

# STACKED 4-POLYTOPES WITH BALL PACKABLE GRAPHS

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ABSTRACT. After investigating the ball-packability of some small graphs, we give a full characterisation, in terms of forbidden induced subgraphs, for the stacked 4-polytopes whose 1-skeletons can be realised by the tangency relations of a ball packing.

## 1. INTRODUCTION

A ball packing is a collection of balls with disjoint interiors. A graph is said to be ball packable if it can be realized by the tangency relations of a ball packing. Formal definitions will be given later.

The combinatorics of disk packings (2-dimensional ball packings) is well understood thanks to the Koebe–Andreev–Thurston’s disk packing theorem, which says that every planar graph is disk packable. However, we know little about the combinatorics of ball packings in higher dimensions.

In this paper we study the relation between Apollonian ball packings and stacked polytopes: An Apollonian ball packing is formed by repeatedly filling new balls into holes in a ball packing. A stacked polytope is formed, starting from a simplex, by repeatedly gluing new simplices onto facets. Detailed and formal introductions can be found respectively in Section 2.3 and in Section 2.4.

There is a 1-to-1 correspondence between 2-dimensional Apollonian ball packings and 3-dimensional stacked polytopes: a graph can be realised by the tangency relations of an Apollonian disk packing if and only if it is the 1-skeleton of a stacked 3-polytope.

As we will see, this relation does not hold in higher dimensions:

On one hand, the 1-skeleton of a stacked polytope may not be realizable by the tangency relations of any Apollonian ball packing. Our main result, proved in Section 4, give a condition on stacked 4-polytopes to restore the correspondence in this direction:

**Theorem 1.1** (Main result). *The 1-skeleton of a stacked 4-polytope is 3-ball packable if and only if it does not contain six 4-cliques sharing a 3-clique.*

For higher dimensions, we propose Conjecture 4.11 at the end of this paper.

On the other hand, the tangency graph of a ball packing may not be the 1-skeleton of any stacked polytope. However, we will show (Theorem 4.9) that this can only happen for 3-dimensional ball packings. So the correspondence remains in this direction for ball packings of dimension higher than 3.

The proofs are based on a method used by Graham et al. in [16]. Before proving the main result, we will investigate the ball packability of some small graphs in Section 3 as a preparation. These graphs can all be written in form of graph joins. The advantage is that ball packings can be explicitly constructed. Since ball

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packability is a property closed under induced subgraph operations, this investigation will provide many forbidden induced subgraphs for ball packable graphs. The forbidden subgraph in our main result is one of them.

## 2. DEFINITIONS AND PRELIMINARIES

**2.1. Ball packings.** We work in the  $d$ -dimensional extended Euclidean space  $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ . A  $d$ -ball of curvature  $\kappa$  means one of the following sets:

- $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{c}\| \leq 1/\kappa\}$  if  $\kappa > 0$ ;
- $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{c}\| \geq -1/\kappa\}$  if  $\kappa < 0$ ;
- $\{\mathbf{x} \mid \langle \mathbf{x}, \hat{\mathbf{n}} \rangle \geq b\} \cup \{\infty\}$  if  $\kappa = 0$ ,

where  $\|\cdot\|$  is the Euclidean norm, and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product. In the first two cases,  $\mathbf{c} \in \mathbb{R}^d$  is called the *center* of the ball. In the last case, the unit vector  $\hat{\mathbf{n}}$  is called the *normal vector* of a half-space, and  $b \in \mathbb{R}$ . The boundary of a  $d$ -ball is a  $(d-1)$ -sphere.

Two balls are tangent at a point  $\mathbf{t} \in \hat{\mathbb{R}}^d$  if  $\mathbf{t}$  is the only element of their intersection. We call  $\mathbf{t}$  the *tangency point*, which can be the infinity point  $\infty$  if it involves two balls of curvature 0.

