacent* if they are vertices of two chambers sharing a panel of color α such that $\mathcal{B}(\omega, \alpha) = 1$. One verifies that pairs of adjacent surreal vertices are also edges in the tangency graph.

With the definitions above (compare [CL14, §3.3]), the following theorem follows by modifying the proof of (2) in the same way as in the proof of [CL14, Theorem 3.7].

Theorem 3.12 (extending [CL14, Theorem 3.7]). *The tangency graph of the ball packing generated by a Lorentzian root system of level 2 takes the real vertices of the Coxeter complex as vertices. Two vertices u and v are connected in the tangency graph if and only if one of the following is fulfilled:*

- *uv is a real edge of the Coxeter complex, in which case u and v are of different colors,*
- *u and v are adjacent surreal vertices, in which case u and v are of the same color.*

Corollary 3.13 (extending [CL14, Corollary 3.8]). *The projective Tits cone of a Lorentzian Coxeter system of level 2 is an edge-tangent infinite polytope, i.e. its edges are all tangent to the projective light cone. Furthermore, the 1-skeleton of the projective Tits cone is the tangency graph of the ball packing generated by the root system.*

4. Partial classification of level-2 Coxeter polytopes

To provide examples of new infinite ball packings, we devote this section to a partial enumeration of Coxeter polytopes of level 2. Recall that a Coxeter polytope $\mathcal{C}$ is of level 2 if every edge (1-face) of $\mathcal{C}$ is time-like or light-like while some vertex (0-face) of $\mathcal{C}$ is space-like. The enumeration is implemented by computer programs. In this paper, we only present the main ideas and sketch the procedures.

4.1. **Preparation.** It is convenient to represent a simple system Δ in $(V, \mathcal{B})$ by the Coxeter graph G. Simple roots are represented by vertices of G. If two simple roots $\alpha, \beta \in \Delta$ are not orthogonal, they are connected by an edge, which is solid with label $3 \leq m < \infty$, if $\mathcal{B}(\alpha, \beta) = -\cos(\pi/m)$; with label ∞ if $\mathcal{B}(\alpha, \beta) = -1$; or dashed with label $-c$ if $\mathcal{B}(\alpha, \beta) = -c < -1$. The label 3 on solid edges are omitted. If we consider the Coxeter polytope $\mathcal{C}$, then vertices of the Coxeter graph G correspond to facets of $\mathcal{C}$. A solid edge of G with integer label means that the intersection of two facets is time-like; a solid edge with label ∞ means that the intersection is light-like; and a dashed edge means that the intersection is space-like. The level of a Coxeter graph is the level of the corresponding Coxeter system.

Let G be a Coxeter graph, G_1 and G_2 be two subgraphs of G. In the following, we use $G_1 + G_2$ to denote the subgraph induced by the vertices of G_1 and G_2, use $G_1 - G_2$ to denote the subgraph induced by the vertices of G_1 that are not in G_2. A subgraph with only one vertex is denoted by the vertex.

For a geometric Coxeter system $(W, S)_B$, the *corank* of it's Coxeter polytope $\mathcal{C}$ is defined as the nullity of the matrix B. The same notion is also used for the corresponding Coxeter graphs. A Coxeter polytope of dimension d and corank k has $d + k + 1$ facets. In particular, a Coxeter polytope of corank 0 is a simplex. In this case, our definition of level agrees with the definition in [Max82]: a Coxeter graph of corank 0 is of level ℓ if deletion of any ℓ vertices leaves a graph of level 0 while deletion of certain $\ell - 1$ vertices leaves a graph of level > 0. For convenience, a Coxeter graph of level ℓ and corank k is said to be a (ℓ, k)-graph, or (ℓ^s, k)-graph if the level is strict. Same abbreviation is used for Coxeter polytopes.

The time-like part of the d-dimensional projective Lorentz space is the Kleinian model of the d-dimensional hyperbolic space. For a Lorentzian Coxeter system, the part of the Coxeter polytope $\mathcal{C}$ in the hyperbolic space is a hyperbolic polytope named *Vinberg polytope*. It is the fundamental domain of the hyperbolic reflection group generated by the reflections in its facets [Vin85]. By Proposition 3.3, Coxeter polytopes of level 1 (resp. strictly of level 1) correspond to finite-volume (resp. compact) Vinberg polytopes, while Coxeter polytopes of level > 2 correspond to infinite-volume Vinberg polytopes.

Vinberg [Vin84] proved that there is no strict level-1 Coxeter polytopes of dimension 30 or higher, and Prokhorov [Pro86] proved that there is no level-1 Coxeter polytopes in hyperbolic spaces of dimension 996 or higher. On the other hand, Allcock [All06] proved that there are infinitely many level-1 (resp. strictly level-1) Coxeter polytopes in every hyperbolic space of dimension 19 (resp. 6) or lower, which suggests that a complete enumeration of level-1 Coxeter polytopes is hopeless. Nevertheless, there are many interesting partial enumerations. The $(1,0)$-polytopes have been completely enumerated by Chein [Che69]. They are hyperbolic simplices of finite volume. The list of Chein also comprises $(1^s,0)$-polytopes, which was first enumerated by Lannér [Lan50]. The $(1,1)$-polytopes have been enumerated by Kaplinskaja [Kap74] for simplicial prisms, Esselmann [Ess96] for compact polytopes and Tumarkin [Tum04] for finite-volume polytopes. Tumarkin also studied $(1^s,2)$- and $(1^s,3)$-polytopes [Tum07, TF08]. Mcleod [Mcl13] finished the classification of all pyramids of level 1.

In this section, we study Coxeter polytopes of level 2. In view of Corollary 3.5, we deduce immediately from the result of [Pro86] that there is no level-2 Coxeter polytopes in hyperbolic spaces of dimension 996 or higher. However, in the shadow of [All06], there might be infinitely many level-2 Vinberg polytopes in lower dimensions, so a complete classification may be hopeless. A $(2,0)$-graph is either a connected graph, or a disjoint union of an isolated vertex and a $(1,0)$-graph. The enumeration of connected $(2,0)$-graphs was initiated in [Max82] and completed in [CL14]. In this section, we would like to enumerate $(2,1)$-graphs.

A k-face of a d-polytope is said to be *simple* if it is the intersection of $d-k$ facets, or *almost simple* if every face containing it is simple. A polytope is said to be k-simple (resp. almost k-simple) if all its k-faces are simple (resp. almost simple). For a Coxeter polytope, the stabilizer of a time-like (resp. light-like) face is of spherical (resp. Euclidean type). Therefore, every time-like (resp. light-like) face is simple (resp. almost simple). We then conclude the following proposition from the definition of level.

