SCRIBABILITY PROBLEMS FOR POLYTOPES

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Abstract. In this paper we study various scribability problems for polytopes. We begin with the classical k-scribability problem proposed by Steiner and generalized by Schulte, which asks about the existence of d-polytopes that can not be realized with all k-faces tangent to a sphere. We answer this problem for stacked and cyclic polytopes for all values of d and k. We then continue with the weak scribability problem proposed by Grünbaum and Shephard, for which we are able to complete the work by Schulte by presenting non weakly circumscribable 3-polytopes. Finally, we propose new (i, j)-scribability problems, in a strong and a weak version, which generalize the classical ones. They ask about the existence of d-polytopes that can not be realized with all their i-faces "avoiding" the sphere and all their j-faces "cutting" the sphere. We are able to provide such examples for all the cases where $j - i \leq d - 3$.

Contents

Key words and phrases. Scribability; Stacked polytopes; Cyclic polytopes; Ball packing.

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Introduction

In this paper we consider various *scribability problems*, which include the classical k-scribability considered by Schulte [Sch87], the weak version introduced by Grünbaum and Shephard [GS87], and new concepts of strong and weak (i, j)-scribability that generalize the previous. We present a collection of techniques for either (i) showing that certain polytopes are non-scribable or (ii) constructing scribed realizations for prescribed families of combinatorial types. Scribability problems show the interplay between polytope theory, sphere configurations, and hyperbolic geometry, and the techniques used in our paper draw from all these topics.

The first scribability problem was asked by Steiner in 1832 [Ste81], when he proposed the inscribability problem for 3-dimensional polytopes. It can be formulated as follows: Does there exist a 3-dimensional polytope that is not *inscribable*? In other words, is there any 3-polytope that can not be realized with all its vertices on a sphere?

A counterexample was discovered nearly 100 years later, when Steinitz [Ste28] presented a technique to construct infinitely many non-inscribable 3-polytopes. For instance, the so-called *triakis tetrahedron*, obtained by gluing a tetrahedron on top of each facet of a tetrahedron, is not inscribable. By polarity, the polytope obtained by truncating all the vertices of a tetrahedron is not *circumscribable*, i.e. has no realization with all the facets tangent to the sphere. Inscribable 3-polytopes were characterized by Rivin [Riv96] (see also [HRS92, Riv93, Riv94, Riv03]) in terms of the hyperbolic dihedral angles, although a fully combinatorial description is still missing (cf. [DS96]).

Schulte [Sch87] considered *k-scribability problems*, higher-dimensional analogues of Steiner's problem: For $0 \leq k \leq d-1$, is there a d-polytope that can not be realized with all its k-faces tangent to a sphere? Leaving aside the trivial cases of $d \leq 2$, he constructed non-k-scribable examples for all the cases except for $d = 3$ and $k = 1$, i.e. 3-polytopes that are not *edge-scribable*. Surprisingly, it turns out that, as a consequence of the remarkable Koebe–Andreev–Thurston disk packing theorem, every 3-polytope does have a realization with all the edges tangent to the sphere (see [Zie07, Section 1.3] and references therein).

Our investigation on k-scribability problems, which we call *strong* to differentiate from the upcoming *weak* version, focuses on two important families of polytopes: *stacked polytopes* and *cyclic polytopes*.

By Barnette's Lower Bound Theorem [Bar71, Bar73], stacked polytopes have the minimum number of faces among all simplicial polytopes with the same number of vertices. Some stacked polytopes are among the first examples of non-inscribable 3-polytopes in Steinitz's work [Ste28]. Gonska and Ziegler [GZ13] fully characterized inscribable stacked polytopes, while Eppstein, Kuperberg and Ziegler [EKZ03] showed that stacked 4-polytopes are essentially not edge-scribable. In Section 4, we look at the other side of the story and prove the following result, which completely answers the strong k-scribability problems for stacked polytopes.

Theorem 1. *For any $0 \leq k \leq d-3$, there are stacked d-polytopes that are not k-scribable. However, every stacked polytope is $(d-1)$-scribable (i.e. circumscribable) and $(d-2)$-scribable (i.e. ridge-scribable).*

The proof of Theorem 1 is divided into three parts: Proposition 4.1 for $0 \leq k \leq d-3$, Proposition 4.3 for $k = d-1$ and Proposition 4.5 for $k = d-2$.

On the other hand, McMullen's Upper Bound Theorem [McM70] states that cyclic polytopes have the maximum number of faces among all polytopes with the same number of vertices. All cyclic polytopes are inscribable; see [Car11], [GS87, p. 67] [Sei91, p. 521] and [GZ13, Proposition 17]. The following theorem completely answers the strong k-scribability problems for cyclic polytopes.

Theorem 2. *For any $1 \leq k \leq d-1$, a cyclic d-polytope with sufficiently many vertices is not k-scribable.*

Theorem 2 is derived from more general results on (i, j)-scribability problems, namely Corollary 5.8 and Proposition 5.12.

The *weak k-scribability problems* were originally proposed by Grünbaum and Shephard [GS87] for 3-dimensional polytopes. It asks for realizations of polytopes with the affine hull of every k-face tangent to the sphere. Schulte [Sch87] considered, again, the analogues of this problem in higher dimensions. However, because the definition does not behave well under polarity, Schulte was not able to construct counterexamples for weak k-scribability of d-polytopes when $k \geq d-2$. In the current paper, we identify convex polytopes with pointed polyhedral cones in Lorentzian space; see Section 1 for details. By adopting this new point of view, we slightly modify the definition of weak k-scribability, and avoid the problem of polarity. This allows us to tie up the loose ends left by Schulte, and prove the following:

Theorem 3. *For $d \geq 3$ and $0 \leq k \leq d-1$ with the exception of $(d,k) = (3,1)$, there is a d-polytope that is not weakly k-scribable.*

We also propose new *(i,j)-scribability* problems, which ask about realizations of polytopes with all their i-faces "avoiding" the sphere and all their j-faces "cutting" the sphere. The definitions of cutting and avoiding, which come in a *strong* and a *weak* form, imply that (i,j)-scribability is well-behaved under polarity, and that (k,k)-scribability is nothing but classical k-scribability. This makes (i,j)-scribability, an interesting topic in its own right, also into a useful tool for classical k-scribability problems, as we will see with cyclic polytopes. In dimension 3, $(0,1)$-scribed polytopes have been studied as "hyperideal polyhedra" in hyperbolic space [BB02, Sch05].

The weak (i,j)-scribability turns out to be indeed quite weak, as we can see in the following theorem.

Theorem 4. *Every d-polytope is weakly (i,j)-scribable for $0 \leq i < j \leq d-1$.*

As for the strong (i,j)-scribability, we are able to construct examples that prove:

Theorem 5. *For $d > 3$ and $0 \leq i \leq j \leq d-1$, there are d-polytopes that are not strongly (i,j)-scribable for $j-i \leq d-2$ when d is even, or $j-i \leq d-3$ when d is odd.*

Theorem 5 follows from Proposition 5.6, which asserts that even dimensional cyclic polytopes with sufficiently many vertices are not strongly $(1,d-1)$-scribable. The proof of Proposition 5.6 uses the sphere separator theorem [MTTV97]. Similar techniques also prove that j-neighborly d-polytopes with many vertices are not $(1,j)$-scribable; see Proposition 5.12. This implies that:

Theorem 6. *For $d \geq 4$ and any $1 \leq k \leq d-2$, there are f-vectors such that no d-polytope with those f-vectors are k-scribable.*

For $k = d-3$, examples of such f-vectors are already given by stacked d-polytopes with more than $d+2$ vertices. See [EKZ03] for $d = 4$ and Proposition 4.1 for higher dimensions.

As for the (i,j)-scribability of stacked polytopes, we prove that

Theorem 7. *For any $d > 3$ and $0 \leq i \leq d-4$, there is a stacked d-polytope that is not $(i,i+1)$-scribable.*

An alternative construction for polytopes that are not strongly $(0,d-3)$-scribable is given in Proposition 6.3, which uses the Stamp Theorem of Below [Bel02] and Dobbins [Dob11].

The paper is organized as follows: Section 1 is dedicated to introduce the set-up for our scribability problems: polyhedral cones in Lorentz space and spherical polytopes. The different scribability problems that we work with are presented in Section 2. Section 4 contains our results about stacked and truncated polytopes, and Section 5 contains those about cyclic and neighborly polytopes. An alternative technique for constructing polytopes that are not (i,j)-scribable is given in Section 6. Finally, in Section 7, we present some problems that we leave open.

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1. Lorentzian view of polytopes

The classical scribability problems only consider bounded convex polytopes in Euclidean space. To define polarity properly in this set-up, one must assume that the polytope contains the origin in its interior. This presents the major difficulty in Schulte's work on weak k-scribability, and also leads to a minor flaw in his proof regarding strong k-scribability (see Remark 2.3).

We find it more natural and convenient to work with spherical polytopes, which arise from pointed polyhedral cones in Lorentzian space. The current section is dedicated to the introduction of this set-up. The main advantage is that, for spherical polytopes, polarity is always well-defined and well-behaved. This facilitates the study of weak scribability, which enables us to obtain Theorem 3. At the same time, as we will see in Lemma 2.2, strong scribability in spherical space and in Euclidean space are equivalent, so the new setting is compatible with previous studies. In fact, the presence of spherical geometry is necessary only in few occasions (e.g. Example 2.5).

1.1. **Convex polyhedral cones in Lorentzian space.** A (closed) non-empty subset of $\mathbb{R}^{d+1}$ is a *convex cone* if it is closed under positive linear combinations. A convex cone is *pointed* if it does not contain any subspace of $\mathbb{R}^{d+1}$. A convex cone is *polyhedral* if it is the conical hull of finitely many vectors in $\mathbb{R}^{d+1}$, i.e. a set $\mathcal{K}$ of the form

$$\mathcal{K} = \operatorname{cone}(V) := \left\{ \sum \lambda_i v_i \,\middle|\, \lambda_i \geq 0, v_i \in V \right\}.$$

for some finite set $V \subset \mathbb{R}^{d+1}$.

Let $\mathcal{K}$ be a convex cone and let $H \subset \mathbb{R}^{d+1}$ be a linear hyperplane disjoint from the interior of $\mathcal{K}$. We say that H is *supporting* for $\mathcal{K}$ if $H \cap \mathcal{K}$ contains non-zero vectors. In this case, the closed half-space H^- that contains $\mathcal{K}$ is called a *supporting half-space*.

If $\mathcal{K}$ is polyhedral, the intersections of $\mathcal{K}$ with its supporting hyperplanes are its *faces*. The set of faces ordered by inclusion forms the *face lattice* of $\mathcal{K}$. A polyhedral cone $\mathcal{K}'$ is *combinatorially equivalent* to $\mathcal{K}$ if their face lattices are isomorphic. In this case, we also say $\mathcal{K}$ and $\mathcal{K}'$ have the same *combinatorial type*, or that $\mathcal{K}'$ is a *realization* of $\mathcal{K}$.

The *Lorentzian space* $\mathbb{L}^{1,d}$ is $\mathbb{R}^{d+1}$ endowed with the Lorentzian scalar product:

$$\langle \mathbf{x}, \mathbf{y} \rangle := -x_0 y_0 + x_1 y_1 + \cdots + x_d y_d, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{d+1}.$$

The *light cone* of $\mathbb{L}^{1,d}$ is the pointed convex cone

$$\mathcal{L} := \left\{ \mathbf{x} \in \mathbb{L}^{1,d} \,\middle|\, \langle \mathbf{x}, \mathbf{x} \rangle \leq 0,\ x_0 \geq 0 \right\}.$$

Remark 1.1. In the literature, the term "*light cone*" usually denotes the set of vectors $\mathbf{x}$ such that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, i.e. the boundary of $\mathcal{L} \cup -\mathcal{L}$, which is not a convex cone.

