RESEARCH STATEMENT

HAO CHEN

Initially educated as a physicist, I received doctoral training in mathematics, and started my career as a discrete geometer. My research interest was Coxeter groups, polytopes, and sphere packings, etc.

Around 2016/2017, I switched my topic to differential geometry, focusing on Triply Periodic Minimal Surfaces (TPMSs), a fantastic geometric object that combines the beauty of bubble films with locally minimized area, and crystals with repeating patterns. The topic finds applications in material sciences of soft matter physics.

I recognize two golden ages in the history of TPMS. The first examples were constructed by Schwarz in the 1860s [Sch72]. Later, Schoen's Gyroid [Sch70] and Meeks' 5-parameter family [Mee90], both constructed in the 1970s, inspired fruitful transdisciplinary collaborations until the 1990s.

I made surprising discoveries of new TPMSs of genus 3 (TPMSg3s), **for the first time in almost 20 years**, with rigorous mathematical proofs. In the process, I built up **an international and interdisciplinary research network**. In particular, **long-term collaborations with physicists and material scientists** have been established.

At the moment, I am dedicated to **promote another golden age of TPMSs**. The goal is to make progress **towards an ultimate classification** of all TPMSs of genus three. For this purpose, I'm trying to expand my expertise to **complex dynamics and bifurcation theory** in order to deal with the singularities that are known to exist in the moduli space of TPMSg3s.

1. Recent and ongoing projects

1.1. **Motivations from physics.** In 2011, Lu Han et al. [HXBC11] synthesized silica mesoporous crystals with polyhedral hollow cavities. Crystallographic analysis reveals TPMS structures. Most interestingly, they observe twinning structure at the domain boundaries: The TPMSs on different sides are symmetrically related by a reflection in the boundary plane. My physicist wife, Dr. Chenyu Jin, was collaborating with Han et al. to explain the observed twinning structure. I was helping her with differential geometry, but soon developed better insight into their problem, partially thanks to my physics background.

In the beginning, I was conducting this research in my spare time for fun. My main tool was Surface Evolver [Bra92], a software that minimizes various energies on triangulated surfaces. It allowed me to numerically reproduce the twinning structure observed [Che21a, HFC+20]. With an innovative use of the software, I also implemented an energetic model of TPMSs that was long expected by physicists [CJ17]. At the same time, I used the software to play with TPMSg3s and their deformations (see [Che21a]) and noticed clues for many TPMSg3s that I discovered later or plan to work on in the future.

1.2. **Mathematical treatment of defects.** Twinning is an example of crystal defects, i.e. interruptions of the periodic pattern. As a mathematician, I was not satisfied with the numerical results. Hence I wrote a proposal aiming at rigorous mathematical treatments of crystal defects in TPMSs. This project received an individual funding from Deutsche Forschungsgemeinschaft (DFG).

In search of a rigorous proof of the twinning phenomenon, I began to talk with mathematicians including Karsten Große-Brauckmann, David Hoffman, Matthias Weber, and Martin Traizet. It became clear that Traizet's node-opening technique is the appropriate tool for my purpose.

I then collaborated with Traizet to open nodes between infinitely many flat 2-tori in a non-periodic way [CT21a]. Due to the complexity, we decided to open only one node between each adjacent tori to ease the setting up. Despite this considerable simplification, the technique turns out to be much more powerful than I originally thought. Most of our examples would be considered by crystallographers as disordered stackings of nodes. Twinning and other planar defects are merely special cases. This is surprising for mathematicians as well as for physicists.

A side product of this work was a complete description of the catenoid limits of TPMSg3s. It turns out that the catenoid limits are described by pairs of solutions of the equation

$$F(q;\tau) = \zeta(q;\tau) - \xi(q;\tau) = C$$

for fixed $(\tau, C) \in \mathbb{C}^2$, where ζ is the Weierstrass zeta function and

$$\xi(x + y\tau;\tau) = 2x\zeta(1/2;\tau) + 2y\zeta(\tau/2;\tau).$$

This established an unexpected connection to PDEs [LW10, CKLW18], complex dynamics [BE16], and mathematical physics.

1.3. **New TPMSg3s.** In physics, knowledge of defects often leads to deeper insights into crystals. This is also the case for TPMSg3s. My study on defects leads to the discovery of new TPMSg3s.