Proposition 4.1. *A Coxeter polytope of level ℓ (resp. strictly of level ℓ) is almost ℓ-simple (resp. ℓ-simple).*

From the Gale diagram [Grü03, §6.3; Tum04, §2] and by Proposition 4.1, we know that there are three possibilities for the combinatorial type of a $(2,1)$-polytope:

- a product of two simplices, abbreviated as $\triangle \times \triangle$;
- the pyramid over a product of two simplices, abbreviated as $\mathrm{Pyr}(\triangle \times \triangle)$;
- the 2-fold pyramid over a product of two simplices, abbreviated as $\mathrm{Pyr}^2(\triangle \times \triangle)$.

We now analyse the three types separately.

4.2. $\mathcal{C}$ **has the type of** $\triangle \times \triangle$**.** In this case, vertices of $\mathcal{C}$ are all simple. The Coxeter graph G consists of two parts, say G_1 and G_2, corresponding to the two simplices. A k-face of $\mathcal{C}$ correspond to a subgraph of G obtained by deleting $k+2$ vertices, including at least one vertex from both G_1 and G_2.

For a $(1,0)$-graph, a vertex is said to be *ideal* if its removal leaves an Euclidean Coxeter graph. For a $(2,0)$-graph, a vertex is said to be *real* if its removal leaves a $(1,0)$-graph. If the $(2,0)$-graph is not connected, then the isolated vertex is the only real vertex.

Lemma 4.2. *If a $(2,1)$-polytope has the type of $\triangle \times \triangle$, then*

(i) *Its Coxeter graph G consists of two $(1,0)$-graphs G_1 and G_2 and they are connected to each other.*
(ii) *For any $v_1 \in G_1$ and $v_2 \in G_2$, the graphs $G_1 + v_2$ and $G_2 + v_1$ are $(2,0)$-graphs.*
(iii) *If one of the simplices, say the one represented by G_2, is of dimension > 1, then the level of G_1 and $G_1 + v_2$ are strict, and v_2 is the only real vertex of $G_1 + v_2$.*

Proof. The two simplices are represented by Coxeter graphs G_1 and G_2 respectively. By [Vin85, Theorem 3.1], a subgraph of G correspond to a time-like face of $\mathcal{C}$ if and only if it is spherical. By a slight extension of [Vin85, Theorem 3.2], a subgraph of G correspond to a light-like face of $\mathcal{C}$ if and only if it is of corank 1 and Euclidean. Since G_1 and G_2 are of corank 0 and do not correspond to any face of $\mathcal{C}$, they are not of level 0.

For any vertex v_1 of G_1, the graph $G_1 - v_1$ is obtained from G by removing at least 3 vertices. Therefore, it corresponds to a simple face of $\mathcal{C}$ of dimension ≥ 1. We then conclude that G_1 is a

$(1,0)$-graph. Same argument applies to G_2. G_1 and G_2 must be connected, otherwise the bilinear form B has two negative eigenvalues, and the Coxeter system is not Lorentzian.

For any $v_2 \in G_2$, the graph $G_1 + v_2$ is not of level 1 because G_1 is. It is of corank 0 since its positive polytope is a simplex. It is of level 2 because further deletion of any two vertices leaves a level-0 Coxeter graph corresponding to a simple face of $\mathcal{C}$ of dimension ≥ 1. Same argument applies to $G_2 + v_1$.

If the simplex represented by G_2 is of dimension > 1, then G_2 has more than two vertices. In this case, the dimensions of the faces mentioned above are all strictly > 1, so the levels are all strict. Furthermore, for any vertex $v_1 \in G_1$, the graph $G_1 + v_2 - v_1$ is of level 0 since it corresponds to a face of $\mathcal{C}$ of dimension ≥ 1. So v_2 is the only real vertex of $G_1 + v_2$. □

Remark. The same type of argument applies for many other lemmas in this section, and we will not repeat them in detail.

We now sketch the procedure for enumerating Coxeter polytopes of this type. We need to distinguish two sub-cases.

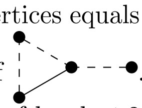
4.2.1. *One of the simplices is of dimension* 1. In this case, $\mathcal{C}$ has the combinatorial type of a simplicial prism. In the Coxeter graph, we may assume that vertices of G_2 correspond to the base facets of $\mathcal{C}$, while vertices of G_1 correspond to lateral facets. By Lemma 4.2(i), G_2 is a $(1,0)$-graph, so the vertices of G_2 are connected by a dashed edge, meaning that the two base facets do not intersect inside the light cone.

A prism is *orthogonally based* if one of the base facets is orthogonal to all the lateral facets. Any prism of level-2 can be cut into two orthogonally based prisms, and two orthogonally based prisms can be spliced into one prism if they share a same base. Therefore, we only need to consider orthogonally based prisms, as also argued by Kaplinskaja [Kap74]. In the Coxeter graph of an orthogonally based prism, one vertex of G_2 is not connected to any vertex of G_1. By Lemma 4.2(ii), deletion of this vertex leaves a $(2,0)$-graph. Therefore, we construct a candidate graph for $(2,1)$-simplicial prism by attaching a vertex to a real vertex of a $(2,0)$-graph with a dashed edge.

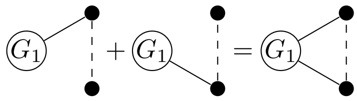
Any candidate graph obtained in this way represents a Coxeter polytope of level 1 or 2. In fact, by attaching a vertex u to a real vertex v with a dashed edge, we are truncating the Coxeter simplex at a space-like vertex. The truncating facet intersects all the lateral faces orthogonally. In the Coxeter graph G, the two vertices u and v belong to G_2, and the other vertices belong to G_1. Vertices of the truncating facet correspond to graphs of the form $G_1 + u - v_1$ where $v_1 \in G_1$ and u is an isolated vertex. Such a graph is of level 0 because G_1 is of level 1. Consequently, if the truncated vertex is the only space-like vertex, the Coxeter polytope we obtain has no space-like vertex, and its level is 1. Otherwise, the polytope is of level 2.

However, for each candidate graph, we still need calculate the label for the dashed edge. For this, we make use of the fact that the determinant of the matrix B is 0. The Coxeter polytope has the combinatorial type of a orthogonally based simplicial prism if this label is < -1. Otherwise, if the label $= -1$, the dashed edge should be replaced by a solid edge with label ∞. In this case, the truncating facet truncates "too much" and meets another vertex, so the Coxeter polytope has the combinatorial type of a pyramid over a simplicial prism, which will be classified later.