For a ball  $S \subset \hat{\mathbb{R}}^d$ , the *curvature-center coordinates* is introduced by Lagarias, Mallows and Wilks in [22]

$$\mathbf{m}(S) = \begin{cases} (\kappa, \kappa\mathbf{c}) & \text{if } \kappa \neq 0; \\ (0, \hat{\mathbf{n}}) & \text{if } \kappa = 0. \end{cases}$$

Here, the term ‘‘coordinate’’ is an abuse of language, since the curvature-center coordinates do not uniquely determine a ball when  $\kappa = 0$ . A real global coordinate system would be the *augmented curvature-center coordinates* [22]. However, the curvature-center coordinates are good enough for our use.

**Definition 2.1.** A  $d$ -ball packing is a collection of  $d$ -balls with disjoint interiors.

For a ball packing  $\mathcal{S}$ , its *tangency graph*  $G(\mathcal{S})$  takes the balls as vertices and the tangency relations as the edges. It is invariant under Möbius transformations and reflections.

**Definition 2.2.** A graph  $G$  is said to be  $d$ -ball packable if there is a  $d$ -ball packing  $\mathcal{S}$  whose tangency graph is isomorphic to  $G$ . In this case, we say that  $\mathcal{S}$  is a  $d$ -ball packing of  $G$ .

*Disk packings*, or 2-ball packings, are well understood thanks to the following famous theorem:

**Theorem 2.3** (Koebe–Andreev–Thurston theorem [19, 32]). *Every connected simple planar graph is disk packable. If the graph is a finite triangulated planar graph, then it has a unique disk packing up to Möbius transformations.*

We know little about the combinatorics of ball packings in higher dimensions. Some attempts of generalizing the disk packing theorem to higher dimensions include [3, 9, 21, 24].

An induced subgraph of a  $d$ -ball packable graph is also  $d$ -ball packable. In other words, the class of ball packable graphs is closed under the induced subgraph operation.

Throughout this paper, ball packings are always in dimension  $d$ . The dimensions of other objects will vary correspondingly.

**2.2. Descartes configurations.** A *Descartes configuration* in dimension  $d$  is a  $d$ -ball packing consisting of  $d + 2$  pairwise tangent balls. The tangency graph of a Descartes configuration is the complete graph on  $d + 2$  vertices. This is the basic element for the construction of many ball packings in this paper.

The following relation was first established for dimension 2 by René Descartes in a letter [11] to Princess Elizabeth of Bohemia, then generalized to dimension 3 by Soddy in the form of a poem [31], and finally generalized to any dimension by Gossett [14].

**Theorem 2.4** (Descartes–Soddy–Gossett Theorem). *In dimension  $d$ , if  $d + 2$  balls  $S_1, \dots, S_{d+2}$  form a Descartes configuration, let  $\kappa_i$  be the curvature of  $S_i$  ( $1 \leq i \leq d + 2$ ), then*

$$(1) \quad \sum_{i=1}^{d+2} \kappa_i^2 = \frac{1}{d} \left( \sum_{i=1}^{d+2} \kappa_i \right)^2$$

Equivalently,  $\mathbf{K}^\top \mathbf{Q}_d \mathbf{K} = 0$ , where  $\mathbf{K} = (\kappa_1, \dots, \kappa_{d+2})^\top$  is the vector of curvatures, and  $\mathbf{Q}_d := \mathbf{I} - \frac{1}{d} \mathbf{e} \mathbf{e}^\top$  is a square matrix of size  $d + 2$ , where  $\mathbf{e}$  is the all-one column vector, and  $\mathbf{I}$  is the identity matrix.

A more generalized relation on the curvature-center coordinates was proved in [22]:

**Theorem 2.5** (Generalized Descartes–Soddy–Gossett Theorem). *In dimension  $d$ , if  $d + 2$  balls  $S_1, \dots, S_{d+2}$  form a Descartes configuration, then*

$$(2) \quad \mathbf{M}^\top \mathbf{Q}_d \mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & 2\mathbf{I} \end{pmatrix}$$

where  $\mathbf{M}$  is the curvature-center matrix of the configuration, whose  $i$ -th row is  $\mathbf{m}(S_i)$ .