During the preliminary numerical work [Che19], I noticed a 1-parameter family that shares many symmetries with the classical tetragonal deformation family tD of Schwarz's D surface, yet completely different. Most surprisingly, it seems to intersect with tD, suggesting a singularity in the moduli space of TPMSg3s.

Shortly after receiving the DFG grant, I became aware that the new family can be constructed using the techniques of Fujimori–Weber [FW09]. Weber was quite surprised by my discovery, as this is the first new TPMSg3s in almost 20 years. Most surprisingly, the new TPMSg3s is not in Meeks' family.

We then collaborated to expand the family into a 2-parameter family oΔ [CW21a]. Later, we collaborated to discover another 2-parameter deformation family oH using similar techniques [CW21b]. They are all outside Meeks' family, but intersect Meeks' family in 1-parameter subfamilies. So they exhibit, for the first time concretely, singularities in the moduli space of TPMSg3s.

These new families renewed the interest in explicit constructions of new TPMSs. Soon later, new TPMSs of genus four were found using similar techniques [FWYY18].

Weber suggested that, for an eventual classification of TPMSg3s, one needs to fully understand the deformations of the gyroid. Two deformations families were numerically discovered by physicists in the 1990s [Fog93, FH99]. An attempted of rigorous proof only succeeded in small neighborhoods [Wey08]. With an innovative use of elliptic functions, I managed to provide a rigorous existence proof [Che21a].

1.4. **Gluing constructions.** I'm currently working on projects to construct minimal surfaces by gluing saddle towers and helicoids. These are again motivated by physicists [MSSTM18] and biologists [BCH+16]. Such constructions were previously implemented assuming extra symmetry [Tra96, You09, TW05]. My purpose is to break these symmetries.

The constructions of TPMSs and Singly Periodic Minimal Surfaces (SPMSs) by gluing saddle towers are recently published as preprints [CT21b, Che21b]. It turns out that, in addition to the horizontal balancing condition that was previously known, the saddle towers must also be balanced under a very subtle vertical interaction. This interaction vanishes in the presence of an horizontal reflection plane, hence was not perceived in previous constructions that assumes symmetry. As a consequence, we obtain new 5-parameter families of TPMSg3s near saddle tower limits. Follow-up papers are coming soon.

A manuscript that glues helicoids into minimal surfaces is under preparation. The main message is an interdisciplinary connection between balancing configurations of helicoids in minimal surfaces and rigid configurations of point vortices in 2D Euler fluids. This manuscript will focus on minimal surfaces with screw or translational symmetries and planar or helicoidal ends. A similar construction for TPMSs will be delayed to a future project about weak limits of TPMSg3s (see below).

These ongoing projects already shows that the helicoid limits and saddle-tower limits of TPMSg3s have a much simpler structure than the catenoid limits. This motivates many of my future projects below.

1.5. **Interdisciplinary collaborations.** On the interdisciplinary aspects, I established a long-term collaboration with the team of Lu Han. They are observing interesting structures in the laboratory, and I can explain many of them with my knowledge on minimal surfaces [BCCH21, SCM+21]. Inversely, these observations guide me in the search for new TPMSs. Many other experimental teams are also making contact with me, hoping for mathematical insights and numerical inputs for their experiments. At the same time, I have been in close contact with physicists that care about TPMSs, including Gerd Schröder-Turk and Stephen Hyde. Because of my proof for the deformations of Gyroid, I'm established as the go-to person about Gyroid and related topics.

2. Future projects

In the coming years, I plan to work towards an ultimate classification of all embedded TPMSs of genus 3 (TPMSg3s). There are several motivations for such an ambitious goal:

- There have been many classifications of complete, embedded minimal surfaces of finite topology in Euclidean space forms. TPMSg3s are the natural next target.
- In all previous classifications, the moduli spaces were found to be smooth manifolds. The moduli space of TPMSg3s, on the other hand, is known to have a very rich structure that was never seen before. In particular, my recent works demonstrated various types of singularities. This presents great yet manageable challenges.
- TPMS structures are ubiquitous in nature and laboratories. Many recent observations in material science can only be explained by deformations of classical TPMSs. Hence the impact expands across disciplines.