We work with the polarity induced by the Lorentzian scalar product. Hence, the *polar* of a set $X \subseteq \mathbb{L}^{1,d}$ is the convex cone

$$X^* := \left\{ \mathbf{x} \in \mathbb{L}^{1,d} \,\middle|\, \langle \mathbf{x}, \mathbf{y} \rangle \leq 0 \text{ for all } \mathbf{y} \in \mathcal{K} \right\}.$$

We also define the *orthogonal companion* $X^\perp$ as the polar of its linear span

$$X^\perp := \operatorname{span}(X)^* = \{ \mathbf{x} \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in X \}.$$

The light cone is self-polar, i.e. $\mathcal{L}^* = \mathcal{L}$. If $\mathcal{V} \subset \mathbb{L}^{1,d}$ is a linear subspace, then $\mathcal{V}^* = \mathcal{V}^\perp$.

If $\mathcal{K}$ is a pointed polyhedral cone, then $\mathcal{K}^*$ is *combinatorially dual* to $\mathcal{K}$. That is, the face lattice of $\mathcal{K}^*$ is obtained from that of $\mathcal{K}$ by reversing the inclusion relations. Indeed, to each face F of $\mathcal{K}$ is associated a face $F^\diamond$ of $\mathcal{K}^*$, which we call the *associated face* of F, given by

$$F^\diamond := F^\perp \cap \mathcal{K}^*.$$

We recall some standard properties of polarity.

Lemma 1.2. *A subspace $\mathcal{V}$ is disjoint from $\mathcal{L}$ if and only if $\mathcal{V}^\perp$ intersects the interior of $\mathcal{L}$, and $\mathcal{V}$ is tangent to $\mathcal{L}$ (that is, $\mathcal{V} \cap \mathcal{L}$ consists of a single ray) if and only if $\mathcal{V}^\perp$ is tangent to $\mathcal{L}$.*

Lemma 1.3. *If* $\mathbf{x}$ *is a non-zero vector on the boundary of pointed polyhedral cone* $\mathcal{K}$ *and belongs to a face* F*, then* $\mathbf{x}^\perp$ *is a supporting hyperplane for* $\mathcal{K}^*$ *that contains the associate face* $F^\diamond$*, and* $\mathbf{x}^*$ *is the corresponding supporting half-space.*

Let F be a k-dimensional face of a pointed polyhedral cone $\mathcal{K}$. The *face figure* of F, denoted by $\mathcal{K}/F$, is the projection of $\mathcal{K}$ onto the quotient space $\mathcal{K}/\operatorname{span} F$. The combinatorial type of $\mathcal{K}/F$ is induced by the faces of $\mathcal{K}$ that contain F as a proper face. So $\mathcal{K}/F$ is combinatorially dual to the associated face $F^\diamond$.

1.2. **Spherical, Euclidean and hyperbolic polytopes.** Let $\mathbb{S}^d$ be the d-dimensional spherical space, identified with the set

$$\mathbb{S}^d = \left\{\mathbf{x} \in \mathbb{R}^{d+1} \,\middle|\, \|\mathbf{x}\|_2 = 1\right\}.$$

A *spherical polytope* in $\mathbb{S}^d$ is an intersection of finitely many hemispheres that does not contain any antipodal points. Every pointed polyhedral cone $\mathcal{K} \subset \mathbb{R}^{d+1}$ corresponds to a *spherical polytope* $\mathcal{P}$ in $\mathbb{S}^d$ given by $\mathcal{P} = \mathcal{K} \cap \mathbb{S}^d$, and every spherical polytope arises this way. The face lattice of $\mathcal{P}$ is inherited from $\mathcal{K}$, and the polar of $\mathcal{K}$ induces the polar of $\mathcal{P}$ given by $\mathcal{P}^* := \mathcal{K}^* \cap \mathbb{S}^d$. Note that a hyperplane in $\mathbb{S}^d$ is the intersection $H \cap \mathbb{S}^d$ where H is a linear hyperplane in $\mathbb{R}^{d+1}$, and a half-space in $\mathbb{S}^d$ is a hemisphere. The light cone appears as a spherical cap $\mathcal{B} := \mathcal{L} \cap \mathbb{S}^d$, whose boundary is a $(d-1)$-sphere "at latitude 45°N", which we denote by $\mathcal{S} := \partial\mathcal{B}$ to avoid confusion with the ambient d-sphere $\mathbb{S}^d$.

Every pointed polyhedral cone $\mathcal{K} \subset \mathbb{R}^{d+1}$ admits a *transversal hyperplane*, i.e. an affine hyperplane H intersecting every ray of $\mathcal{K}$. If we identify H with the Euclidean space $\mathbb{E}^d$, then the intersection $\mathcal{P} = \mathcal{K} \cap H$ is a bounded convex d-polytope in $\mathbb{E}^d$. Conversely, every d-dimensional Euclidean polytope can be *homogenized* to the $(d+1)$-dimensional pointed polyhedral cone $\hom(\mathcal{P}) := \{(\lambda, \lambda\mathbf{x}) \,|\, \mathbf{x} \in \mathcal{P}, \lambda \geq 0\}$. Again, the face lattice of $\mathcal{P}$ is inherited from $\mathcal{K}$, and the polar of $\mathcal{K}$ induces the polar of $\mathcal{P}$. Hence, from the combinatorial point of view, there is no difference between spherical polytopes and Euclidean polytopes.

We usually identify $\mathbb{E}^d$ with the fixed hyperplane $H_0 = \{\mathbf{x} \mid x_0 = 1\}$, so that the light cone $\mathcal{L}$ appears as the standard unit ball in $\mathbb{E}^d$. We abuse the notation and denote the Euclidean unit ball by $\mathcal{B}$ and its boundary by $\mathcal{S}$, as its spherical counterparts. Bounded Euclidean d-polytopes in $\mathbb{E}^d$ correspond to spherical d-polytopes contained in the hemisphere $x_0 > 0$ of $\mathbb{S}^d$ through central (gnomonic) projection from the origin. Furthermore, the polarity induced by the Lorentzian scalar product coincides with the classical polarity in Euclidean space. That is:

$$\mathcal{P}^* := \{\mathbf{x} \,|\, (\mathbf{x}, \mathbf{y}) \leq 1 \text{ for all } \mathbf{y} \in P\}$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product. For an affine subspace $H \subset \mathbb{E}^d$, we use $H^\perp$ to denote the *polar subspace*

$$H^\perp := \{\mathbf{x} \in \mathbb{E}^d \mid (\mathbf{x}, \mathbf{y}) = 1 \text{ for all } \mathbf{y} \in H\},$$

which corresponds to the orthogonal companion in $\mathbb{L}^{1,d}$.

The unit ball $\mathcal{B}$ in the Euclidean space $\mathbb{E}^d$ can also be seen as the Klein model of the hyperbolic space $\mathbb{H}^d$, then $\mathcal{P} \cap \mathcal{B}$ is a hyperbolic polytope. In the current paper, polytopes of interest have no vertex in the interior of $\mathcal{B}$, so they are not compact, or even of infinite volume in the hyperbolic space. The hyperbolic view is very useful for the study of stacked polytopes in Section 4. In particular, we will make use of hyperbolic reflection groups and hyperbolic dihedral angles for our proofs. Readers unfamiliar with hyperbolic polytopes are referred to [Vin93] or [Rat06].

Lorentz transformations are those linear transformations of $\mathbb{R}^{d+1}$ that preserve $\mathcal{L}$. Lorentz transformations of $\mathbb{L}^{1,d}$ correspond to projective transformations of $\mathbb{E}^d$ that preserve $\mathcal{S}$, or Möbius transformations of $\mathbb{S}^d$. If we regard the interior of $\mathcal{B}$ as the hyperbolic space $\mathbb{H}^d$, then the Möbius transformations correspond to hyperbolic isomorphisms of $\mathbb{H}^d$.

Remark 1.4. The term "*Lorentz transformation*" is usually used to denote those linear transformations that preserve the Lorentzian inner product. What we call a Lorentz transformation here, i.e. those preserving $\mathcal{L}$, correspond to what is usually known as the *orthochronous Lorentz transformations.*

We use the notation span(X) to denote the linear span in $\mathbb{L}^{1,d}$, the spherical span in $\mathbb{S}^d$, and the affine span in $\mathbb{E}^d$, depending on the context.

2. Definitions and properties

2.1. **Strong k-scribability.** The classical scribability problems go back to Steiner [Ste81], and were studied in full generality by Schulte [Sch87]. We will present them in terms of spherical polytopes, but as we will see soon, this formulation is equivalent to the classical setup.

Consider a polytope $\mathcal{P} \subset \mathbb{S}^d$ and let F be a face of $\mathcal{P}$. We say that F is *tangent* to $\mathcal{S}$ if $\operatorname{relint}(F) \cap \mathcal{S}$ consists of a single point, which is called the *tangency point* of F and denoted by t_F.

Definition 2.1. A spherical polytope $\mathcal{P}$ is *(strongly) k-scribed* if every k-face of $\mathcal{P}$ is tangent to $\mathcal{S}$, and *(strongly) k-scribable* if it has a k-scribed realization.

Here, the adjective *strong* is used to distinguish from the weak scribability, which will be defined later. We often omit the adjective since this type of scribability problem is of the earliest and greatest interest. We also say that a polytope $\mathcal{P}$ is *inscribable*, *edge-scribable*, *ridge-scribable* or *circumscribable* if it is 0-, 1-, $(d-2)$- or $(d-1)$-scribable, respectively.

If $\mathcal{P} \subset \mathbb{S}^d$ is contained in the upper hemisphere $x_0 > 0$, then it corresponds to a bounded polytope in $\mathbb{E}^d$ (identified with H_0), and our definition of "*k-scribed*" coincides with that of Schulte [Sch87]. However, since not every spherical polytope is in $x_0 > 0$, it is not straightforward to see that every k-scribable spherical polytope has a k-scribed realization in Euclidean space.

Lemma 2.2. *If a d-polytope $\mathcal{P} \subset \mathbb{S}^d$ is k-scribable, then $\mathcal{P}$ admits a k-scribed realization in the Euclidean space $\mathbb{E}^d$ that is bounded and contains the center of $\mathcal{S}$ in its interior.*

Remark 2.3. Before showing that our definition is indeed equivalent to that of Schulte [Sch87], let us first point out a related problem in Schulte's paper.

In [Sch87, p. 507f.], Schulte uses a footnote of [Grü03, p. 285] which says that, for any point x in the interior of $\mathcal{B}$, there is a projective transformation T_x that preserves $\mathcal{S}$ and sends x to the center of $\mathcal{S}$. Schulte argued that, if a polytope $\mathcal{P}$ does not contain the center of $\mathcal{S}$, one can send a point $x \in \mathcal{P} \cap \mathcal{B}$ to the center of $\mathcal{S}$ using T_x, and then $T_x\mathcal{P}$ is a polytope containing the center.

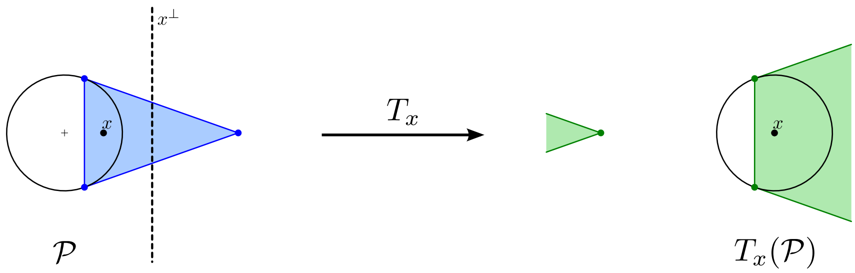


Figure 1. T_x destroys the boundedness of the polytope.