The list of $(2,0)$-graphs with at least five vertices can be found in [CL14], where real vertices are colored in white or grey. By attaching a vertex to each of these real vertices, we obtain 655 candidate Coxeter graphs. Among them, 129 graphs correspond to pyramids over simplicial prisms; 17 graphs correspond to simplicial prisms of level 1, as also enumerated in [Kap74]; the remaining 509 graphs correspond to orthogonally based prisms of level 2. Due to the large number of graphs, we do not give the list in this paper.

A Coxeter graph with three vertices is always of level ≤ 2, and the number of real vertices equals the number of dashed edges. We then obtain level-2 Coxeter graphs in the form of . It corresponds to a two dimensional square. A Coxeter graph with four vertices is of level ≤ 2 as long as there is no dashed edge, and the number of real vertices equals the number triangles representing hyperbolic triangle subgroups. This completes the classification of orthogonally based simplicial prisms of level 1 or 2.

The complete list of prisms of level 1 or 2 is obtained by splicing two orthogonally based prisms of level 1 or 2 if they share a same orthogonal base. In other words, if two Coxeter graphs of orthogonally based prism of level 1 or 2 share the same subgraph G_1, we can identify this subgraph, and merge the dashed edge into one, as shown below.



The result is of level 1 if the two orthogonally based prisms are both $(1,1)$-polytopes, or of level 2 otherwise.

4.2.2. *The two simplices are both of dimension* > 1. A vertex u of a $(1,0)$-graph H is a *port* of H if there is a $(2,0)$-graph in the form of $H+v$ in which v is the only real vertex and u is a neighbor of v.

We construct a candidate $(2,1)$-graph by connecting the ports of two $(1^s,0)$-graphs in all possible ways that satisfy Lemma 4.2. For each candidate, we calculate its corank, and verify its level of G by checking the level of $G - v_1 - v_2$ for each $v_1 \in G_1$ and $v_2 \in G_2$. Recall that G is of level 2 if the level of $G - v_1 - v_2$ is always ≤ 1 but not always 0.

In practice, ports are detected with the help of the following lemma.

Lemma 4.3 (Extending [Ess96, Lemma 4.2])**.** *If u is a port of H, then there is a $(2,0)$-graph in the form of $H+v$ in which v is the only real vertex and is only connected to u by a solid edge with label* 3*.*

For each port u of H, we find all the $(2^s,0)$-graphs in the form of $H+v$ in which v is the only real vertex and u is a neighbor of v. These $(2^s,0)$ graphs indicate the possible ways for connecting H to other $(1,0)$-graphs.

Some $(1^s,0)$-graphs with ports are listed in Figure 3, where ports are colored in white and marked with numbers. We exclude hyperbolic triangle groups with label 7, 9 or ≥ 11. By the same technique as in [Ess96, § 4.1, Step 3) 4)], we verified by computer that these triangle groups can not be used to form any Coxeter graph of positive corank. In Table 1, we list all the 28 Coxeter polytopes of the type $\triangle \times \triangle$, with both simplices of dimension > 1. For each polytope, we give the position of G_1 and G_2 in Figure 3, and the edges connecting G_1 and G_2 in the format of (port in G_1, port in G_2, label).

4.3. $\mathcal{C}$ **has the type of** $\mathrm{Pyr}(\triangle \times \triangle)$**.** In this case, the Coxeter graph G consists of three parts: a vertex corresponding to the base facet, and two subgraphs G_1 and G_2 corresponding to the two simplices. The base facet has the type of $\triangle \times \triangle$. Vertices on the base facet are all simple. Except for the apex vertex, every other k-face of $\mathcal{C}$ corresponds to a subgraph of G obtained by deleting $k+2$ vertices, including at least one vertex from both G_1 and G_2. The stabilizer of the apex is represented by the graph $G_1 + G_2$. The corank of $G_1 + G_2$ is 1, and its level may be 0 if the apex is light-like, or 1 if the apex is space-like. We now study these two sub-cases separately.

4.3.1. *The apex is light-like.* In this case, $G_1 + G_2$ is a Euclidean graph.

If a $(1,0)$-graph has a unique ideal vertex, we call this vertex the *hinge* of the graph.

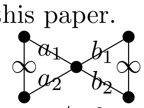
Lemma 4.4. *If $G_1 + G_2$ is a $(0,1)$-graph, then*

(i) *G_1 and G_2 are both Euclidean and are not connected to each other;*
(ii) *$G_1 + v$ and $G_2 + v$ are both $(1,0)$-graphs;*
(iii) *If one of the simplices, say the one represented by G_2, is of dimension > 1, then v is the hinge of $G_1 + v$;*
(iv) *Let $v_2 \in G_2$ be a neighbor of v, then $G_1 + v + v_2$ is a $(2,0)$-graph.*

For a proof, the first point follows from the same argument as in the proof of [Tum04, Lemma 4], and other points follows from the same type of argument as in the proof of Lemma 4.2. We now sketch the procedure for enumerating Coxeter polytopes of this type.

If one of the simplices is of dimension 1, we construct a candidate $(2,1)$-graph as follows. For an ideal vertex v of a non-strict $(1,0)$-graph H, we extend H to a $(2,0)$-graph by attaching a vertex u to v with a solid edge of label a. We allow a to be 2, meaning that u and v are actually not connected. We attach a second vertex u' to v in a second (possibly the same) way with label a'. Then we connect u and u' by a solid edge with label ∞. In the graph we obtain, v correspond to the base facet of $\mathcal{C}$, $H-v$ correspond to G_1 and $u+u'$ correspond to G_2. Since $v+u+u'$ is a $(1,0)$-graph, a and a' can not be both 2. One then verifies Lemma 4.4 on $H+u+u'$, and conversely that any graph obtained in this way is of level 1 or 2. Furthermore, $H+u+u'$ has a positive corank [Tum04, Lemma 3][1] which necessarily equals 1 because $H+u$ is of corank 0. With the same argument as in Section 4.2.1, we see that the graph is of level 2 as long as u and u' are not both the only real vertex of $H+u$ and $H+u'$ respectively.

The list of non-strict $(1,0)$-graph with ≥ 4 vertices can be found in [Che69]. The procedure above then gives, up to graph isomorphism, 358 graphs of level 1 or 2. Among them, 89 are of level 1 as also enumerated in [Tum04]. The remaining 269 graphs correspond to pyramids of level 2, and 129 of them were discovered earlier in Section 4.2.1 when enumerating orthogonally based simplicial prisms. Due to the large number of graphs, we do not give the list in this paper.

If both simplices are of dimension 1, then the Coxeter graph is in the form of 

. It corresponds to a square pyramid. Its level is at most 2, and equals 2 if $1/a_i + 1/b_j < 1/2$ for some $i, j \in \{1, 2\}$.