I would like to compare the space of TPMSg3s to a jigsaw puzzle. To solve a jigsaw puzzle, one usually starts from special pieces or boundary pieces, and expand from there. This is also the strategy for hunting TPMSg3s, both in history and in my planned projects.

Here is a summary of what is known or strongly believed about TPMSg3s

- The moduli space contains a very well-behaving manifold, known as Meeks' family. I constructed a few families of TPMSg3s that do not belong to but intersect with Meeks' family, namely oΔ, oH, tG and rGL. These known examples are special pieces for our puzzle.

- There are 6 possibilities for the limit of a TPMSg3s, namely catenoids, helicoids, saddle-towers, doubly periodic Scherk surfaces, and Karcher–Meeks–Rosenberg (KMR) examples. They are boundary pieces for our puzzle.
- Among the six limits of TPMSg3s, we understand best the helicoid and saddle-tower limits. We know how to describe the space catenoid limits, but do not understand the concrete structure. There is no published result about Riemann limits, doubly periodic Scherk limits, and KMR limits, but they seem easy.
- We know the existence of new 5-parameter families near the catenoid limits and saddle-tower limits.

Hence my plan for the near future are

(1) **Construct new families of TPMSg3s**. In particular, I plan to construct two tetragonal deformation family of Gyroid, and expand them into an orthorhombic deformation family. I also plan to expand my oH family into an orthorhombic deformation family. In the best scenario, I might construct two new 5-parameter families of TPMSg3s, which would be a major step towards an ultimate classification. For this purpose, I need to deepen my knowledge on elliptic and hyper-elliptic functions, and develop new techniques.

(2) **Better understanding the weak limits**. First of all, I plan to write down the gluing construction for the helicoid limits, doubly periodic Scherk limits, and KMR limits of TPMSg3s. These should not be difficult with my previous experience. Then I plan to classify all the catenoid limits. It boils down to investigate the solutions of $F(q;\tau) = C$. I already have numerical results about the solution space. But for a rigorous treatment, I need to expand my expertise to PDEs and bifurcation theory.

(3) **Bifurcation instants and Morse index**. A bifurcation instant refers to a TPMSg3 for which the same deformation of the lattice may lead to different deformations of the surface. They are therefore potential starting points for discovering new TPMSg3s. Bifurcation instants are hinted by a jump in the Morse index. In [KPS14], bifurcation instants were confirmed at odd jumps in the Morse index. But for technical reasons, they were not able to conclude for even jumps. My recent works on oΔ, oH, tG and rGL reveals bifurcation branches that was not known to [KPS14]. Most surprisingly, I also find a surprising branch from the H surface with an even jump in the Morse index. This provides a strong motivation to extend the result of [KPS14] to even jumps.

(4) **Assess the feasibility of an ultimate classification**. I conjecture that the moduli space of TPMSg3s is connected. In particular, all TPMSg3s can be deformed continuously to a Meeks surface, and all TPMSg3s can be deformed continuously to a catenoid limit. Proving any of these conjectures would be a very good news for an ultimate classification.

(5) **Continue the collaboration with physicists**. The physicists are interested in the "energy" of TPMSg3s to explain the frequent appearance of cubic TPMSg3s. In particular, the frustration of Gaussian curvature is proposed as a phenomenological energy to explain the dominance of Gyroid and Schwarz' D and P structures in material sciences [STFH06]. The new TPMSg3s produced within our project make it possible to verify this theory. Moreover, TPMSg3s can be used as photonic crystals and exhibit interesting optical properties, but only cubic TPMSg3s have been investigated in

detail. I plan to study the optical properties of non-cubic TPMSg3s, which might lead to adjustable photonic materials.

(6) **Migration existing codes to open source platforms**. At the moment, the most advanced and precise calculations about TPMSg3s are implemented in Mathematica. The choice is justified by the powerful built-in elliptic functions of Mathematica. However, unfortunately, the code is not accessible to everyone. I plan to migrate the calculations to an open source platform, presumably Python based. This will facilitate the dissemination of our results.

3. Discrete geometry

In the last part this statement, I would like to briefly summarize some of my works on discrete geometry before my switch of topic.