While this argument is correctly used in [Grü03], Schulte did not take precautions for the fact that, if the point x is not carefully chosen, $T_x\mathcal{P}$ might be unbounded (or even not connected). For example, consider the triangle $\mathcal{P}$ in Figure 1, constructed by taking the convex hull of a point together with the intersections of its polar line with the circle. For any point $x \in \mathcal{P} \cap \mathcal{B}$, the *polar hyperplane* $x^\perp$ is a line intersecting the triangle. As T_x sends x to the center, $x^\perp$ is sent to infinity, which destroys the boundedness.

In our proof of Lemma 2.2, which uses the same idea as Schulte's, we will carefully choose a suitable x for a k-scribed polytope.

Proof. Assume that $\mathcal{P}$ is k-scribed and choose an arbitrary k-face F of $\mathcal{P}$ with tangency point $t_F \in \mathcal{P}$. Observe that $F \subset H_F = t_F^\perp$ and $\mathcal{B} \subset H_F^- = {t_F}^*$.

We claim that H_F^- is also a supporting half-space for $\mathcal{P}$. The k-scribedness implies that the interior of every $(k+1)$-face of $\mathcal{P}$ intersects $\mathcal{B}$. So every $(k+1)$-face G of $\mathcal{P}$ containing F has some point in the open halfspace $H_F^- \setminus H_F$. Since $G \cap H_F = F$, this implies that H_F^- is supporting for all these faces, and hence also for $\mathcal{P}$.

Now, if x is a point sufficiently close to t_F on the segment connecting t_F and the center of $\mathcal{B}$, then $x \in \operatorname{int}(\mathcal{P} \cap \mathcal{B})$ and the polar hyperplane $x^\perp$ is disjoint from $\mathcal{P}$ and $\mathcal{B}$. Thus the projective transformation T_x will send $\mathcal{P}$ to a bounded k-scribed polytope containing the center of $\mathcal{S}$. □

As a consequence, k-scribability for Euclidean polytopes is equivalent to k-scribability for spherical polytopes.

2.2. **Weak k-scribability.** Weak k-scribability problems were first asked for 3-polytopes by Grünbaum and Shephard [GS87], and then generalized to higher dimensions by Schulte [Sch87, Section 3]. By considering spherical polytopes, the notion of weak scribability is weakened, but has the desired properties with respect to polarity.

Consider a polytope $\mathcal{P} \subset \mathbb{S}^d$, and let F be a face of $\mathcal{P}$. We say that F is *weakly tangent* to $\mathcal{S}$ if its spherical span is tangent to $\mathcal{S}$, i.e. if $\operatorname{span}(F) \cap \mathcal{S}$ consists of a single point.

Definition 2.4. A spherical polytope $\mathcal{P}$ is *weakly k-scribed* if every k-face of $\mathcal{P}$ is weakly tangent to $\mathcal{S}$, and *weakly k-scribable* if it has a k-scribed realization.

Definition 2.4 is weaker than the Euclidean definition of Grünbaum–Shephard and Schulte. The two definitions of strong k-scribability were equivalent thanks to Lemma 2.2 which show that for every strongly k-scribed polytope $\mathcal{P}$, there is a hemisphere (e.g. H_F^-) containing both $\mathcal{P}$ and $\mathcal{S}$. This is however not true for weakly k-scribed polytopes and in this case the spherical version is weaker.

For example, with the Grünbaum–Shephard definition, any weakly inscribed polytope is also strongly inscribed; see Schulte [Sch87]. This is not the case with Definition 2.4. The first example is, again, given by a stacked polytope. The *triakis tetrahedron*, which is the polytope obtained by stacking a vertex on top of every facet of a 3-dimensional simplex, is a well-known polytope that is not strongly inscribable [Ste28] (cf. [Grü03, Theorem 13.5.3], [GZ13]). However, it is weakly inscribable.

Example 2.5. The triakis tetrahedron is weakly inscribable (in the sense of Definition 2.4).

Proof. Consider the following eight vectors in $\mathbb{R}^4$.

$$v_{1,2} := (+\sqrt{2}, 0, \pm 1, 1) \qquad v_{3,4} := (+\sqrt{3}, \pm\sqrt{2}, 0, 1)$$
$$v_{5,6} := (-\sqrt{2}, \pm 1, 0, 1) \qquad v_{7,8} := (-\sqrt{3}, 0, \pm\sqrt{2}, 1).$$

They all satisfy $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. Hence for all $1 \leq i \leq 8$, the intersection of the line $\operatorname{span}(v_i)$ and the light cone $\mathcal{L} \subset \mathbb{R}^4$ is a single ray. This means that the intersection of $\mathcal{K} = \operatorname{cone}(v_1, \dots, v_8)$ with $\mathbb{S}^3$ is a weakly inscribed 3-polytope $\mathcal{P}$. Its combinatorial type is that of the triakis tetrahedron. □

Figure 2 illustrates the inscribed configuration. On the left, we visualize the intersection of $\mathcal{K}$ with the hyperplane $x_3 = 1$; notice that then the intersection of $\mathcal{L} \cup -\mathcal{L}$ with the hyperplane appears as a 2-sheet hyperboloid. On the right, we visualize it by considering the intersection of $\mathcal{K} \cup -\mathcal{K}$ with the hyperplane $x_0 = 1$; then we see two connected components, corresponding to $\mathcal{K}$ and $-\mathcal{K}$, respectively.

Nevertheless, Definition 2.4 has the desired property: the polar of a weakly k-scribable d-polytope is weakly $(d-1-k)$-scribable; see the upcoming Lemma 2.12(ii). This is precisely the missing piece that prevented Schulte from proving Theorem 3.

2.3. **Strong and weak (i,j)-scribability.** In this section, we present the new concept of (i,j)-scribability, which generalizes the concept of k-scribability presented above. Instead of asking for realizations with the k-faces tangent to the sphere, we are interested in realizations with the i-faces "avoiding" the sphere and the j-faces "cutting" it, in such a way that (i,j)-scribability reduces to i-scribability when $i = j$. As before, the definitions come in strong and weak forms.

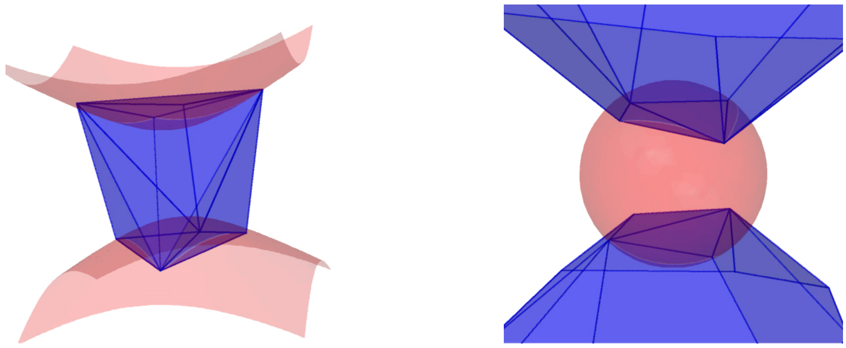


FIGURE 2. The inscribed triakis tetrahedron $\mathcal{P}$, view of $\pm\mathcal{P}$ and $\pm\mathcal{S}$ projected on the hyperplane $x_3 = 1$ (left), and on $x_0 = 1$ (right).

Definition 2.6. Consider a spherical polytope $\mathcal{P} \subset \mathbb{S}^d$, and let F be a proper face of $\mathcal{P}$.

We say that F

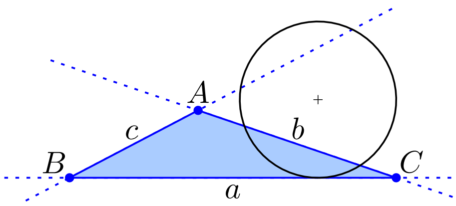
- *strongly cuts* $\mathcal{S}$ if $\operatorname{relint}(F) \cap \mathcal{B} \neq \emptyset$;
- *weakly cuts* $\mathcal{S}$ if $\operatorname{span}(F) \cap \mathcal{B} \neq \emptyset$;

We say that F

- *strongly avoids* $\mathcal{S}$ if there is a supporting hyperplane H of $\mathcal{P}$ such that $F = H \cap \mathcal{P}$ and $\mathcal{B} \subset H^-$.
- *weakly avoids* $\mathcal{S}$ if $\operatorname{span}(F) \cap \operatorname{int} \mathcal{B} = \emptyset$;

where $\operatorname{int} \mathcal{B} = \mathcal{B} \setminus \mathcal{S}$ is the interior of the ball $\mathcal{B}$.

Example 2.7. Consider the following triangle.



The edge a strongly avoids and cuts $\mathcal{S}$. The edge b strongly cuts $\mathcal{S}$ and does not avoid $\mathcal{S}$ in any sense. The edge c weakly cuts $\mathcal{S}$ as shown by the dashed line. The vertex A weakly avoids $\mathcal{S}$. The vertices B and C strongly avoid $\mathcal{S}$ and do not cut $\mathcal{S}$ in any sense.

In the following, in order to ease the text, the phrase "in the strong (resp. weak) sense" means that the adverb "strongly" (resp. "weakly") is implied wherever applicable in the context.

Definition 2.8. Let $0 \leq i \leq j \leq d-1$. In the *strong* or *weak* sense, a spherical d-polytope $\mathcal{P} \subset \mathbb{S}^d$ is *i-avoiding* if every i-face of $\mathcal{P}$ avoids $\mathcal{S}$, and *j-cutting* if every j-face of $\mathcal{P}$ cuts $\mathcal{S}$. We say that $\mathcal{P}$ is *(i,j)-scribed* if it is i-avoiding and j-cutting, and *(i,j)-scribable* if it has an (i,j)-scribed realization.

Remark 2.9. These notions should not be confused with a (m,d)-scribable polytope in the sense of Schulte [Sch87], which means a d-polytope that is m-scribable.

Remark 2.10. There is a third even weaker version of scribability that seems reasonable at first sight, where faces are *feebly cutting* if they are not strongly avoiding and *feebly avoiding* if they are not strongly cutting. However, observe that we can always find such a feebly $(d-1,0)$-scribed realization of any polytope P by simply taking a realization of P with all the vertices in $-\mathcal{B} \subset \mathbb{S}^d$, and thus this feeble version of scribability is trivial.

2.4. **Properties of (i,j)-scribability.** A first easy observation is that the strong versions are indeed stronger than the weak versions. That is, the strong forms of cutting, avoiding and scribiba-bility imply the weak forms.

Lemma 2.11. *In the strong or weak sense, a face that cuts and avoids $\mathcal{S}$ is tangent to $\mathcal{S}$. Consequently, a (k,k)-scribed polytope is k-scribed.*

We say that a face is *strictly* cutting (resp. avoiding) $\mathcal{S}$ if it is cutting (resp. avoiding) $\mathcal{S}$ but not tangent to $\mathcal{S}$. In the strong sense, it is possible that a face is neither cutting nor avoiding $\mathcal{S}$ at the same time. This is however not possible in the weak sense.

The following lemma collects the essential properties of (i,j)-scribability, which are repeatedly used in the upcoming proofs. It is a generalized version of the main results in [Sch87], Theorems 1 and 2. It implies also the weak version of [Sch87, Theorem 1], needed for proving Theorem 3.