If both simplices are of dimension > 1, we construct a candidate $(2,1)$-graph by taking two non-strict $(1,0)$-graphs with hinges and identifying their hinges. For any graph G constructed in this way, one easily verifies Lemma 4.4. The corank of G is positive by [Tum04, Lemma 3][1], and necessarily equals 1 by applying [Vin84, Proposition 12] on $G-v$ for any v different from the hinge. Finally, we verify the level of G by checking the level of $G-v_1-v_2$ for each $v_1 \in G_1$ and $v_2 \in G_2$. Recall that G is of level 2 if the level of $G-v_1-v_2$ is always ≤ 1 but not always 0.

All non-strict $(1,0)$-graphs with a hinge and ≥ 4 vertices are listed in Figure 4. In Table 2, we list all the 65 polytopes of this class by giving the position of G_1+v and G_2+v in Figure 4 respectively.

4.3.2. *The apex is space-like.* In this case, G_1+G_2 is a $(1,1)$-graph.

For a $(3,0)$-graph G, a vertex is said to be *surreal* if its removal leaves a $(2,0)$-graph.

Lemma 4.5. *If G_1+G_2 is a $(1,1)$-graph, then*

(i) *G_1 and G_2 are both $(1^s,0)$-graphs, and they are connected;*
(ii) *G_1+v and G_2+v are $(2^s,0)$-graphs in which v is the only real vertex;*

and for any $v_2 \in G_2$,

(iii) *the graph G_1+v_2 is a $(2^s,0)$-graph in which v_2 is the only real vertex;*
(iv) *the graph G_1+v+v_2 is a $(3^s,0)$-graph for which v and v_2 are the only surreal vertices.*

For a proof, the first point is [Tum04, Lemma 2(I)], and other points follows from the same type of argument as in the proof of Lemma 4.2. We now sketch the procedure for enumerating Coxeter polytopes of this type.

If one of the simplices is of dimension 1, G_1+G_2 represents a $(1,1)$-prism. Coxeter graphs for $(1,1)$-prisms are classified in [Kap74], where a list of orthogonally based $(1,1)$-prisms is given. Each graph in the list is obtained by attaching a dashed edge to the unique real vertex of a $(2^s,0)$-graph, see also [Vin85, § 5.4]. Therefore, if we ignore the dashed edges, the list in [Kap74] essentially classified all connected $(2^s,0)$-graphs with a unique real vertex.

Given a $(1^s,0)$-graph H, we construct a candidate $(2,1)$-graph as follows. We extend H to three $(2^s,0)$-graphs (possibly same) $H+u$, $H+v$ and $H+w$, in which u, v and w are respectively the unique real vertex, therefore $H+u+v+w$ satisfies Lemma 4.5(iii). We now choose u as the base facet. We combine the three graphs by by identifying H, then connect v and w by a dashed edge, such that G_1+v+w represent a $(1,1)$ prism. We also connect u,v and u,w respectively by a

[0] This lemma follows from Proposition 12 of [Vin84], instead of [Vin67].

solid edge, and give it all possible labels (necessarily between 2 and 6) such that $H+u+v$ and $H+u+w$ are $(3^s,0)$-graphs. One verifies that u,v and u,w are respectively the only two surreal vertices, so $H+u+v+w$ satisfies Lemma 4.5(iv). Finally, we verify the corank and the level of the candidate graph.

In order to satisfy Lemma 4.5(i), v and w can not be both disjoint from H. But the vertex u can be an isolated vertex, in which case $H+u+v+w$ is indeed a $(2,1)$-pyramid. Otherwise, we list in Figure 1 all the 18 connected $(2,1)$-pyramid in dimension ≥ 5. For 4-dimensional pyramids

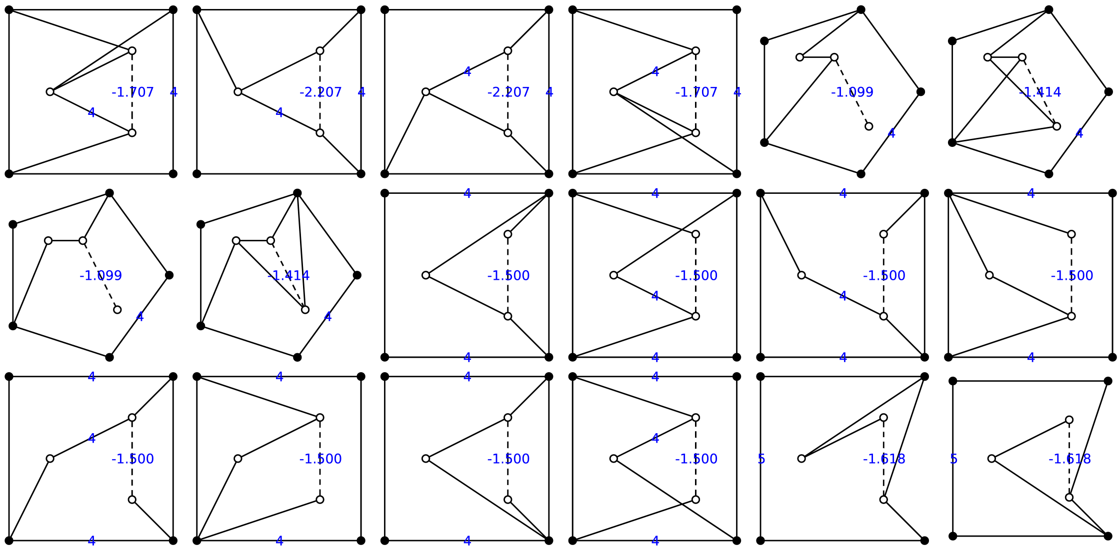
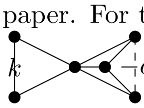
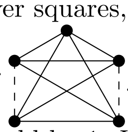


FIGURE 1. The 18 connected $(2,1)$-graphs of rank ≥ 7 whose Coxeter polytope has the type of a pyramid over a prism with space-like apex.

over triangular prisms, we obtain 266 connected $(2,1)$-graphs from triangle graphs with labels at most 6. Because of the large number of graphs, we do not list them in this paper. For triangle graphs with a label $k \geq 7$, the Coxeter graph is necessarily in the form of 

. The unlabeled edges can not have label ≥ 7, so for a given k, there are only finitely many possibilities for the labels. For each of them, the value of $-c$ is determined by k using the fact that G_1+G_2 is a $(1,1)$-graph. We then use Sage to find the expressions of the determinant in terms of k, and find no integer root that is ≥ 7 for these expressions. So we believe that the labels on solid edges are at most 6 for this type of $(2,1)$-graphs. However, the author thinks that this is the point to question the reliability of computer enumeration, and an analytic explanation is welcomed.