My doctoral thesis [Che14] was about the combinatorics and symmetries of ball packings. The goal was to generalize the circle packing theorem into higher dimensions, that is, to study the tangency graph of higher dimensional ball packings. My major achievement was a generalization of the famous Apollonian ball packing using a large variety of hyperbolic reflection groups [CL15, Che16a]. My research naturally extended to other related topics, including root systems [CL17] and polytopes [Che16b].

After obtaining my PhD degree in 2014, I continued working on some related problems in discrete geometry.

3.1. **Chromatic number of ball packings.** Inspired by a MathOverflow problem, I studied the chromatic number $\chi(d)$ for the tangency graph of d-dimensional ball packings. At the time, only trivial bounds are known for $\chi(d)$, i.e. a linear lower bound and an exponential upper bound. I recognized a similarity of the problem to the famous Borsuk conjecture. A recent construction of Bondarenko [Bon14] leads to a counter-example of the Borsuk conjecture of dimension 64. I used his technique and found many ball packings with strongly regular tangency graphs, whose chromatic number is significantly higher than the trivial lower bound. In particular, for every prime power q, I found a unit ball packing of dimension $d = q^3 - q^2 + q$ with chromatic number $\chi(d) = q^3 + 1$ [Che17].

3.2. **Selectively balancing unit vectors.** In my doctoral thesis [Che14], I studied the *dot product representation* of graphs. Such a representation maps vertices of a graph to vectors in $\mathbb{R}^d$ such that two vertices are connected by an edge if and only if the dot product of the corresponding vectors is bigger than 1. I disproved a conjecture [LC14] asserting that a $(d+1)$-cube graph has no dot product representation in $\mathbb{R}^d$. Hoping to improve this result, I collaborated with Blokhuis on a distantly related topic: A set of unit vectors in $\mathbb{R}^n$ is said to be *selectively balancing* if the Euclidean norm of some linear combination of them, with only coefficients -1, 0 or 1, is at most 1. Let $\sigma(n)$ be the minimum integer m such that any m unit vectors in $\mathbb{R}^n$ are selectively balancing. We proved that $\sigma(n) \sim \frac{1}{2} n \log n$ [BC18]. Based on this result, I conjecture the existence of a constant c such that the $(cn \log n)$-cube admits a dot product representation in $\mathbb{R}^n$ for infinitely many n.

3.3. **Scribability of polytopes.** Apart from the projects above, my research has been focusing on scribability problems of polytopes. Such problems trace back to 1832, when Steiner [Ste81] asked whether every polyhedron is inscribable. It received a negative answer only in 1928, when Steinitz [Ste28] constructed the first non-inscribable polyhedron.

Schulte [Sch87] proposed higher dimensional analogues of the problem. He asked about realizations of d-dimensional polytopes with all their k-faces tangent to the sphere (k-scribed). For all $d \geq 2$ and $(d,k) \neq (3,1)$, he found polytopes with no such realization (not k-scribable). He also proposed a weak version of the problem, and find non-scribable examples for most but not all cases. Joint with Padrol, we considered Schulte's problems and the more general (i,j)-scribabilities in the more proper setting of projective space [CP17]. We completely settled Schulte's weak problems. Then we studied scribabilities of stacked polytopes and cyclic polytopes, the two extremes of polytopes in terms of their f-vectors. On the one hand, we proved that every stacked d-polytope is k-scribable for $k = d-1$ or $d-2$ but non-examples exist for every smaller k. On the other hand, we proved that a cyclic d-polytope with sufficiently many vertices is not k-scribable.

Back to Steiner's problem, inscribable polyhedra are completely characterized by Rivin et al. [HRS92] in 1992. But Steiner's original question also asked about quadratic surfaces other than the sphere. Recently, Danciger–Maloni–Schlenker [DMS20] obtained Riven-type characterizations for polyhedra inscribable to a cylinder or a hyperboloid, under the assumption that the interiors of the polyhedra do not intersect the quadratic surface. Joint with Schlenker, we obtained complete characterizations for polyhedra inscribed to two-sheet hyperboloids, including, remarkably, a purely graph theoretical characterization [CS21]. In this project, we had to consider the projective space $\mathbb{R}P^3$ as a combination of the hyperbolic space and the de-Sitter space.

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