Lemma 2.12. *Let $d \geq 1$ and $0 \leq i,j \leq d-1$. In the strong or weak sense, if a d-polytope $\mathcal{P}$ is (i,j)-scribable, then:*

(i) *$\mathcal{P}$ is (i',j')-scribable for any $i' \leq i$ and $j' \geq j$;*
(ii) *the polar polytope $\mathcal{P}^*$ is $(d-1-j, d-1-i)$-scribable;*
(iii) *if $j \leq d-2$, each facet of $\mathcal{P}$ is an (i,j)-scribable $(d-1)$-polytope;*
(iv) *if $i \geq 1$, each vertex-figure of $\mathcal{P}$ is an $(i-1,j-1)$-scribable $(d-1)$-polytope.*

Proof.

(i) Assume that $\mathcal{P}$ is strongly j-cutting, we need to prove that $\mathcal{P}$ is also j'-cutting for any $j' \geq j$. Indeed, for any $(j+1)$-face F, take a point in $\operatorname{relint} F' \cap \mathcal{B}$ for each j-face F' incident to F, then the barycenter of these points is in $\operatorname{relint} F \cap \mathcal{B}$. By polarity we see that if $\mathcal{P}$ is strongly i-avoiding, then $\mathcal{P}$ is also i'-avoiding for any $i' \leq i$. This proves the strong version of the statement. The weak version follows similarly, just replace relint by span.
(ii) The weak version follows directly from Lemma 1.2. For the strong version, one also needs Lemma 1.3.
(iii) For either strong or weak version, notice from the proof of (i) that $\mathcal{P}$ is strictly $(d-1)$-cutting in the weak sense. Let F be a facet. If we identify $\operatorname{span} F \subset \mathbb{S}^d$ with $\mathbb{S}^{d-1}$, then $F \subset \mathbb{S}^{d-1}$ is an (i,j)-scribed $(d-1)$-polytope with respect to the sphere $\operatorname{span} F \cap \mathcal{S}$.
(iv) By polarizing (iii). □

In the remaining of the paper, our mission is two-fold: for each triple (d,i,j) we either try to prove that every d-polytope is (i,j)-scribable, or we try to construct an example of a d-polytope that is not (i,j)-scribable. In view of Lemma 2.12(i), we seek to construct polytopes that are not (i,j)-scribable with $j-i$ as large as possible, or to prove that every d-polytope is (i,j)-scribable for $j-i$ as small as possible. The remaining items of Lemma 2.12 are used to do induction on the dimension d.

We end this section by proving that the Euclidean and spherical definitions of strong (i,j)-scribability are equivalent. When $i<j$, the analogue of Lemma 2.2 does not guarantee simultaneously that $\mathcal{P}$ is bounded and contains the origin. However, boundedness suffices for the equivalence between the Euclidean and the spherical setup.

Lemma 2.13. *If a d-polytope $\mathcal{P}$ is strongly (i,j)-scribable, with $0 \leq i \leq j \leq d-1$ then $\mathcal{P}$ admits a strongly (i,j)-scribed bounded realization in Euclidean space $\mathbb{E}^d$.*

Proof. We prove the dual statement, i.e. $\mathcal{P}$ admits a strongly (i,j)-scribed realization with the center of $\mathcal{S}$ in its interior. Since $\mathcal{P}$ is strongly (i,j)-scribed, each facet contains a point of $\mathcal{B}$. The barycenter of these points is interior to both $\mathcal{P}$ and $\mathcal{B}$, and can be sent to the center of $\mathcal{S}$ with a Möbius transformation. This gives a realization of $\mathcal{P}$ containing the center, then the polarity yields a bounded realization of $\mathcal{P}^*$. □

3. Weak scribability

In this section we concentrate on the investigation of weak (i,j)-scribability. On the one hand, we show in Theorem 3 that there are d-polytopes that are not weakly k-scribable except for $d=3$ and $k=1$ or $d \leq 2$, since a 3-polytope is already strongly 1-scribable. On the other hand, we show in Theorem 4 that when $i<j$, every polytope is weakly (i,j)-scribable.

3.1. **Weak k-scribability.** Despite Example 2.5, there are 3-polytopes that are not weakly inscribable. We provide two constructions.

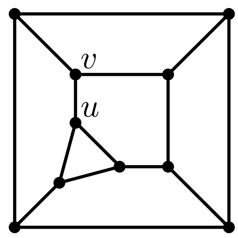


FIGURE 3. The truncated cube is not weakly inscribable.

Example 3.1. The 3-polytope $\mathcal{P}$ obtained by truncating one vertex of a 3-cube is not weakly inscribable.

Proof. Let $\mathcal{K}$ be the polyhedral cone spanned by a spherical realization of $\mathcal{P}$ and let H be a transversal hyperplane of $\mathcal{K}$. If we identify the Euclidean space $\mathbb{E}^d$ with H (instead of with H_0 as usual), the polytope $\mathcal{P}' = H \cap \mathcal{K}$ is a bounded polytope, and the "sphere" $\mathcal{S} = H \cap (\partial\mathcal{L} \cup -\partial\mathcal{L})$ appears as a quadric in $\mathbb{E}^d$. Since $\mathcal{P}$ is weakly inscribed, the vertices of $\mathcal{P}'$ are all on the quadric $\mathcal{S}$.

It is well known that if seven vertices of a (combinatorial) 3-cube lie on a quadric, so does the eighth one [BS08, Section 3.2]. We can recover a 3-cube by removing the truncating facet of $\mathcal{P}'$. Let w be the 8th vertex of this cube (the one that does not belong to $\mathcal{P}'$). Then w and the points u and v from Figure 3 all lie on the quadric. If a quadric contains 3 collinear points, then it contains a whole line (see [BS08, Ex 3.7]). However, since $0 \notin H$, our conic section does not contain lines; a contradiction. □

We are now ready to prove Theorem 3, which we repeat below.

Theorem 3. *Except for the case $d = 3, k = 1$, for every $d \geq 3$ and $0 \leq k \leq d - 1$, there is a d-polytope that is not weakly k-scribable.*

Schulte [Sch87, Sec. 3] gave the proof for $k \leq d - 3$, but remarked that the remaining cases would follow from the existence of polytopes that are not weakly circumscribable. The polar of the truncated cube is such a polytope.

Proof. The proof is by induction on d and follows the same steps as Schulte's.

In dimension 3, the truncated cube is not weakly inscribable, and its polar is not circumscribable by 2.12(i) (the weak version with $i = j = k$). The pyramid over the truncated cube is a 4-polytope that has a truncated cube as a facet and as a vertex figure, so it is not weakly k-scribable for $k = 0, 1$ by Lemma 2.12(ii) and (iii) (the weak version with $i = j = k$). Its polar has a stacked octahedron as a facet and as a vertex figure, so it is not weakly k-scribable for $k = 2, 3$.

In higher dimensions, the pyramid over a $(d - 1)$-polytope that is not weakly k-scribable gives a d-polytope that is neither k- nor $(k + 1)$-scribable. So the theorem follows by induction. □

Recall that although we work in spherical space, our definition of weak k-scribability is weaker than the one for the Euclidean setting used by Schulte. Consequently, examples for Theorem 3 are not weakly k-scribable in Euclidean space, neither. This finishes Schulte's work.

We now present an alternative construction. Despite the absence of Lemma 2.12(ii) in Euclidean space, it is possible to bypass the spherical geometry, and construct polytopes that are not weakly circumscribable directly within the Euclidean setup of [Sch87]. Our construction is based on the following lemma:

Lemma 3.2. *Any weakly circumscribed Euclidean polygon with more than 4 edges is also strongly circumscribed.*

Proof. Let $\mathcal{P}$ be a weakly circumscribed polygon in Euclidean space. A half-plane L^- supporting an edge of $\mathcal{P}$ is either of the form $\{\langle a, x\rangle \leq -1\}$ for some unit vector a, in which case we call the edge *separating* because $\mathcal{P}$ and $\mathcal{B}$ lie at opposite sides of the supporting line, or of the form $\{\langle a, x\rangle \leq 1\}$, and then $\mathcal{B} \cup \mathcal{P} \subset L^-$ and we call the edge *non-separating*. Observe that $\mathcal{P}$ is not strongly circumscribed if and only if it has a separating edge.

Assume that $\mathcal{P}$ has three separating edges. If the unit normal vectors were positively dependent, the intersection of the half-planes would be empty. Hence, we may assume that one of them can be written as a linear combination of the other two with positive coefficients, but then one can easily see that this inequality is redundant.

We also claim that, with the presence of a separating edge, $\mathcal{P}$ can have at most two non-separating edges. To see this, consider one separating edge with unit vector a_0 and two non-separating edges with vectors a_1, a_2. A linear relation of the form $a_1 = \lambda a_0 + \mu a_2$ with positive coefficients is not possible. Otherwise, a small computation shows that the inequality defined by a_1 is redundant. The forbidden linear relation is however inevitable if $\mathcal{P}$ has three non-separating edges.

Hence, if $\mathcal{P}$ has a separating edge, then it can have at most four edges, two of each kind. The different situations are illustrated in Figure 4. □

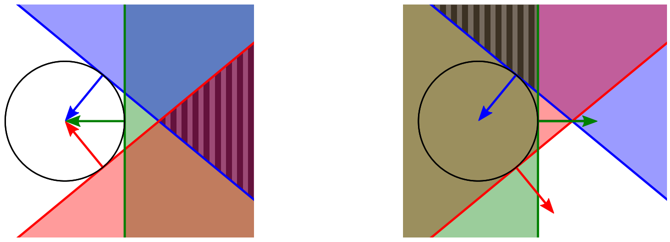


FIGURE 4. Redundant inequalities for the proof of Lemma 3.2. The depicted vectors are outer normal vectors and point away from the half-space. The stripped region is the intersection of all the half-spaces. Notice that in both situations there is a half-space containing the intersection of the other two.

The same proof carries over almost directly to spherical polytopes. The correct statement in the spherical setting would be: If $\mathcal{P} \subset \mathbb{S}^2$ is a weakly circumscribed spherical polygon with more than 4 edges, then either $\mathcal{P}$ or $-\mathcal{P}$ is strongly circumscribed.

Example 3.3. Let $\mathcal{P}$ be the polytope obtained by truncating all the vertices of a tetrahedron, then stacking on each of the newly created facets, and then stacking again on each of the newly created facets (Figure 5). Then $\mathcal{P}$ is not weakly circumscribable.

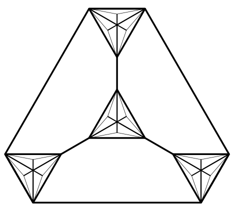


FIGURE 5. This polytope is not weakly circumscribable.

Proof. We start by showing that $\mathcal{P}$ is not strongly circumscribable. First of all, the truncated tetrahedron is not strongly circumscribable (it is polar to the triakis tetrahedron). For any facet arising from the truncation, we can replace the supporting half-space by any supporting half-space of one of the simplices stacked on the facet, and the result is still a truncated tetrahedron. Hence $\mathcal{P}$ is not strongly circumscribable.

Now assume that $\mathcal{P}$ is weakly circumscribed but not strongly circumscribed. Then there is a facet F whose supporting half-space does not contain $\mathcal{B}$. Every facet is incident to a vertex of degree 6. Let v be such a vertex incident to F, then the vertex figure at v is a weakly circumscibed hexagon that is not strongly circumscribed, contradicting Lemma 3.2. $\square$

The same method can be applied to many other 3-polytopes proven to be non-circumscribable by Steinitz (cf. [Grü03, Theorem 13.5.2]) to get an infinite family of 3-polytopes that are not weakly circumscribable. More specifically, if a simple 3-polytope has more vertices than facets, then truncating the vertices yields a polytope that is not strongly circumscribable, and stacking twice on the truncated facets provides a polytope that is not weakly circumscribable.

3.2. **Weak (i, j)-scribability.** We end by dealing with the remaining cases and showing that weak (i, j)-scribability is indeed very weak.

