BALL PACKINGS WITH HIGH CHROMATIC NUMBERS FROM STRONGLY REGULAR GRAPHS

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Abstract. Inspired by Bondarenko's counter-example to Borsuk's conjecture, we notice some strongly regular graphs that provide examples of ball packings whose chromatic numbers are significantly higher than the dimensions. In particular, from generalized quadrangles we obtain unit ball packings in dimension $q^3 - q^2 + q$ with chromatic number $q^3 + 1$, where q is a prime power. This improves the previous lower bound for the chromatic number of ball packings.

1. The problem and previous works

A *ball packing* in d-dimensional Euclidean space is a collection of balls with disjoint interiors. The *tangency graph* of a ball packing takes the balls as vertices and the tangent pairs as edges. The *chromatic number* of a ball packing is defined as the chromatic number of its tangency graph.

The Koebe–Andreev–Thurston disk packing theorem says that every planar graph is the tangency graph of a 2-dimensional ball packing. The following question is asked by Bagchi and Datta in [BD13] as a higher dimensional analogue of the four-colour theorem:

Problem. What is the maximum chromatic number $\chi(d)$ over all the ball packings in dimension d?

The authors gave $d + 2 \leq \chi(d)$ as a lower bound since it is easy to construct $d + 2$ mutually tangent balls. By ordering the balls by size, the authors also argued that $\kappa(d) + 1$ is an upper bound, where $\kappa(d)$ is the kissing number for dimension d.

However, the case of $d = 3$ has already been investigated by Maehara [Mae07], who proved that $6 \leq \chi(3) \leq 13$. His construction for the lower bound uses a variation of Moser's spindle, which is the tangency graph of an *unit* disk packing in dimension 2 with chromatic number 4, and the following lemma:

Lemma. *If there is a unit ball packing in dimension d with chromatic number χ, then there is a ball packing in dimension $d + 1$ with chromatic number $\chi + 2$.*

The technique of Maehara [Mae07] can be easily generalized to higher dimensions and gives $d + 3 \leq \chi(d)$.

Another progress is made by Cantwell in an answer on MathOverflow [Can], who proved that the graph of the halved 5-cube (also called the Clebsch graph) is the tangency graph of a 5-dimensional *unit* ball packing with chromatic number 8. Then the Lemma implies that $10 \leq \chi(6)$. This argument can be generalized to higher dimensions using a result of Linial, Meshulam and Tarsi [LMT88, Theorem 4.1], and gave $d + 4 \leq \chi(d)$ for $d = 2^k - 2$.

As we have seen, both constructions study the chromatic number of unit ball packings and invoke the Lemma. We will do the same. A unit ball packing can be regarded as a set of points such that the minimum distance between pairs of points is at least 1, then the tangency graph of the packing is the unit-distance graph for these points. The *finite version* of the Borsuk conjecture can be formulated as follows: the chromatic number of the unit-distance graph for a set of points with maximum distance 1 is at most $d+1$. So the chromatic number problem for unit ball packings is the "opposite" of the Borsuk conjecture. By ordering the unit balls by height, we see that the chromatic number of a unit ball packing is at most one plus the one-side kissing number.

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The Borsuk conjecture was first disproved by Kahn and Kalai [KK93]. Recently, Bondarenko [Bon14] found a counter-example for Borsuk conjecture in dimension 65. His construction was then slightly improved by Jenrich [JB14] to dimension 64, which is the current record for the smallest counter-example. Their construction is based on geometric representations of strongly regular graphs.

In this note, we use the technique of Bondarenko to find unit ball packings with strongly regular tangency graphs, whose chromatic numbers are significantly higher than their dimensions. In particular

Theorem. *For every prime power q, there is a unit ball packing of dimension $d = q^3 - q^2 + q$ whose tangency graph is strongly regular with chromatic number $\chi(d) = q^3 + 1$.*

Examples are given by the graphs of generalized quadrangles with parameters (q, q^2). This yields the first non-constant lower bound for the difference $\chi(d) - d$.

In view of [KK93], we propose the following conjecture, and hope that examples in this note may help further improvement of the lower bound.

Conjecture. *There is a constant c such that $\chi(d) \geq c^{\sqrt{d}}$.*

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2. Strongly regular graphs

We use [CvL91] for general references on strongly regular graphs.