For 3-dimensional pyramids over squares, both simplices are of dimension 1, and the Coxeter graph is necessarily in the form of . To be a $(2,1)$-graph, the dashed edges need to bear correct labels and the corank should be 1. We do not have a complete characterisation for this case.

If both simplices are of dimension >1, G_1+G_2 falls in the list in [Ess96] and [Tum04]. The list contains eight graphs. Each graph G in the list is obtained by connecting two $(1^s,0)$-graphs G_1 and G_2. We extend G_1 and G_2 to two $(2^s,0)$ graphs G_1+u_1 and G_2+u_2 in which u_1 and u_2 are respectively the unique real vertices (possibly isolated). We then obtain a candidate $(2,1)$-graph G by identifying u_1 and u_2 as a single vertex u. Finally, we calculate the corank, and verify the level of the candidate by checking the level of $G-v_1-v_2$ for $v_1 \in G_1$ and $v_2 \in G_2$. This time, we only need the level of $G-v_1-v_2$ to be always ≤ 1. If u is an isolated vertex, the result is clearly a

$(2,1)$-graph. Otherwise, there are three connected $(2,1)$-graph. They are listed in Figure 2, where the white vertex correspond to the base facet.

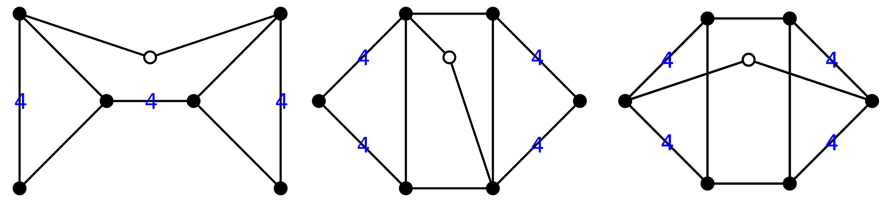


FIGURE 2. The three connected $(2,1)$-graphs whose Coxeter polytope has the type of $\mathrm{Pyr}(\triangle \times \triangle)$ with space-like apex.

4.4. $\mathcal{C}$ **has the type of** $\mathrm{Pyr}^2(\triangle \times \triangle)$**.** In this case, $\mathcal{C}$ can be viewed as a pyramid in two different ways with different apexes and bases. The two base facets are of the type $\mathrm{Pyr}(\triangle \times \triangle)$, and are represented by two vertices u and v in the Coxeter graph. The rest of the graph consists of two parts, say G_1 and G_2, representing the two simplices. The intersection of the two base facets is a ridge of $\mathcal{C}$ with the combinatorial type $\triangle \times \triangle$. Vertices on this base ridge are all simple. Except for the two apexes and the edge connecting them, every k-face of $\mathcal{C}$ corresponds to a subgraph of G obtained by deleting $k+2$ vertices, including at least one vertex from both G_1 and G_2. The stabilizer of the edge connecting the two apexes is represented by $G_1 + G_2$. Its level is 0 since it corresponds to an edge of $\mathcal{C}$.

If a $(2,0)$-graph H has only two real vertices u and v, and $H - u - v$ is of Euclidean type, then we say that $u + v$ is the hinge of H.

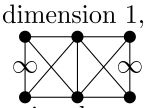
Lemma 4.6. *If $\mathcal{C}$ has the combinatorial type of a 2-fold pyramid over a product of two simplices, then*

(i) *G_1 and G_2 are both Euclidean and are not connected to each other;*
(ii) *$G_1 + u + v$ and $G_2 + u + v$ are both $(2,0)$-graphs with hinge $u + v$;*
(iii) *for any $v_2 \in G_2$, the subgraph $G_1 + u + v + v_2$ is a $(3,0)$-graph, in which v_2 is a surreal vertex, while no vertex of G_1 is surreal.*

The proof use the same type of arguments as before, and use the fact that $G_1 + G_2 + u$ and $G_1 + G_2 + v$ are $(1,1)$-pyramids enumerated in [Tum04]. We now sketch the procedure for enumerating Coxeter polytopes of this type.

If one of the simplices is of dimension 1, we construct a candidate $(2,1)$-graph as follows. Let H be an Euclidean graph and $H + u + v$ be a $(2,0)$-graph with hinge $u + v$. We extend H to a $(3,0)$-graph $H + w$ such that w is a surreal vertex but no vertex in H is surreal. We extend H to another $(3,0)$-graph $H + w'$ in a second (possibly the same) way, and connect w and w' by a solid edge with label ∞. We require further that $u + v + w$ and $u + v + w'$ are of level 0, and $u + w + w'$ and $v + w + w'$ are connected. This guarantees that $u + v + w + w'$ is a $(2,0)$-graph and $u + v$ is the hinge. What we obtain is then a candidate $(2,1)$-graph.

All $(2,0)$-graphs with a hinge and ≥ 5 vertices are listed in Figure 5. Based on this list, the procedure above gives 221 candidate $(2,1)$-graphs. After verification of corank and level, 49 of them are confirmed as $(2,1)$-graphs. They are listed in table 3, in which we give the position of the $(2,0)$-graph $H + u + v$ in Figure 5, and the four labels on the edges connecting w and w' to u and v.

If both simplices are of dimension 1, the Coxeter graph of the 4-dimensional 2-fold pyramid over a square is in the form of . To be a $(2,1)$-graph, each of the four unlabeled triangles should be of level 0, either triangle on the left and either triangle on the right should form a graph of level ≤ 1, and the corank should be 1. We do not have a complete characterisation for this case.

If both simplices are of dimension > 1, we construct a candidate $(2,1)$-graph by taking two $(2,0)$-graphs with hinges and identifying their hinges (possibly in two different ways). We then verify the corank and the level of each candidate. The latter is done by checking the level of $G - v_1 - v_2$ for each $v_1 \in G_1$ and $v_2 \in G_2$. Recall that G is of level 2, if the level of $G - v_1 - v_2$ is always ≤ 1 but not always 0. In Table 4, we list all the 36 polytopes of this class by giving the position of $G_1 + u + v$ and $G_2 + u + v$ in Figure 5 respectively. It turns out that, for every pair in the table, there is a unique way to identify the hinges up to graph isomorphism.

4.5. **Remark and discussion.** We have seen a lot of level-2 Coxeter graphs. The algorithms for classification are implemented in the computer algebra system Sage [S^+14]. For some cases of low rank, because of the large number (even infinite) of graphs, we gave characterisations and construction methods instead of explicit lists. For pyramid (space-like apex) and 2-fold pyramids over squares, our characterisation is not satisfactory. For pyramids over triangular prisms with space-like apex, we ruled out labels of large value by computer program, but the reliability of computer can be questioned.