Theorem 4. *Every d-polytope is weakly (i, j)-scribable for $0 \leq i < j \leq d-1$.*

Proof. It suffices to show that every d-polytope $\mathcal{P}$ is weakly $(i, i+1)$-scribable for all $0 \leq i \leq d-2$. Consider a Euclidean realization of $\mathcal{P}$ and a generic affine subspace L of dimension $d-i$ that does not intersect $\mathcal{P}$. Then L intersects the affine span of every $(i+1)$-face of $\mathcal{P}$ at a single point, but does not intersect the affine span of any i-face. We can find, in the neighborhood of L, an ellipsoid $\mathcal{E}$ that contains all these intersection points but remains disjoint from the affine span of every i-face. The affine transformation that sends $\mathcal{E}$ to the unit ball $\mathcal{B}$ sends $\mathcal{P}$ to a weakly $(i, i+1)$-scribed realization. $\square$

4. Stacked polytopes

From now on, we focus only on strong scribability. In the remaining of the paper, we often omit the adjective "*strong*" , which has to be understood whenever we talk about (i, j)-scribability unless explicitly stated otherwise.

The goal of the current section is to answer the scribability problems for stacked polytopes. That is, we want to know which are the values of k such that every stacked d-polytope is k-scribable.

We recall the definitions of stacking and stacked polytope. For a d-polytope $\mathcal{P}$ with a simplicial facet F, we *stack* a vertex onto F by taking the convex hull of $\mathcal{P} \cup p$ for some point p close enough to the barycenter of F. In terms of the *connected sum* (cf. Section 4.2), this corresponds to gluing $\mathcal{P}$ and a simplex d-simplex Δ along F. A *stacked polytope* is any polytope obtained from a simplex by repeatedly stacking vertices onto facets. The dual of a stacked polytope is a *truncated polytope*, obtained from a simplex by repeatedly cutting off vertices.

A stacked polytope $\mathcal{P}$ of dimension $d \geq 3$ has a unique triangulation $\mathcal{T}$ with no interior faces of dimension $< d-1$, the *stacked triangulation*. The *dual tree* of $\mathcal{T}$ takes the maximal simplices in $\mathcal{T}$ as vertices and connects two vertices if they share a face of dimension $d-1$.

Already in dimension 3, stacked polytopes provide the first examples of polytopes that are not inscribable [Ste28]. Gonska and Ziegler [GZ13] proved that a stacked polytope is inscribable if and only if all the nodes of its dual tree have degree ≤ 3. In higher dimensions, Eppstein, Kuperberg and Ziegler [EKZ03] proved that no stacked 4-polytope on more than 6 vertices is edge-scribable.

While it is well known stacked polytopes present obstructions to inscribability and edge-scribability, the other side of the story seems to have escaped the attention of the community. In this section, we show that stacked polytopes are actually always circumscribable and ridge-scribable. On the other hand, stacked polytopes that are not k-scribable exist for any other smaller k.

Proposition 4.1. *For any $d > 3$ and $0 \leq k \leq d-3$, there is a stacked d-polytope that is not k-scribable.*

Proof. In [GZ13] it is proved that, for every $d \geq 3$, there is a stacked d-polytope that is not inscribable, which solves the case $k = 0$. We conclude the proposition by induction using Lemma 2.12(ii) and the fact that every stacked d-polytope is the vertex-figure of a stacked $(d+1)$-polytope, and a k-face figure of a stacked $(d+1+k)$-polytope. $\square$

We can strengthen this statement for the case $d > 3$ and $k = d - 3$.

Proposition 4.2. *For $d > 3$, no stacked d-polytope with more than $d + 2$ vertices is $(d - 3)$-scribable.*

Proof. Any stacked d-polytope $\mathcal{P}$ with more than $d + 2$ vertices admits a vertex figure with the combinatorial type of a stacked $(d - 1)$-polytope with more than $d + 1$ vertices. To see this, first notice that the vertex figure at any vertex of $\mathcal{P}$ has the combinatorial type of a stacked $(d - 1)$-polytope. Since $\mathcal{P}$ has more than $d+2$ vertices, it is not a simplex nor a bypiramid, hence the dual tree of $\mathcal{P}$ has a node of degree ≥ 2. Thus there are three simplices in the stacked triangulation sharing a ridge of $\mathcal{P}$. For any vertex in this ridge, the vertex figure contains at least $d+2$ vertices.

By Lemma 2.12(ii), if $\mathcal{P}$ is $(d-3)$-scribable, its vertex figures are all $(d-4)$-scribable. The theorem then follows by induction since no 4-polytope on more than 6 vertices is edge-scribable [EKZ03, Corollary 9]. □

4.1. **Circumscribability.**

Proposition 4.3. *Every stacked polytope is circumscribable.*

Proof. We prove by explicit construction the dual version of the proposition, namely that every truncated polytope is inscribable.

Let $\mathcal{P}$ be a truncated polytope, obtained from a d-simplex $\mathcal{P}_0$ by repeatedly truncating vertices. We start with an inscribed realization of $\mathcal{P}_0$.

In the first step, we perform simultaneously all the truncations that remove vertices of $\mathcal{P}_0$. This is carried out as follows: For every vertex v to be truncated, we pull v towards the exterior of $\mathcal{S}$ by a sufficiently small distance. Let $\mathcal{P}_0'$ be the adjusted simplex. Then the sphere $\mathcal{S}$ intersects the edges of $\mathcal{P}_0'$ near the adjusted vertices, while the other vertices remain on the sphere. The convex hull of these intersection points gives the desired truncated polytope $\mathcal{P}_1$ with all its vertices on $\mathcal{S}$. Observe that every vertex not present in $\mathcal{P}_0$ is incident to a (essentially unique) simplicial facet of $\mathcal{P}_1$.

Now, let $\mathcal{P}_k$ be the truncated polytope we obtain after the first k steps. In the $(k + 1)$-th step, we perform simultaneously all the truncations that remove vertices of $\mathcal{P}_k$. The proof is by induction on k. We assume that all the vertices of $\mathcal{P}_k$ are situated on the sphere $\mathcal{S}$, and that every vertex v not present in $\mathcal{P}_{k-1}$ is incident to a simplicial facet F_v of $\mathcal{P}_k$. These assumptions have been verified for $k = 1$.

Consider all the vertices v of $\mathcal{P}_k$ to be truncated in the $(k + 1)$-th step. We pull every such v by a sufficiently small distance towards the exterior of the sphere $\mathcal{S}$ along the edge e_v that is incident to v but does not belong to F_v. These movements do not change the combinatorial type of $\mathcal{P}_k$. Let $\mathcal{P}_k'$ be the adjusted polytope. Then the sphere $\mathcal{S}$ intersects the 1-skeleton of $\mathcal{P}_k'$ near the adjusted vertices, while the other vertices remain on the sphere. The convex hull of these intersection points gives the desired truncated polytope $\mathcal{P}_{k+1}$ with all its vertices on $\mathcal{S}$, and every newly created vertex is incident to a (essentially unique) simplical facet of $\mathcal{P}_{k+1}$.

It then follows by induction that $\mathcal{P}$ has an inscribed realization. The procedure is sketched in Figure 6. □

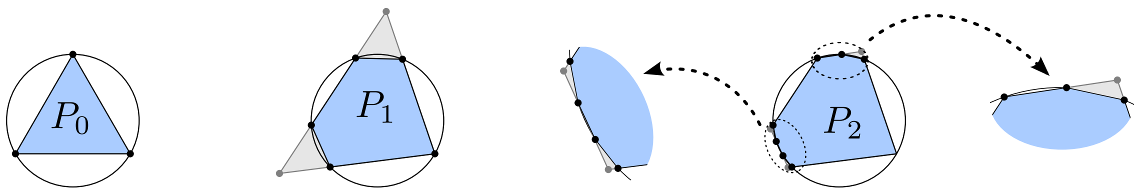


FIGURE 6. Steps of the proof of Proposition 4.3

The fact that every newly created vertex is incident to a simplex facet is critical for this proof. It allows us to pull the vertices independently without changing the combinatorial type of the polytope.

Observe that this proof also works if we replace the sphere by any strictly convex surface. This means that truncated polytopes are *universally inscribable* in the sense of [GP15].

Proposition 4.4. *Every truncated polytope has a realization with all its vertices on any given strictly convex surface.*

And the construction also works in more general settings. For example, it works if $\mathcal{P}_0$ is an inscribable simple polytope (e.g. cube) and every vertex of $\mathcal{P}_0$ is truncated in the first step. In this case, the first step can be carried out by shrinking the sphere by a sufficiently small amount. From this moment, every vertex is adjacent to a a simplex facet and the remaining steps remain as described in the proof.

4.2. Ridge-scribability.

Proposition 4.5. *Every stacked polytope is ridge-scribable.*

Proof. Let Δ be a ridge-scribed realization of the simplex. The interior of $\mathcal{B}$ can be regarded as the Klein model of d-dimensional hyperbolic space. The tangency points of the ridges are all ideal in hyperbolic space, so the facets are all parallel or ultraparallel. Then the hyperbolic reflections in the facets of $\mathcal{P}$ generate a universal Coxeter group W. The associated Coxeter diagram is a complete graph with label ∞ on all the edges. The fundamental domain of W is Δ. See [Vin67, Vin71] for more details on hyperbolic Coxeter groups. Simplices in the orbit $W(\Delta)$ form a simplicial complex called the *Coxeter complex*; see [AB08].

Stacked polytopes can be seen as strongly connected subcomplexes of the Coxeter complex. To see this, notice that for any $w \in W$, the simplex $w(\Delta)$ is a ridge-scribed simplex in the Euclidean view. The dual graph of $W(\Delta)$ is the Cayley graph of W, which is a tree. The convexity is guaranteed by the fact that, for any hyperplane H that is tangent to $\mathcal{B}$ and contains a ridge of $W(\Delta)$, $W(\Delta)$ is contained in the halfspace H^-. Ridge-scribed realizations for stacked polytopes are therefore given by the Coxeter complex. □

Inspired by this proof, we now extend the proof to a generalization of stacked polytopes. Let $\mathcal{P}$ be a ridge-scribed polytope. The reflections in the facets of $\mathcal{P}$ again form a hyperbolic Coxeter group. The Coxeter complex is a polytopal cell complex, each cell being a copy of $\mathcal{P}$. Every strongly connected subcomplex of the Coxeter complex again forms a convex polytope, which we call a *stacked $\mathcal{P}$-polytope*. Then the same argument proves that

Proposition 4.6. *Stacked $\mathcal{P}$-polytopes are ridge-scribable if $\mathcal{P}$ is.*

We can further extend the proof to connected sums of polytopes. Recall that two polytopes are *projectively equivalent* if there is a projective transformation that sends one to the other. Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes with facets projectively equivalent to F, then the *connected sum* of $\mathcal{P}$ and $\mathcal{Q}$ through F, denoted $\mathcal{P}\#_F\mathcal{Q}$, is obtained by "gluing" $\mathcal{P}$ and $\mathcal{Q}$ by identifying the projectively equivalent facets (see [RG96, Section 3.2]). So the operation of stacking is actually taking connected sum with a simplex.