Let G be a strongly regular graph with parameters (v, k, λ, μ). That is, G is a k-regular graph on v vertices such that every pair of adjacent vertices have λ neighbors in common and every pair of non-adjacent vertices have μ neighbors in common. We assume that

$$\lambda - \mu \geq -2k/(v-1). \tag{1}$$

If this is not the case, we may replace G by its complement $\bar{G}$, which is a strongly regular graph with parameters $(v, v-k-1, v-2k-2+\mu, v-2k+\lambda)$. For our study of ball packings, we may focus on connected graphs, therefore $\mu > 0$. For any vertex of G, the graphs induced by its neighbors in G and by its neighbors in $\bar{G}$ are respectively the *first* and the *second subconstituent* of G.

The adjacency matrix A of G has three eigenvalues k, r, s with multiplicities 1, f, g, respectively. They can be expressed in terms of the parameters as follows:

$$r, s = (\lambda - \mu \pm \delta)/2,$$
$$f, g = (v - 1 \pm \Delta)/2,$$

where $\delta = \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}$ and $\Delta = ((v-1)(\mu-\lambda) - 2k)/\delta \leq 0$. The eigenvalues of $\bar{G}$ are $v-k-1$, $-s-1$, $-r-1$ with multiplicities 1, g, f, respectively. Note that $r > 0 > s+1$ and $f \leq g$.

Let I be the identity matrix and J the all-ones matrix. Then

$$E = (A - sI)(I - J/v)$$

is an eigenmatrix of A corresponding to the eigenvector r, and the column vectors of E (labeled by vertices of G) form a unit spherical 2-distance set on the sphere $\mathbb{S}^{f-1} \subset \mathbb{R}^f$, with angles $\cos\alpha = r/k$ for adjacent vertices and $\cos\beta = -(r+1)/(v-k-1)$ for non-adjacent vertices [Bon14]; see also [BCN89, Theorem 4.1.4]. By putting a ball of radius $\sin(\alpha/2) = \sqrt{(1-r/k)/2}$ at each point of the 2-distance set, we obtain a ball packing whose tangency graph is G.

By Hoffman's bound [Hof70] (see also [Del73] [BCN89, Proposition 1.3.2]), the clique number of the complement $\omega(\bar{G})$ is at most $1 + (v-k-1)/(1+r)$, so the chromatic number $\chi(G)$ is at least

$$v/\big(1 + \frac{v-k-1}{1+r}\big) = 1 - k/s.$$

Remark 1. As B. Bagchi pointed to the author, the same argument works for any regular graph. More specifically, if the adjacency matrix of a k-regular graph G of v vertices has the minimum eigenvalue s of multiplicity g, then G is the tangency graph of a ball packing in dimension $v-g-1$, and the chromatic number of G is at least $1-k/s$. The advantage of strongly regular graphs is that g tends to be big, which would decrease the dimension.

3. Colorful strongly regular ball packings

From Brouwer's online list of strongly regular graphs [Bro], we notice some graphs such that $f+3<1-k/s$. In Table 1, we list their parameters, eigenvalues with multiplicities, and the Hoffman bound $1-k/s$. For comparison, we highlight the dimension f and the Hoffman bound $1-k/s$. For the complement of McLaughlin graph, the Hoffman bound gives the exact value of the chromatic number [HT96].

Remark 2. For many graphs in the table, the conventional parameters in the literature do not satisfy our assumption (1). For these graphs, we use the parameters of their complements, as explained in the beginning of Section 2.

name or ref	v	k	λ	μ	r	f	s	g	$1-k/s$
Higman–Sims	100	77	60	56	7	22	−3	77	80/3
[DC08, LR13]	105	72	51	45	9	20	−3	84	25
[DC08, LR13]	120	77	52	44	11	20	−3	99	80/3
[HK02]	126	75	48	39	12	20	−3	105	26
2nd subconst. of McL	162	105	72	60	15	21	−3	140	36
[Hae81]	175	102	65	51	17	21	−3	153	35
[Hae81, DC08]	176	105	68	54	17	21	−3	154	36
[Hae81]	176	85	48	34	17	22	−3	153	88/3
[Del73]	243	132	81	60	24	22	−3	220	45
[DC08, LR13]	253	140	87	65	25	22	−3	230	143/3
McLaughlin	275	162	105	81	27	22	−3	252	55
[GS75]	276	135	78	54	27	23	−3	252	46
[BR13]	729	520	379	350	22	112	−5	616	621/5