All these graphs correspond to infinite ball packings that are generated by inversions. For explicit images of ball packings, the readers are referred to the artworks of Leys' [Ley05]. The 3-dimensional ball packings in Leys' paper (and also on his website) are inspired from [BH04]. Similar idea was also proposed by Bullett and Mantica [BM92, MB95], who also noticed generalizations in higher dimensions.

However, the packings considered in these literatures are very limited. In our language, the Coxeter polytopes associated to these packings only have the combinatorial type of pyramid over regular polytopes. In [BM92], the authors were aware of Maxwell's work, but explained that:

> Our approach via limit sets of Kleinian groups is more naive, replacing arguments about weight vectors in Minkowsky N-space by elementary geometric arguments involving polygonal tiles on the Poincare disc: it mirrors the algorithm we use to construct the circle-packings and seems well adapted to computation of the exponent of the packing and other scaling constants.

On the contrary, weight vectors are very useful for investigations. In fact, weight vectors only make the computation of the *exponent* (the growth rate of the curvatures) much easier. One easily verifies that the height of a weights is asymptotically equivalent to the curvature of the corresponding ball.

Remark. The Hausdorff dimension of the residual set of infinite ball packings are usually approximated by computing the exponent. In the literature, Boyd's works (e.g. [Boy74]) are often cited to support this numeric estimate. However, this was not fully justified until recently by Oh and Shah [OS12].

Allcock [All06] proved that there are infinitely many Coxeter polytopes in lower dimensional hyperbolic space. However, we would like to point out that the situation is not completely dark. We notice that the "doubling trick" used in Allcock's construction produces Coxeter subgroups of finite index, so the infinitely many hyperbolic Coxeter groups constructed in [All06] are all commensurable. It has been noticed in [Max82, § 4] that commensurable Coxeter groups of level-2 correspond to the same ball packing. Indeed, if two Coxeter groups are commensurable, their Coxeter complex is the subdivision of the same coarser Coxeter complex.

Therefore, it makes more sense to enumerate commensurable classes of Coxeter groups, as Maxwell did in [Max82, Table II]. For Coxeter systems of corank 0, the commensurable classes and subgroup relations have been studied for level 1 and 2 in [Max98], and are completely determined for level 1 by Johnson et al. [JKRT02]. Despite of Allcock's result, we may still ask: Are there infinitely many commensurable classes for level-ℓ Coxeter groups acting on lower dimensional hyperbolic spaces? For level 1 Coxeter groups, the answer is "yes" in dimension 2 (triangle groups), 3 [MR03, § 4.7.3], 4 and 5 [Mak68; Vin85, § 5.4]. The constructions in dimension 3–5 made use of level-1 polytopes of low corank.

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G_1	G_2	Edges between G_1 and G_2	G_1	G_2	Edges between G_1 and G_2
2	13	(1,0,3), (2,2,3)	4	19	(0,0,3), (3,1,3)
6	11	(0,1,3)	6	13	(0,0,3), (0,2,3)
10	17	(0,1,3), (3,1,3)	10	22	(0,0,3), (3,1,3)
11	14	(1,0,3)	11	22	(1,2,3)
12	12	(0,1,3)	12	12	(2,2,3)
12	15	(0,0,3), (0,2,3)	12	15	(2,1,3)
12	19	(0,0,3), (0,1,3)	12	19	(1,2,3)
12	24	(2,0,3), (2,1,3)	12	27	(0,0,3), (2,1,3)
13	14	(0,0,3), (2,0,3)	13	22	(0,0,3), (2,1,3)
13	26	(0,0,3), (2,2,3)	15	15	(0,0,3), (2,2,3)
15	15	(1,1,3)	15	19	(0,0,3), (2,1,3)
16	30	(0,0,3), (2,1,3)	17	22	(0,0,3), (2,1,3)
17	26	(0,0,3), (2,2,3)	18	18	(0,1,3), (1,0,3)
24	24	(0,0,3), (1,1,3), (2,2,3)	25	25	(0,0,3), (0,0,4), (1,1,3), (1,1,4)

TABLE 1. The first two columns are the positions of G_1 and G_2 in Figure 3, and the third columns are the edges connecting G_1 and G_2. The ports in Figure 3 are numbered, so the edges are represented in the format of (port in G_1, port in G_2, label). By connecting G_1 and G_2 by the indicated edges, we obtain the $(2,1)$-graphs for the products of two simplices (both of dimension > 1).

1–2	1–5	1–9	1–12	1–15	1–16	1–19	1–22	1–24	1–27
2–3	2–7	2–8	2–23	3–5	3–13	3–15	3–16	3–20	3–28
4–9	4–12	4–19	4–22	4–24	4–27	5–7	5–8	5–23	7–9
7–12	7–15	7–16	7–19	7–22	7–24	7–27	8–9	8–12	8–15
8–16	8–19	8–22	8–24	8–27	9–26	10–12	10–19	10–27	11–12
11–19	11–27	12–18	12–26	13–23	15–23	16–23	18–19	18–27	19–26
20–23	22–26	23–28	24–26	26–27					

TABLE 2. For each pair i–j in the list, by identifying the white vertices of the i-th and the j-th graph in Figure 4, we obtain the $(2,1)$-graph of a pyramid over the product of two simplices (both of dimension > 1).

4:(2,3)(3,2)	4:(2,4)(4,2)	5:(2,3)(3,2)	6:(2,3)(3,2)	15:(2,2)(3,3)
15:(2,3)(3,2)	15:(2,4)(4,2)	15:(3,3)(3,3)	15:(3,4)(4,3)	24:(2,3)(4,3)
28:(2,2)(3,3)	28:(2,3)(3,2)	28:(3,3)(3,3)	32:(2,2)(3,3)	32:(2,3)(3,2)
32:(2,4)(4,2)	32:(3,3)(3,3)	37:(2,3)(3,2)	38:(2,2)(3,3)	38:(2,3)(3,2)
38:(3,3)(3,3)	39:(2,3)(3,2)	40:(2,2)(3,3)	40:(2,3)(3,2)	40:(3,3)(3,3)
41:(2,2)(3,3)	41:(3,3)(3,3)	42:(2,2)(3,3)	42:(3,3)(3,3)	48:(3,2)(3,3)
49:(2,2)(4,3)	49:(2,3)(4,2)	49:(3,2)(3,4)	49:(4,3)(4,3)	57:(2,3)(4,3)
59:(2,2)(3,3)	59:(3,3)(3,3)	61:(2,2)(3,3)	61:(2,3)(3,2)	61:(2,4)(4,2)
61:(3,3)(3,3)	65:(2,3)(3,2)	66:(2,2)(3,3)	66:(2,3)(3,2)	66:(3,3)(3,3)
67:(2,3)(3,2)	68:(2,2)(3,3)	68:(2,3)(3,2)	68:(3,3)(3,3)	

TABLE 3. For each entry i:$(a,b)(c,d)$ in the list, take the i-th graph $H+u+v$ in Figure 5, where u is the gray vertex and v is the white vertex. Introduce two new vertices w and w', and connect them to H such that wu has label a, wv has label b, $w'u$ has label c, $w'v$ has label d, and finally label the edge ww' by ∞. The result is the $(2,1)$-graph of a 2-fold pyramid over a prism.