We say that two polytopes are *Möbius equivalent* if there is a Möbius transformation (projective transformation preserving $\mathcal{S}$) that sends one to the other. Then

Proposition 4.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be ridge-scribed polytopes with facets Möbius equivalent to F, then the connected sum $\mathcal{P}\#_F\mathcal{Q}$ is ridge-scribable.*

Proof. With a Möbius transformation if necessary, we may assume that $\mathcal{P}$ and $\mathcal{Q}$ are ridge-scribed and $\mathcal{P}\cap\mathcal{Q} = F$. For any ridge R adjacent to F, the hyperplane that is tangent to $\mathcal{S}$ and contains R is supporting both for $\mathcal{P}$ and for $\mathcal{Q}$ (see the proof of Lemma 2.2). So the polytope $\mathcal{P}\#_F\mathcal{Q} = \mathcal{P}\cup\mathcal{Q}$ is convex and ridge-scribed by construction. □

Ridge-scribed simplices are all Möbius equivalent; see [EKZ03, Lemma 7] for the dual statement. We can therefore regard Proposition 4.5 as a special case of Proposition 4.7.

Finally, we prove an interpretation of Proposition 4.5 in terms of ball packings. For every point $\mathbf{x} \in \mathbb{E} \setminus \mathcal{B}$, the part of $\mathcal{S}$ visible from $\mathbf{x}$ is a spherical cap on $\mathcal{S}$. For an edge-scribed polytope, the

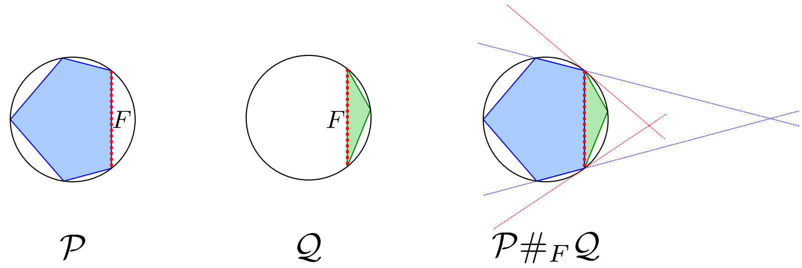


FIGURE 7. The relevant tangent hyperplanes and facet defining hyperplanes in a connected sum of ridge-scribed polytopes through a Möbius equivalent facet.

caps corresponding to the vertices have disjoint interiors. After a stereographic projection, they form a ball packing in Euclidean space whose tangency graph is isomorphic to the 1-skeleton of the polytope; see [Che13]. The dual version of Proposition 4.5 says that every truncated polytope is edge-scribable. Therefore,

Corollary 4.8. *The* 1*-skeleton of every truncated d-polytope is the tangency graph of a ball packing in dimension $d-1$.*

Here we provide a self-consistent proof independent to Proposition 4.5. In fact, the two proofs are essentially the same, as inversions correspond to hyperbolic reflections; see [Che14].

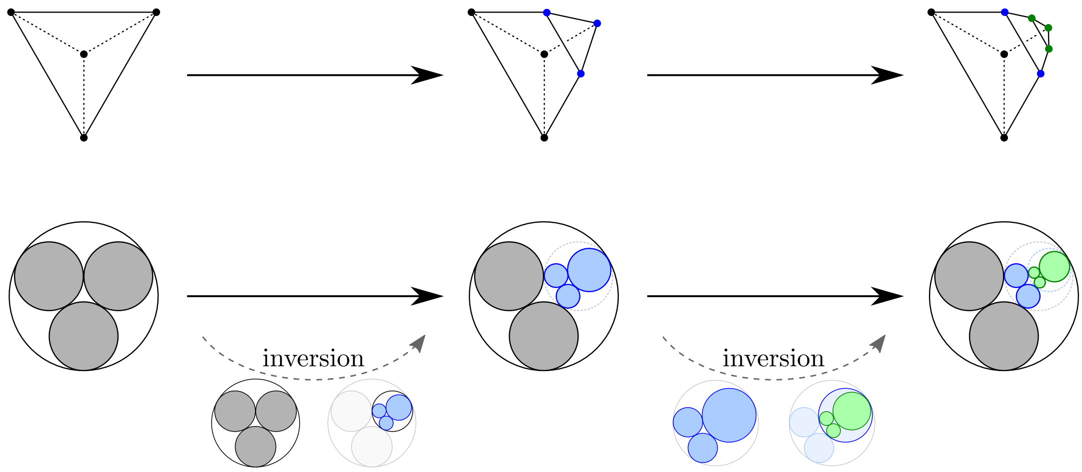


FIGURE 8. The graph of a truncated polytope as a ball packing through a series of inversions.

Proof. We construct the truncated polytope and the ball packing in parallel, and keep the 1-skeleton and the tangency graph isomorphic. We begin with an edge-tangent d-simplex Δ and the corresponding configuration of $d+1$ pairwise tangent balls $B_0, \dots, B_d$ of dimension $d-1$.

We now proceed by induction, truncating the vertices one by one. Let B be the ball corresponding to the vertex to be truncated.

If B is in the initial packing of $d+1$ balls, say $B = B_0$, then the inversion in ∂B sends $B_1, \dots, B_d$ into pairwise tangent balls $B'_1, \dots, B'_d$ in the interior of B, and B'_i is tangent to B_i, $1 \le i \le d$. Since the tangency points are preserved, replacing B by these primed balls gives the desired packing.

Otherwise, B appeared in some latter truncation step, when a ball B'_d was replaced by d pairwise tangent balls $B'_0, \dots, B'_{d-1}$. Say $B = B'_0$. They are all tangent from the interior to $\partial B'_d$, which we can regard the exterior of B'_d as a ball of negative curvature. The inversion in ∂B sends $B'_1, \dots, B'_d$ into pairwise tangent balls $B''_1, \dots, B''_d$ in the interior of B. Again, the tangency points of these balls with B coincide with those of B with its neighbors. Replacing B by these doubly primed balls gives the desired packing.

The corollary then follows by induction. See also Figure 8. □

4.3. **The $(i, i+1)$ scribability.** Finally, we prove Theorem 7 by constructing stacked d-polytopes that are not $(i, i+1)$-scribed for $0 \le i \le d-4$.

We still regard the interior of $\mathcal{B} \subset \mathbb{E}^d$ as the Klein model of the hyperbolic space $\mathbb{H}^d$. To study the vertex figure $\mathcal{P}/v$, consider a surface Σ that intersects perpendicularly all the hyperbolic geodesics that passes through v (in $\mathbb{E}^d$). There are three cases: If v is a point of $\mathbb{H}^d$ (in the interior of $\mathcal{B}$), Σ is a $(d-1)$-sphere $\mathbb{S}^{d-1}$ centered at v; this is more clear if we assume v is the center of $\mathcal{B}$, or simply use the Poincaré ball model for $\mathbb{H}^d$. If v is an ideal point of $\mathbb{H}^d$ (on the boundary of $\mathcal{B}$), then Σ is a horosphere based at v, which can be identified to the Euclidean space $\mathbb{E}^{d-1}$; this can be easily seen with Poincaré half-space model, cf. [Rat06, § 6.4]. Finally, if v is hyperideal for $\mathbb{H}^d$ (in the exterior of $\mathcal{B}$), then Σ is the totally geodesic surface given by the intersection $\mathcal{B} \cap v^\perp$, which can be identified to the hyperbolic space $\mathbb{H}^{d-1}$.

Now the vertex figure $\mathcal{P}/v$ can be realized as the polytope $\mathcal{P} \cap \Sigma$, which lies in $\mathbb{S}^{d-1}$, $\mathbb{E}^{d-1}$ or $\mathbb{H}^{d-1}$ if v is in the interior, boundary or exterior of $\mathcal{B}$, independently. Since Σ is perpendicular to all the hyperbolic geodesics through v, the dihedral angles of $\mathcal{P}$ is preserved in $\mathcal{P} \cap \Sigma$; cf. [Rat06]. More generally, a face figure $\mathcal{P}/F$ can be obtained by consecutively taking vertex figures at each vertices of F. So we can realize $\mathcal{P}/F$ as a spherical, Euclidean or hyperbolic polytope if F is (in the strong sense) strictly cutting, tangent or strictly avoiding $\mathcal{S}$, independently.

We will need the following lemma:

Lemma 4.9. *Let $\mathcal{P}$ be a $(0, d-3)$-scribed d-simplex, and F be a facet of $\mathcal{P}$. If we regard $\mathcal{B}$ as the Klein model of the hyperbolic space $\mathbb{H}^d$, then the hyperbolic dihedral angles at the ridges incident to F sum up to at least π.*

Proof. Since the $(d-3)$-faces cut $\mathcal{S}$, their links are spherical or Euclidean triangles, i.e. the dihedral angles at the ridges incident to a $(d-3)$-face sum up to at least π. If we consider only the $(d-3)$-faces incident to F, the dihedral angles at the ridges incident to them sum up to at least $\binom{d}{2}\pi$.

However, this summation also includes some ridges not incident to F. These ridges are all incident to the vertex v that is not in F. Since v avoids $\mathcal{S}$, the vertex figure is a hyperbolic or Euclidean $(d-1)$-simplex. By a result of Höhn [Höh53] (see also [Gad56]) and [ALS08]), the dihedral angles of these ridges sum up to at most $\binom{d-1}{2}\pi$. Subtracting this from the summation above yields at least $\binom{d}{2}\pi - \binom{d-1}{2}\pi = (d-1)\pi$.

Furthermore, every ridge is counted $d-1$ times, which is the number of $(d-3)$-faces incident to each ridge. So the sum of the dihedral angles at the ridges incident to F sum up to at least π. □

By Lemma 2.12, Theorem 7 is derived from the following proposition.

Proposition 4.10. *There is a stacked 4-polytope that is not $(0, 1)$-scribable.*

Proof. Consider the stacked 4-polytope obtained by stacking on every facet of a simplex, and then stacking again on every facet. The stacked triangulation of the resulting polytope consists of $1+5+20 = 26$ 4-simplices. Twenty of them have boundary facets, we call them *exterior simplices*. The remaining six only have interior facets, we call them *interior simplices*.

There are 40 ridges incident to the interior facets, and we want to estimate the sum of the hyperbolic dihedral angles at these ridges. Each interior simplex contributes at least $10\pi/3$. To see this, notice that the link of each edge is a spherical or Euclidean triangle, and so the adjacent dihedral angles sum up to at least π, and that each ridge is incident to 3-edges, so each angle is counted three times. On the other hand, each exterior simplex shares a facet with an interior simplex. Hence, by Lemma 4.9, it contributes with at least π to the sum of the dihedral angles of ridges incident to interior simplicies. Hence, the dihedral angles at these 40 ridges sum up to at least 40π.

Therefore, there is at least one ridge at which the hyperbolic dihedral angle is at least π, which destroys the convexity of the polytope. □

Dihedral angles for $(0,1)$-scribed 3-polytopes are fully characterized by Bao and Bonahon [BB02], who refer these polytopes as "hyperideal polyhedra", and also proved the uniqueness up to hyperbolic isometry; see also [Sch05] for the connection to circle configurations.

5. Cyclic polytopes

The main result of this section is that even-dimensional cyclic polytopes with sufficiently many vertices are not $(1,d-1)$-scribable. We also investigate odd-dimensional cyclic polytopes and neighborly polytopes in general.

A d-polytope is *k-neighborly* if every k vertices form a face. Since the only k-neighborly d-polytope with $k > \lfloor d/2 \rfloor$ is the simplex, we call a d-polytope simply *neighborly* if it is $\lfloor d/2 \rfloor$-neighborly.

The most important examples of neighborly polytopes are cyclic polytopes. Consider a curve γ of *order d*, which means that each hyperplane intersects γ in at most d points, such as the d-dimensional moment curve $(t, t^2, \ldots, t^d)$. Take the convex hull of n distinct points on γ. That is,

$$\operatorname{conv}(\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n))$$

for n distinct parameters $t_1 < t_2 < \cdots < t_n$. Then the combinatorial type of this polytope (and, even more, the oriented matroid defined by the points $\gamma(t_i)$) does not depend on the choice of the parameters t_i. We call any polytope of this combinatorial type a *cyclic d-polytope* with n vertices, and denote it by $\mathcal{C}_d(n)$. If we identify the vertices of $\mathcal{C}_d(n)$ with the indices $[n] = \{1, \ldots, n\}$, the combinatorics of $\mathcal{C}_d(n)$ are described by the followed criterion, called *Gale's evenness condition* (cf. [Grü03, Section 4.7], [Zie95, Theorem 0.7]).