Table 1. Strongly regular graphs with high chromatic numbers

Two infinite families are not included in the table. One is the complements to the C20 family from Hubaut [Hub75], recovered by Godsil [God92, Lemma 5.3], with the parameters

$$(q^3,(q+1)(q^2-1)/2,(q+3)(q^2-3)/4+1,(q+1)(q^2-1)/4)$$

where q is an odd prime power. They can be represented as *unit* ball packings in dimension $f=q^2-q$ with chromatic number at least $1-k/s=q^2$, which already provide a non-constant difference $\chi(d)-d$.

The other family is the complements to the point graphs of the generalized quadrangles with parameters (q,q^2), which provides an even better difference.

A *generalized quadrangle* [PT09] with parameters (σ,τ), denoted by $GQ(\sigma,\tau)$, is an incidence structure $(P,L,\in)$, where P is the set of points and L is the set of lines, satisfying the following axioms:

- Each point is incident with $\tau+1$ lines and two distinct points are incident with at most one line.
- Each line is incident with $\sigma+1$ points and two distinct lines are incident with at most one point.
- For a point p and a line ℓ such that $p\notin\ell$, there is a unique pair (p',ℓ') such that $p\in\ell'\ni p'\in\ell$.

It is known that $GQ(q,q^2)$ exists when q is a prime power, and is unique for $q=2,3$.

The *point graph* of a generalized quadrangle, also denoted by $GQ(\sigma,\tau)$, has P as the set of vertices, and two vertices are joined by an edge if they are incident to the same line. It is a strongly regular graph with parameters

$$((\sigma\tau+1)(\sigma+1),\sigma(\tau+1),\sigma-1,\tau+1).$$

The complement of $GQ(q,q^2)$, denoted by $\overline{GQ}(q,q^2)$, is a strongly regular graph with parameters

$$((q+1)(q^3+1), q^4, q(q-1)(q^2+1), (q-1)q^3).$$

In particular, $\overline{GQ}(2,4)$ is the Schläfli graph and $GQ(3,9)$ is the first subconstituent of the McLaughlin graph (112 vertices). $\overline{GQ}(q,q^2)$ can be represented as a ball packing in dimension $f = q^3-q^2+q$, and the chromatic number of $\overline{GQ}(q,q^2)$ is at least q^3+1. Moreover,

Proposition. *The chromatic number of $\overline{GQ}(q,q^2)$ is exactly q^3+1.*

Proof. A *spread* of a generalized quadrangle is a set of lines such that each point is incident with a unique line in the set. By [PT09, Theorem 3.4.1(ii)], a generalized quadrangle $GQ(q,q^2)$ has spreads, meaning that the vertices of the graph $GQ(q,q^2)$ can be partitioned into q^3+1 cliques of size $q+1$. So the chromatic number of $\overline{GQ}(q,q^2)$ is at most q^3+1. □

This proves the Theorem. By the Lemma in Section 1, we have also constructed a ball packing in dimension q^3-q^2+q+1 with chromatic number q^3+3. This is, to the knowledge of the author, the first construction with non-constant difference $\chi(d)-d$.

Remark 3. Up to complement, the Clebsch graph, the Higman–Sims graph, the McLaughlin graph, the second subconstituent of the McLaughlin graph (162 vertices), and $\overline{GQ}(q,q^2)$ are the only known examples of Smith graphs [Smi75, CGS78]. They can be constructed from a rank 3 permutation group such that the stabilizer of a vertex has rank ≤ 3 on both subconstituents. It turns out that Smith graphs tend to have high chromatic number (the Clebsch graph is noticed by Cantwell [Can]).

Remark 4. Hoffman's bound is not always a good bound for the chromatic number. Our method does not recover, for example, Bondarenko's counter-example to the Borsuk conjecture. The author believes that there are other strongly regular graphs with high chromatic number. However, for the conjecture in Section 1, the power of strongly regular graphs might be very limited.

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