4–4	8–15	8–22	8–56	8–62	13–13
13–49	15–15	15–22	15–32	15–35	15–54
15–56	15–61	15–62	22–22	22–32	22–35
22–54	22–56	22–61	22–62	32–56	32–62
35–56	35–62	38–56	49–49	54–56	54–62
56–56	56–61	56–62	56–66	61–62	62–62

TABLE 4. For each pair i–j in the list, by identifying the white/light-gray vertices of the i-th and the j-th graph in Figure 5, we obtain the $(2,1)$-graph of a 2-fold pyramid over the product of two simplices (both of dimension > 1).

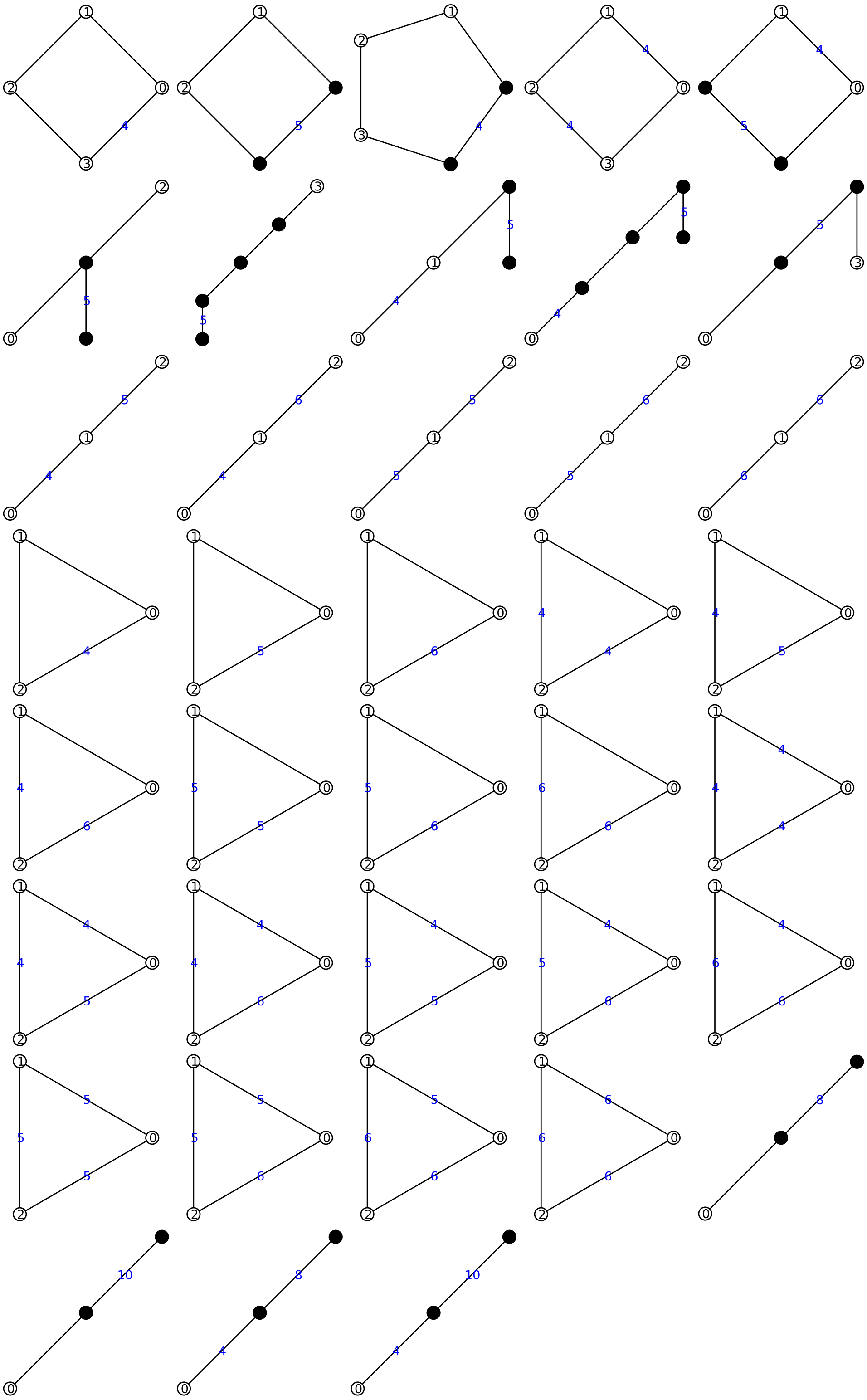

FIGURE 3. $(1^s, 0)$-graphs of ≥ 3 vertices with ports (numbered white vertices)