Given a Descartes configuration  $S_1, \dots, S_{d+2}$ , we can construct another Descartes configuration by replacing  $S_1$  with an  $S_{d+3}$ , such that the curvatures  $\kappa_1$  and  $\kappa_{d+3}$  are the two roots of (1) treating  $\kappa_1$  as unknown. So we have the relation

$$(3) \quad \kappa_1 + \kappa_{d+3} = \frac{2}{d-1} \sum_{i=2}^{d+2} \kappa_i$$

We see from (2) that the same relation holds for all the entries in the curvature-center coordinates, i.e.

$$(4) \quad \mathbf{m}(S_1) + \mathbf{m}(S_{d+3}) = \frac{2}{d-1} \sum_{i=2}^{d+2} \mathbf{m}(S_i)$$

These equations are essential for the calculations in the paper.

By recursively replacing  $S_i$  with a new ball  $S_{i+d+2}$  in this way, we obtain an infinite sequence of balls  $S_1, S_2, \dots$ , in which any  $d + 2$  consecutive balls form a Descartes configuration. This is Coxeter's loxodromic sequences of tangent balls [10].

### 2.3. Apollonian cluster of balls.

**Definition 2.6.** A collection of  $d$ -balls is said to be *Apollonian* if it can be built from a Descartes configuration by repeatedly introducing, for  $d + 1$  pairwise tangent balls, a new ball that is tangent to all of them.

Please note that a ball, when added, is allowed to touch more than  $d + 1$  balls, and may intersect some other balls. So the result may not be a packing. Coxeter's loxodromic sequence is Apollonian, for example.

We reformulate the replacing operation in the previous part by inversions: Given a Descartes configuration  $\mathcal{S} = \{S_1, \dots, S_{d+2}\}$ , let  $R_i$  be the inversion in the sphere that orthogonally intersects the boundary of  $S_j$  for all  $1 \leq j \neq i \leq d+2$ , then  $R_i\mathcal{S}$  forms a new Descartes configuration, which keeps all the balls except that  $S_i$  is replaced by  $R_iS_i$ .

With this point of view, a Coxeter's sequence can be obtained from an initial Descartes configuration  $\mathcal{S}_0$  by recursively constructing a sequence of Descartes configurations by  $\mathcal{S}_{n+1} = R_{j+1}\mathcal{S}_n$  where  $j = n \pmod{d+2}$ , and taking the union.

More generally, the group  $W$  generated by the  $R_i$ 's is called the *Apollonian group*. The union of the orbits  $\cup_{S \in \mathcal{S}_0} WS$  is called the *Apollonian cluster* (of balls) [16].

The Apollonian cluster is an infinite ball packing in dimensions two [15] and three [4]. That is, the interiors of any two balls in the cluster are either identical or disjoint.

This is unfortunately not true for higher dimensions. However, if we do not require  $\mathcal{S}_0$  to be a Descartes configuration, i.e. if  $S_1, \dots, S_{d+2}$  are not forced to be pairwise tangent, then a similar generating method yields a non-Apollonian infinite ball packing under certain conditions [4]. Maxwell [23] related this fact to hyperbolic reflection groups, and showed that this generating method works only up to dimension nine.

Define

$$\mathbf{R}_i := \mathbf{I} + \frac{2}{d-1}\mathbf{e}_i\mathbf{e}_i^\top - \frac{2d}{d-1}\mathbf{e}_i\mathbf{e}_i^\top$$

where  $\mathbf{e}_i$  is a  $(d+2)$ -vector whose entries are 0 except the  $i$ -th entry being 1. So  $\mathbf{R}_i$  coincide with the identity matrix at all rows except the  $i$ -th row, where the diagonal entry is  $-1$  and the off-diagonal entries is  $2/(d-1)$ .

One can then verify that  $\mathbf{R}_i$  induces a representation of the Apollonian group. In fact, if  $\mathbf{M}$  is the curvature-center matrix of a Descartes configuration  $\mathcal{S}$ , then  $\mathbf{R}_i\mathbf{M}$  is the curvature-center matrix of  $R_i\mathcal{S}$ .