Proposition 5.1. *Let $I \subset [n]$ be a set of d vertices. Then I indexes a facet of $\mathcal{C}_d(n)$ if and only if for any two vertices $j < k$ in $[n] \setminus I$, the set $\{i \in I \mid j < i < k\}$ contains an even number of vertices.*

It is well-known that cyclic polytopes are inscribable; see [Car11] [GS87, p. 67] [Sei91, p. 521] [GZ13, Proposition 17]. This implies that they are $(0,j)$-scribable for any $j \geq 0$. We will however see that, in even dimensions, cyclic polytopes provide non-examples for (i,j)-scribability with $i > 0$. In particular, cyclic polytopes behave poorly with respect to k-scribability, as indicated by Theorem 2, which we recall below.

Theorem 2. *For any $1 \leq k \leq d-1$, a cyclic d-polytope with sufficiently many vertices is not k-scribable.*

This follows from the main results of this section, namely Propositions 5.6 for even dimensions, Corollary 5.8 for odd dimensions and $k > 1$, and Proposition 5.12 for $k = 1$.

5.1. k-ply systems and k-sets.

As we have already mentioned in Section 4, any point $\mathbf{x}$ in $\mathbb{E}^d \setminus \mathcal{B}$ can be associated with a closed spherical cap on $\mathcal{S}$, namely the set of points of $\mathcal{S}$ that are visible from $\mathbf{x}$. A set of spherical caps on $\mathcal{S}$ is said to be a *k-ply system* if no point of $\mathcal{S}$ is in the interior of k caps. These systems were studied by Miller et al. [MTTV97], who proved the following Separation Theorem. Here, the *intersection graph* is the graph where every vertex represents a cap, and two caps form an edge if they intersect.

Proposition 5.2 (Sphere Separator Theorem)**.** *The intersection graph of a k-ply system consisting of n caps on a d-dimensional sphere can be separated into two disjoint parts, each of size at most $\frac{d+1}{d+2}n$, by removing $O(k^{1/d}n^{1-1/d})$ vertices.*

To the knowledge of the authors, the best known constant factor in the proposition is

$$c_2 = \sqrt{\frac{2\pi}{\sqrt{3}}}\left(\frac{1+\sqrt{k}}{\sqrt{2(1+k)}} + o(1)\right)$$

for $d = 2$ and

$$c_d = \frac{2A_{d-1}}{A_d^{1-1/d} V_d^{1/d}} + o(1)$$

for $d > 2$; see [ST96]. Here V_d is the volume of a unit d-ball and A_d is the area of a unit d-sphere, so $A_{d-1} = dV_d$.

For a point set $V \subset \mathbb{E}^d$, a subset I of cardinality k is said to be a *k-set* if there is a hyperplane strictly separating I and $V \setminus I$. We will define the k-sets of a polytope to be the k-sets of its set of vertices, and say that a k-set intersects $\mathcal{S}$ if its convex hull intersects $\mathcal{S}$. The following lemma relates k-sets and k-ply systems.

Lemma 5.3. *A point set $V \subset \mathbb{E}^d \setminus \mathcal{B}$ corresponds to a k-ply system on $\mathcal{S}$ if and only if every k-set intersects the sphere $\mathcal{S}$.*

Proof. Assume that there is a k-set I such that $\operatorname{conv} I \cap \mathcal{S} = \emptyset$. Then there is a hyperplane tangent to $\mathcal{S}$ separating I and the interior of $\mathcal{B}$. The tangency point is visible from every point in I. In other words, it is in the interior of at least k of the associated caps, so the set of caps corresponding to V is not a k-ply system. The other direction is obtained by reversing the argument. □

The following obvious fact can be regarded as a special case of this lemma.

Corollary 5.4. *The caps of $\mathcal{S}$ corresponding to $v, w \in \mathbb{E}^d \setminus \mathcal{B}$ have disjoint interiors if and only if the segment $[v, w]$ strongly cuts $\mathcal{S}$.*

5.2. **Even dimensional cyclic polytopes.** The following is the key for proving our main result. It uses that all even-dimensional cyclic polytopes have the same oriented matroid to make statements about the k-sets of any realization of $\mathcal{C}_d(n)$ (which fail for odd dimensions, cf. Section 5.3).

Lemma 5.5 (k-set Lemma)**.** *For even d and $k \geq 3d/2 - 1$, every k-set of $\mathcal{C}_d(n)$ contains a facet of $\mathcal{C}_d(n)$.*

Proof. Without loss of generality, we can assume that $k = 3d/2 - 1$. Let I be a k-set of $\mathcal{C}_d(n)$.

Every even-dimensional cyclic polytope has its vertices on a order-d curve γ [Stu87]. Every hyperplane H intersects γ in at most d points, and hence there can be at most d changes of sides of H between I and $[n] \setminus I$. So I can be decomposed into at most $d/2 + 1$ consecutive segments of $[n]$. If we ignore the external segments (the ones containing 1 or n), there are at most $d/2 - 1$ internal segments.

Since $k = 3d/2 - 1 \not\equiv d/2 \pmod 2$, at most $d/2 - 1$ of these internal segments have odd length. By removing a vertex from the boundary of each of the odd internal segments, we obtain a set J of at least $k + 1 - d/2$ vertices satisfying Gale's evenness condition. As $k \geq 3d/2 - 1$, J contains at least d vertices. Finally, we take a d-element subset of J by taking even-length subsegments from the internal segments together with external subsegments from the external segments. This set corresponds to a facet since it still fulfills Gale's evenness condition. □

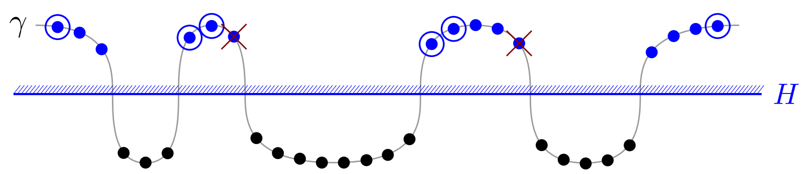


FIGURE 9. Sketch of a 15-set of a cyclic 6-polytope. The curve γ intersects the hyperplane H in 6 points, separating them into the k-set (above) and its complement (below).

The proof of the k-set Lemma is illustrated in Figure 9. It shows a 15-set of a cyclic 6-polytope, which consists of 4 segments, of lengths 3, 3, 5 and 4, respectively. The second and third segments are internal. We remove one extreme point from every internal odd segment (marked with a cross). Then we can select a subset that forms a facet (circled elements).

We are finally ready to prove the main result of this part.

Proposition 5.6. *Let $d \geq 4$ be an even integer and*

$$n > (c_{d-1}(d+1))^{d-1}(3d/2-1),$$

then the cyclic polytope $\mathcal{C}_d(n)$ is not $(1, d-1)$-scribable.

By Lemma 2.12(i), this implies that, in even dimensions, $\mathcal{C}_d(n)$ is not (i, j)-scribable for $1 \leq i \leq j \leq d-1$ if n is large enough. In particular, we have proved Theorem 2 for even dimensions.

Proof. It is enough to prove that $\mathcal{C}_d(n)$ is not $(1, d-1)$-scribable when n is sufficiently large. Assume that $\mathcal{P}$ is a $(1, d-1)$-scribed cyclic polytope with n vertices. Let $k = 3d/2 - 1$. By the k-set Lemma 5.5, every k-set of the vertices of $\mathcal{P}$ contains a facet. Since every facet of $\mathcal{P}$ cuts the sphere $\mathcal{S}$, this implies that every k-set intersects $\mathcal{S}$. Hence, the collection spherical caps corresponding to the vertices of $\mathcal{P}$ form a k-ply system. By the Sphere Separator Theorem 5.2, the intersection graph of the caps admits a separator of size

$$c_{d-1}\lfloor d/2 \rfloor^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} < \frac{n}{d+1}.$$

However, since all the edges of $\mathcal{P}$ strongly avoid the sphere, the intersection graph is a complete graph by Lemma 5.4, and the removal of the separator leaves a complete graph of more than $\frac{d}{d+1}n$ vertices, contradicting Proposition 5.2. □

By Lemma 2.12(iii) and (iv), we obtain the following corollary, which provides the final counterexamples to (i, j)-scribability for the proof of Theorem 5.

Corollary 5.7. *For odd d, the pyramid over a cyclic $(d-1)$-polytope with sufficiently many vertices is a d-polytope that is neither $(1, d-2)$-scribable nor $(2, d-1)$-scribable.*

5.3. **Odd dimensional cyclic polytopes.** The proof of the k-set Lemma 5.5 fails dramatically in odd dimensions. When d is odd, different realizations of $\mathcal{C}_d(n)$ may have different oriented matroids and hence different k-set structures. In particular, the vertices do not necessarily lie on any order-d curve. In fact, the k-set Lemma 5.5 does not hold in odd dimensions, as we will see in Remark 5.11.

Nevertheless, since $\mathcal{C}_d(n)$ has $\mathcal{C}_{d-1}(n-1)$ as a vertex-figure, we obtain the following corollary by Lemma 2.12(iv).

Corollary 5.8. *Let $d \geq 5$ be an odd integer and*

$$n > (c_{d-1}(d+1))^{d-1}(\lfloor 3d/2 \rfloor - 1),$$

then the cyclic polytope $\mathcal{C}_d(n)$ is not $(2, d-1)$-scribable.

By Lemma 2.12(i), this implies that, in odd dimensions, $\mathcal{C}_d(n)$ is not (i, j)-scribable with $2 \leq i \leq j \leq d-1$ if n is large enough. This proves Theorem 2 for $1 < k \leq d-1$. The 1-scribability of odd-dimensional cyclic polytopes will be taken care of in Section 5.4.

However, odd-dimensional cyclic polytopes are $(1, d-1)$-scribable, in contrast to the situation in even dimensions.

Proposition 5.9. *For odd d, the cyclic polytope $\mathcal{C}_d(n)$ is $(1, d-1)$-scribable.*

Proof. Take a realization of $\mathcal{C}_{d-1}(n-1)$ in $\mathbb{R}^{d-1}$ that is inscribed with respect to a sphere $\mathcal{S}'$, and fix a vertex v of $\mathcal{C}_{d-1}(n-1)$. Shrink $\mathcal{S}'$ to $\mathcal{S}''$ with respect to v such that any edge of $\mathcal{C}_{d-1}(n-1)$ that is disjoint from v avoids $\mathcal{S}''$.

Embed $\mathbb{R}^{d-1}$ into $\mathbb{R}^d$ as the hyperplane $x_0 = 0$, and extend $\mathcal{S}''$ to a sphere $\mathcal{S}$ in $\mathbb{R}^d$ so that $\mathcal{S}''$ is the equator of $\mathcal{S}$. Then we split the vertex v into two vertices $v_\pm = (\pm h, v)$. The "height" h is chosen such that, if a light source is placed at $v_\pm$, then the shadow of $\mathcal{S}$ on $\mathbb{R}^{d-1}$ is exactly $\mathcal{S}'$.