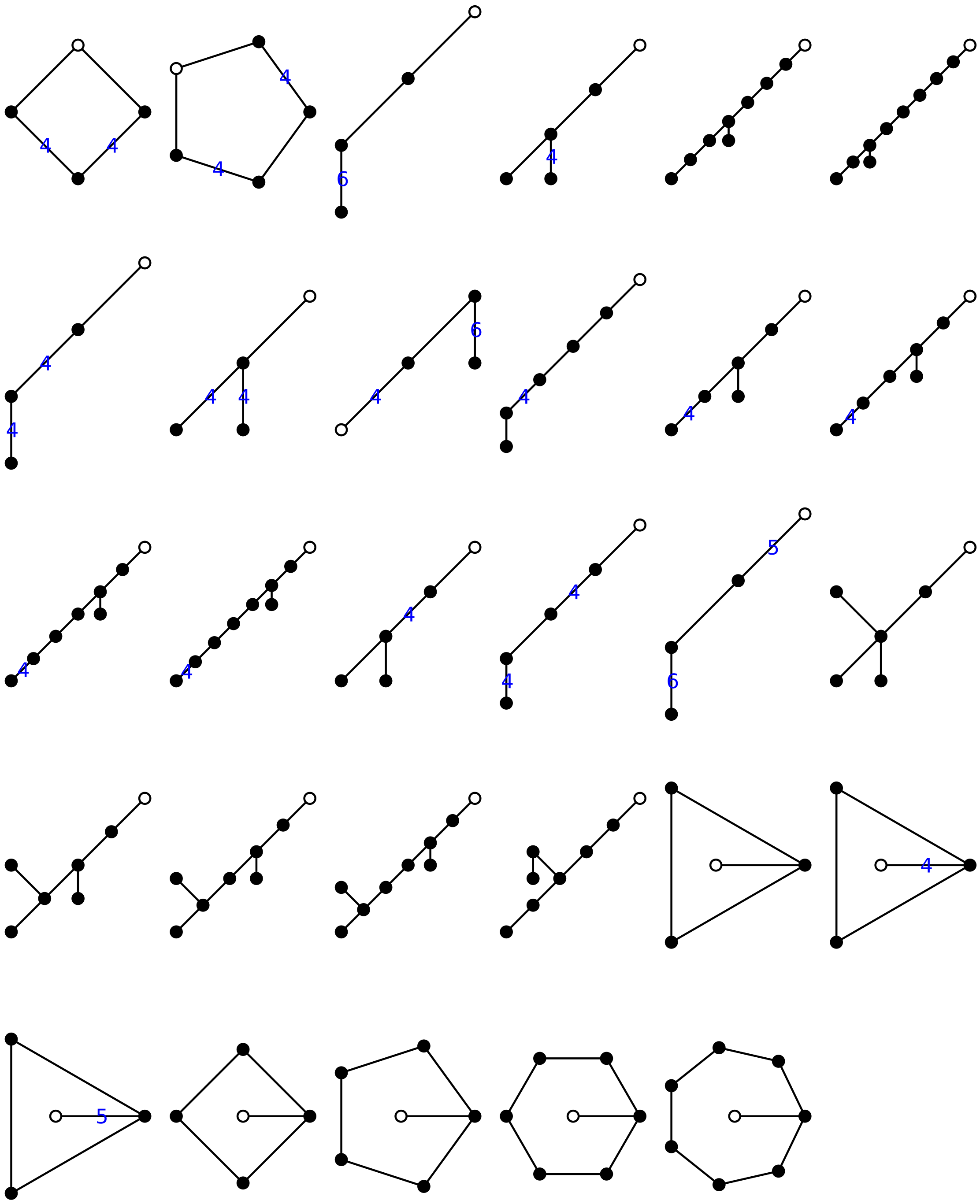

FIGURE 4. non-strict $(1, 0)$-graphs of ≥ 4 vertices with a hinge (the white vertex)

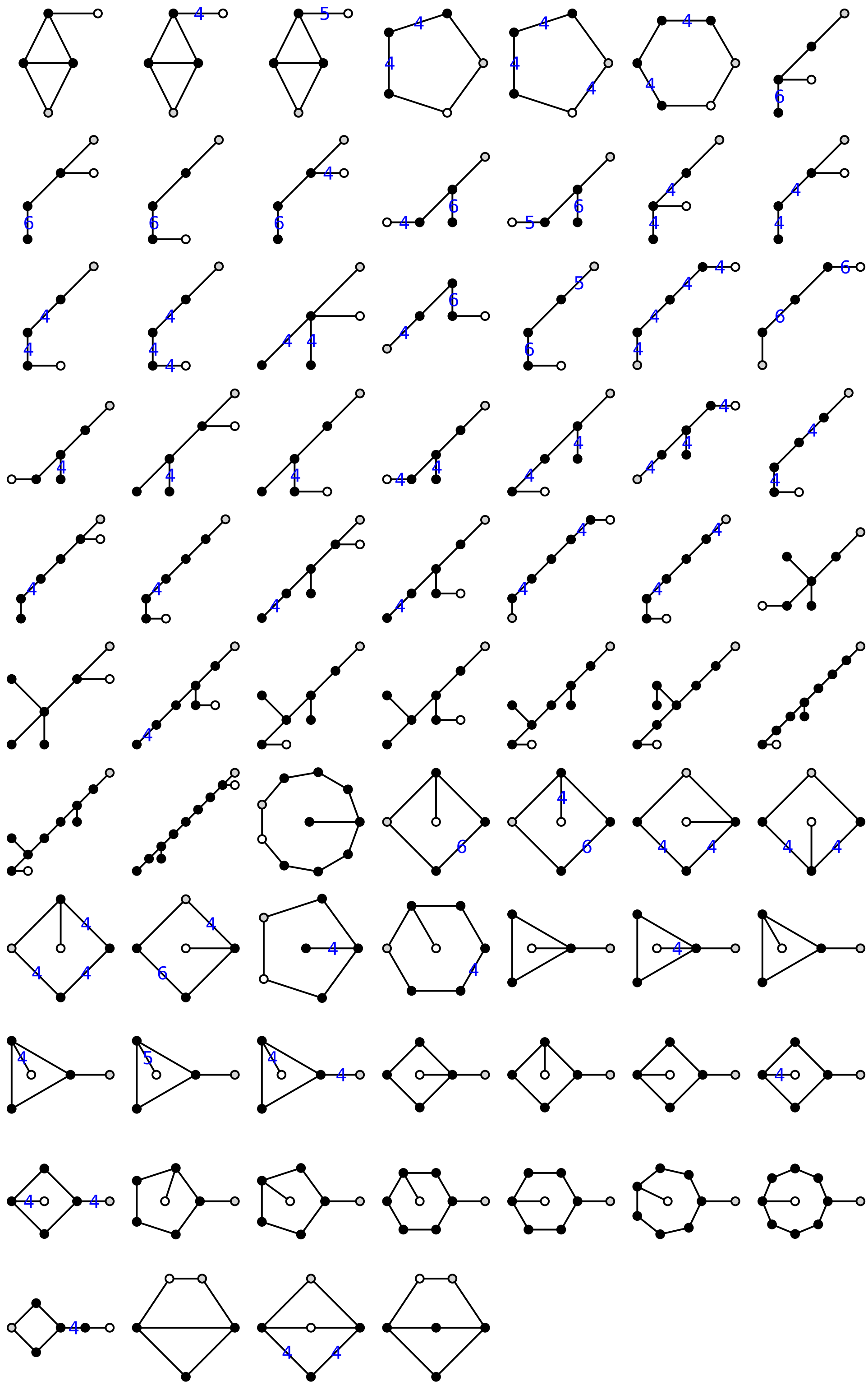

FIGURE 5. (2, 0)-graphs of ≥ 5 vertices with a hinge (the white and the light-gray vertices)