**2.4. Stacked polytopes.** For a simplicial polytope, a *stacking operation* glues a new simplex onto a facet.

**Definition 2.7.** A simplicial  $d$ -polytope is *stacked* if it can be iteratively constructed from a  $d$ -simplex by a sequence of *stacking operations*.

We call the 1-skeleton of a polytope  $\mathcal{P}$  the *graph* of  $\mathcal{P}$ , denoted by  $G(\mathcal{P})$ . For example, the graph of a  $d$ -simplex is the complete graph on  $d+1$  vertices.

The graph of a stacked  $d$ -polytope is a  *$d$ -tree*, that is, a chordal graph whose maximal cliques are of a same size  $d+1$ . Inversely,

**Theorem 2.8** (Kleinschmidt [17]). *A  $d$ -tree is the graph of a stacked  $d$ -polytope if and only if there is no three  $(d+1)$ -cliques sharing  $d$  vertices.*

$d$ -trees satisfying this condition will be called *stacked  $d$ -polytopal graphs*.

A simplicial  $d$ -polytope  $\mathcal{P}$  is stacked if and only if it admits a triangulation  $\mathcal{T}$  with only interior faces of dimension  $(d-1)$ . For  $d \geq 3$ , this triangulation is unique, whose simplices correspond to the maximal cliques of  $G(\mathcal{P})$ . This implies that stacked polytopes are uniquely determined by their graph (i.e. stacked polytopes with isomorphic graphs are combinatorially equivalent). The *dual tree* [13] of  $\mathcal{P}$  takes the simplices of  $\mathcal{T}$  as vertices, and connect two vertices if the corresponding simplices share a  $(d-1)$ -face.

The following correspondence between Apollonian 2-ball packings and stacked 3-polytopes can be easily seen by comparing the construction processes and using Theorem 2.3:

**Theorem 2.9.** *If a disk packing is Apollonian, then its tangency graph is stacked 3-polytopal. If a graph is stacked 3-polytopal, then it is disk packable, and its disk packing is Apollonian and unique up to Möbius transformations.*

The first part of this theorem will be generalized to higher dimensions as Theorem 4.9.

The relation between 3-tree, stacked 3-polytope and Apollonian 2-ball packing can be illustrated as follows:

$$\text{3-tree} \xrightarrow[\text{sharing a 3-clique}]{\text{no three 4-cliques}} \text{stacked 3-polytope} \longleftrightarrow \text{Apollonian 2-ball packing}$$

### 3. BALL-PACKABILITY OF GRAPH JOINS

**Notations.** We use  $G_n$  to denote any graph on  $n$  vertices, and use the following notations for some special graphs.

- $P_n$ : the path on  $n$  vertices (therefore of length  $n - 1$ );
- $C_n$ : the cycle on  $n$  vertices;
- $K_n$ : the complete graph on  $n$  vertices;
- $\bar{K}_n$ : the empty graph on  $n$  vertices;
- $\diamond_d$ : the 1-skeleton of the  $d$ -dimensional orthoplex;

The *join* of two graphs  $G$  and  $H$ , denoted by  $G \star H$ , is the graph obtained by connecting every vertex of  $G$  to every vertex of  $H$ . Most of the graphs in this paper will be expressed in term of graph joins. Notably,  $\diamond_d = \underbrace{\bar{K}_2 \star \cdots \star \bar{K}_2}_d$ .

**3.1. Graphs in form of  $K_d \star P_m$ .** The following theorem is a reformulation of a result first obtained by Wilker [33]. A proof was sketched in [4]. Here we present a very elementary proof, suitable for our further generalization.

**Theorem 3.1.** *Let  $d \geq 2$  and  $m \geq 0$ .*

- (i)  $K_2 \star P_m$  is 2-ball packable for any  $m$ ;
- (ii)  $K_d \star P_m$  is  $d$ -ball packable if  $m \leq 4$ ;
- (iii)  $K_d \star P_m$  is not  $d$ -ball packable if  $m \geq 6$ ;
- (iv)  $K_d \star P_5$  is  $d$ -ball packable if and only if  $2 \leq d \leq 4$ ;

*Proof.* (i) is trivial.