The convex hull of $v_\pm$ and the remaining vertices of $\mathcal{C}_{d-1}(n-1)$ is a realization of $\mathcal{C}_d(n)$ (cf. [CD00] and Remark 5.10). One then easily checks that, in this realization, edges incident to $v_\pm$ are tangent to $\mathcal{S}$; the other edges, belonging to $\mathcal{C}_{d-1}(n-1)$, avoid $\mathcal{S}$ by construction. On the other hand, all the facets are adjacent to either v_+ or v_- and cut $\mathcal{S}$. The construction is sketched in Figure 10. □

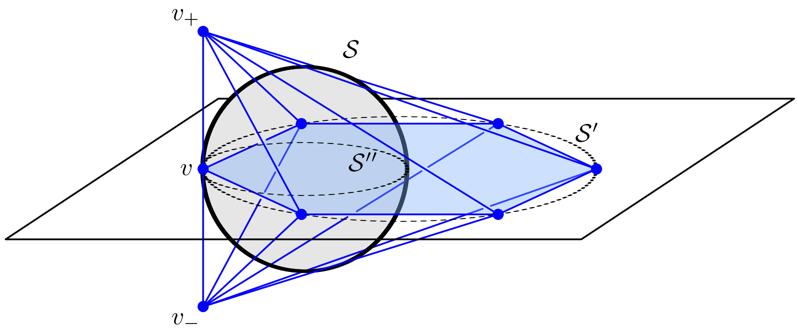


FIGURE 10. Sketch of the construction of Proposition 5.9.

Remark 5.10. Cordovil and Duchet [CD00] proposed a process that can realize any oriented matroid for $\mathcal{C}_d(n)$ when d is odd, but their description does not quite work. They first stack a vertex onto $\mathcal{C}_{d-1}(n-1)$, then split it into an extra dimension, followed by a perturbation. This process does not in general give a cyclic polytope. The correct construction consists of splitting a vertex of $\mathcal{C}_{d-1}(n-1)$ into an extra dimension, as we did in the proof above, and then a perturbation.

Remark 5.11. There can be arbitrarily large k-sets of an odd dimensional cyclic polytope that do not contain a facet. To see this, take the realization of $\mathcal{C}_d(n)$ from Proposition 5.9. From the copy of $\mathcal{C}_{d-1}(n-1)$ lying in $x_0 = 0$ take a subset of vertices not containing any facet, and lift them to height $x_0 = \varepsilon > 0$; and descend the remaining vertices of $\mathcal{C}_{d-1}(n-1)$ to $x_0 = -\varepsilon$. If ε is sufficiently small, then this does not change the combinatorial type, and the points in the open half-space $x_0 > 0$ form a k-set not containing any facet.

5.4. **Neighborly polytopes.** In this part, we apply the ideas leading to Proposition 5.6 to general neighborly polytopes. However, the lack of an analogue to Lemma 5.5 does not allow us to carry over the argument in full generality.

Let $\mathcal{P}$ be a j-neighborly d-polytope. Since every j-set of $\mathcal{P}$ forms a $(j-1)$-face, the argument for Proposition 5.6 proves that if $\mathcal{P}$ has sufficiently many vertices then it is not $(1, j-1)$-scribable. In particular, neighborly polytopes with sufficiently many vertices are not edge-scribable. This provides the last missing piece (namely $k = 1$) for Theorem 2.

We will however prove a slightly stronger result

Proposition 5.12. *For $d \geq 4$, a k-neighborly d-polytope $\mathcal{P}$ with sufficiently many vertices is not $(1, k)$-scribable.*

For a proof, the argument for Proposition 5.6 applies almost directly. But in place of Lemma 5.5, we need the following k-set lemma.

Lemma 5.13 (k-set lemma for neighborly polytopes)**.** *Every $(k+1)$-set of a k-neighborly d-polytope is a k-face.*

For a polytope, a set I of vertices is a *missing face* if I is not a face but every proper subset I is. For a k-neighborly polytope, every $(k+1)$-set either forms a face or a missing face. The k-set Lemma 5.13 is then a special case of the following more general lemma.

Lemma 5.14. *The k-sets of a polytope are not missing faces.*

Proof. Let V be the vertex set of the polytope, and I be a subset of k vertices. If I is a k-set, then $\operatorname{conv}(I) \cap \operatorname{conv}(V \setminus I) \neq \emptyset$. But if I is a missing face, then $\operatorname{conv}(I) \cap \operatorname{conv}(V \setminus I) = \emptyset$. □

Neighborliness is a property that only depends on the f-vector. Hence Proposition 5.12 implies Theorem 6, which we restate here:

Theorem 6. *For $d \geq 4$ and any $1 \leq k \leq d-2$, there are f-vectors such that no d-polytope with those f-vectors are k-scribable.*

Proof. For $1 \leq k \leq \lfloor d/2 \rfloor$, the theorem follows by taking $j = \lfloor d/2 \rfloor$ in Proposition 5.12. The remaining cases, i.e. $\lceil d/2 \rceil \leq k \leq d-2$, are obtained by taking the polar. □

6. Stamps

Polytopes that are not $(0, d-3)$-scribable can be obtained by taking the polar of cyclic polytopes. Here we present another alternative construction based on projectively prescribed faces.

Lemma 6.1. *For every $d \geq 2$, there is a polytope $\mathcal{P}$ with no $(0, d-1)$-scribed projectively equivalent realization.*

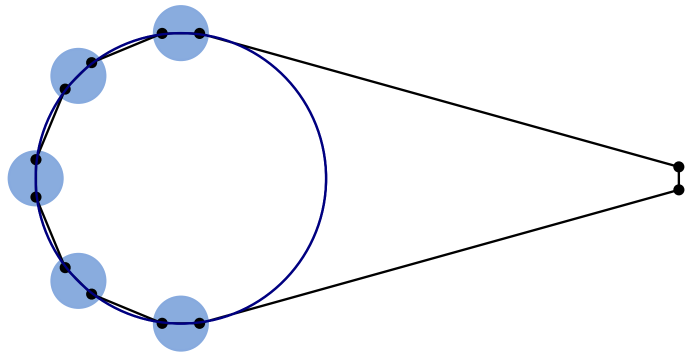


Figure 11. The construction of Lemma 6.1. Any polygon projectively equivalent to this polygon is not $(0, 1)$-scribed.

Proof. Consider $N = \binom{n+2}{2} - 1$ generic points $p_1, \ldots, p_N$ lying on the $x_0 < 0$ hemisphere of $\mathcal{S}$. There is a unique quadric going through $\binom{n+2}{2} - 1$ generic points, and this dependence is continuous since the coefficients of the quadric are the solution of a linear system of equations on the points' coordinates. Hence, there exists an $\varepsilon > 0$ such that for any $q_1, \ldots, q_N$ with $q_i \in B_\varepsilon(p_i)$, the unique quadric that goes through these q_i's is an ellipsoid contained in $2\mathcal{B}$.

Now, consider dN distinct points p_i^j for $1 \leq i \leq N$ and $1 \leq j \leq d$ with $p_i^j \in \mathcal{S} \cap B_\varepsilon(p_i)$. Choose d additional points p_0^j, $1 \leq j \leq d$, on the hyperplane $x_0 = 3$ in the neighborhood $B_\varepsilon(p_0)$ where $p_0 = (3, 0, \ldots, 0)$, such that all the $d(N+1)$ points are in convex position. Let $\mathcal{P}$ be the convex hull of all these points. If ε is small enough, then for each $0 \leq i \leq N$, the corresponding p_i^j's form a facet F_i of $\mathcal{P}$.

For the sake of contradiction, assume a projective transformation T such that $T\mathcal{P}$ is $(0, d-1)$-scribed, then $T^{-1}\mathcal{S}$ is a quadric that intersects all the facets of $\mathcal{P}$. Since $T^{-1}\mathcal{S}$ contains a point $q_i \in B_\varepsilon(p_i)$ for each $1 \leq i \geq N$, the quadric is contained in $2\mathcal{B}$ and hence does not intersect F_0. Hence, such a transformation T cannot exist.

This construction is sketched in Figure 11. □

We need the following result, found by Below [Bel02] and by Dobbins [Dob11, Dob15].

Proposition 6.2 ([Bel02, Ch. 5], see also [Dob11, Thm. 4.1] and [Dob15, Thm. 1])**.** *Let $\mathcal{P}$ be a d-dimensional polytope with algebraic vertex coordinates. Then there is a polytope $\widehat{\mathcal{P}}$ of dimension $d+2$ that contains a face F that is projectively equivalent to $\mathcal{P}$ in every realization of $\widehat{\mathcal{P}}$.*

Such a polytope $\widehat{\mathcal{P}}$ is called a *stamp* for $\mathcal{P}$ in [Dob11, Dob15]. We are now ready to prove the main result of this part.

Proposition 6.3. *A stamp for a $(d-2)$-polytope with no $(0, d-3)$-scribed projectively equivalent realization is not $(0, d-3)$-scribable.*

Proof. Let $\mathcal{P}$ be a $(d-2)$-polytope with no $(0, d-3)$-scribed projectively equivalent realization, whose existence is guaranteed by Lemma 6.1. Observe that in the construction of Lemma 6.1 we can impose that it has algebraic coordinates (and even rational). Now let $\widehat{\mathcal{P}}$ be the stamp polytope from Theorem 6.2. We claim that $\widehat{\mathcal{P}}$ is not $(0, d-3)$-scribable. Otherwise, its $(d-2)$-dimensional face F, which is projectively equivalent to $\mathcal{P}$ in every realization of $\widehat{\mathcal{P}}$, is also $(0, d-3)$-scribed, contradicting our assumption. □

7. Open problems

Several natural question arise from our results. The most intriguing is probably the existence of d-polytopes that cannot be $(0, d-1)$-scribed.

Conjecture 7.1. *For $d > 3$, there are d-polytopes that are not $(0, d-1)$-scribable.*

Although we strongly believe that the conjecture is true, we did not manage to construct examples. A promising strategy to find such a polytope would be using projectively unique polytopes, or polytopes with a very constrained realization space. So far, the largest family of projectively unique polytope that we know of are those constructed by Adiprasito and Ziegler [AZ15, Section A.5.2]. However, they are essentially inscribable, and hence they do not provide counterexamples directly.

Even for the case of $(1, d-1)$-scribability, our results are not complete. We only managed to find polytopes that are not $(1, d-1)$-scribable in even dimensions: for odd dimensions $d \geq 5$, cyclic polytopes do not provide examples; see Proposition 5.9.

Conjecture 7.2. *For every odd $d \geq 3$, there are d-polytopes that are not $(1, d-1)$-scribable.*

For odd dimensional cyclic polytopes, we know that they are $(1, d-1)$-scribable (Proposition 5.9) but not $(1, \lfloor d/2 \rfloor)$-scribable (Proposition 5.12). We would like to know

Question 7.3. *For odd $d \geq 5$ and $\lfloor d/2 \rfloor < k < d-1$, is every cyclic d-polytope $(1, k)$-scribable?*

We showed that cyclic polytopes with sufficiently many vertices are not circumscribable. We conjecture that this holds for any neighborly polytope.

Conjecture 7.4. *Neighborly polytopes with sufficiently many vertices are not circumscribable.*

If the conjecture is true, the dual of cyclic polytopes would give the first examples of f-vectors that are not inscribable (see also [GZ13]), completing our Theorem 6.

On the other hand, every cyclic polytope is inscribable, and so are all neighborly polytopes of a large family [GP15]. Computational results of Moritz Firsching show that every neighborly 4-polytope with at most 11 vertices is inscribable (personal communication). In [GP15] the following question is posed.

Question 7.5. *Is every neighborly polytope inscribable?*

For stacked polytopes, we do not have results on (i, j)-scribability for $j - i \geq 2$. Unfortunately, the angle-sum technique that proves Theorem 7 does not work for dimension 5 or higher.

Question 7.6. *Given i, j such that $j - i \geq 2$, is every stacked d-polytope (i, j)-scribable?*

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