For dimensions  $d > 2$ , we construct a ball packing for the  $(d+1)$ -simplex  $K_{d+2} = K_d \star P_2$  as follows: The two vertices of  $P_2$  are represented by two disjoint half-spaces  $A$  and  $B$  distance 2 apart, and the  $d$  vertices of  $K_d$  are represented by  $d$  pairwise tangent unit balls touching both  $A$  and  $B$ . Figure 1 shows the situation for  $d = 3$ , where red balls represent vertices of  $K_3$ .

The idea of the proof is the following: We construct the ball packing of  $K_d \star P_m$  from the ball packing above of  $K_d \star P_2$ , by appending new balls to the chain of balls representing the path. Every new ball is forced to touch all the  $d$  unit balls of  $K_d$ , therefore must center on a straight line perpendicular to the hyperplanes defining  $A$  and  $B$ . The construction fails if the sum of the diameters exceeds 2.

We now construct  $K_d \star P_3$  by adding a ball  $C$  tangent to  $A$ . By (3), the diameter of  $C$  is  $2/\kappa_C = (d-1)/d < 1$ . So the construction of  $K_d \star P_3$  succeeded since  $C$  is disjoint from  $B$ .

Then we add the ball  $D$  tangent to  $B$ . It has the same diameter as  $C$ , and they sum up to  $2(d-1)/d < 2$ . So the construction of  $K_d \star P_4$  succeeded, which proves the statement (ii).

We now add the ball  $E$  tangent to  $C$ . Still by (3), the diameter of  $E$  is

$$\frac{2}{\kappa_E} = \frac{(d-1)^2}{d(d+1)}$$

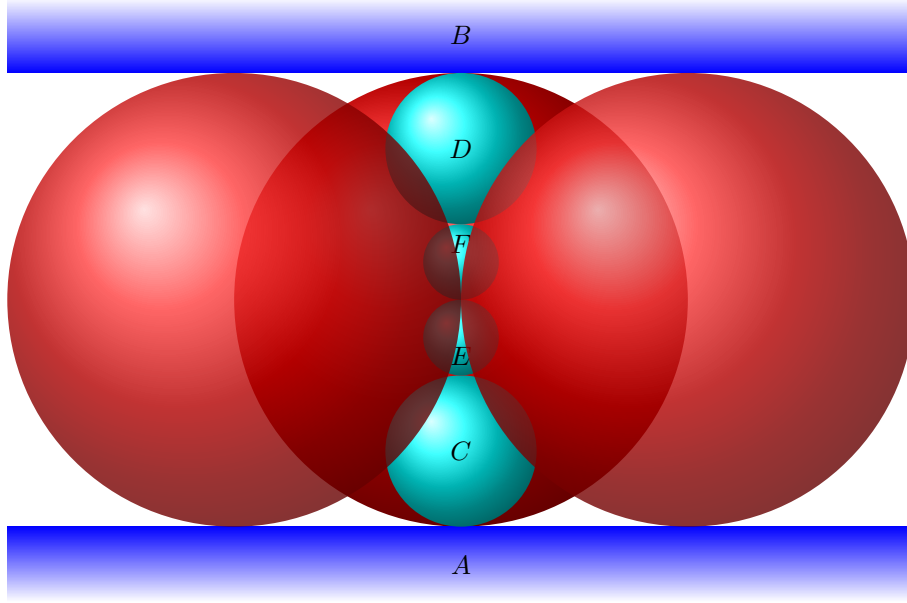


FIGURE 1. An attempt of constructing the ball packing of  $K_3 \star P_6$  results in  $K_3 \star C_6$ .

If we sum up the diameters of  $C$ ,  $D$  and  $E$ , we get

$$(5) \quad 2\frac{d-1}{d} + \frac{(d-1)^2}{d(d+1)} = \frac{3d^2 - 2d - 1}{d(d+1)}$$

which is smaller than 2 if and only if  $d \leq 4$ . Therefore the construction can succeed only if  $2 \leq d \leq 4$ , which proves (iv).

Now for  $2 \leq d \leq 4$ , we continue to add the ball  $F$  tangent to  $D$ . It has the same diameter as  $E$ . If we sum up the diameters of  $C$ ,  $D$ ,  $E$  and  $F$ , we get

$$(6) \quad 2\left(\frac{d-1}{d} + \frac{(d-1)^2}{d(d+1)}\right) = 4\frac{d-1}{d+1}$$

which is smaller than 2 if and only if  $d < 3$ . Therefore (iii) is proved.  $\square$

*Remark.* Figure 1 shows the attempt of constructing the ball packing of  $K_3 \star P_6$  but yields the ball packing of  $K_3 \star C_6$ . This packing is called Soddy's hexlet [30]. It's an interesting configuration since the sum of diameters of  $C$ ,  $D$ ,  $E$  and  $F$  is exactly 2.

*Remark.* Let's point out the main differences between dimension 2 and higher dimensions: If  $d = 2$ , a Descartes configuration divides the space into 4 disjoint regions, and the radius of a circle tangent to the two unit circles of  $K_2$  can tend to 0. However, if  $d > 2$ , the complement of a Descartes configuration is always connected, and the radius of a ball tangent to all the  $d$  balls of  $K_d$  can not be too small. Using the Descartes–Soddy–Gossett theorem, one can verify that the radius is at least  $\frac{d-2}{d+\sqrt{2d^2-2d}}$ , which tends to  $\frac{1}{1+\sqrt{2}}$  as  $d$  tends to infinity.

**3.2. Graphs in form of  $K_n \star G_m$ .** The following is a corollary of Theorem 3.1.

**Corollary 3.2.** *For  $d = 3$  or  $4$ ,  $K_d \star G_6$  are not  $d$ -ball packable, with the exception of  $K_3 \star C_6$ . For  $d \geq 5$ ,  $K_d \star G_5$  are not  $d$ -ball packable.*

*Proof.* For construction of  $K_d \star G_m$ , we just repeat the construction in the proof of Theorem 3.1.

Since the centers of the balls of  $G_m$  are situated on a straight line,  $G_m$  can only be a path, a cycle  $C_m$  or a disjoint union of paths. The first possibility is ruled out by Theorem 3.1. The cycle is only possible when  $d = 3$  and  $m = 6$ , in which case the ball packing of  $K_3 \star C_6$  is Soddy's hexlet.

If  $G_m$  is a disjoint union of paths, we have to leave gaps between balls, which makes it more difficult to avoid self-intersection. For the graphs in the theorem, a ball packing is impossible even without any gap. So the construction must fail.  $\square$

We study in the following some other graphs in form of  $K_n \star G_m$ , using kissing configuration and spherical codes.

A *d-kissing configuration* is a packing of unit  $d$ -balls all touching another unit ball. The *d-kissing number*  $k(d, 1)$  (we use this notation for the convenience of later generalizations) is the maximum number of balls in a  $d$ -kissing configuration.

For lower dimensions, the kissing number are known to be 2 for dimension 1, 6 for dimension 2, 12 for dimension 3 [8], 24 for dimension 4 [25], 240 for dimension 8 and 196560 for dimension 24 [26].

We have immediately the following theorem.

**Theorem 3.3.**  *$K_3 \star G$  is  $d$ -ball packable if and only if  $G$  is the tangency graph of a  $(d - 1)$ -kissing configuration.*

For the proof, just represent  $K_3$  by one unit ball and two disjoint half-spaces distance 2 apart, then the other balls must form a  $(d - 1)$ -kissing configuration.

For example,  $K_3 \star G_{13}$  is not 4-ball packable,  $K_3 \star G_{25}$  is not 5-ball packable, and in general,  $K_3 \star G_m$  is not  $d$ -ball packable if  $m > k(d - 1, 1)$ .

We can generalize this idea as follows: A  *$(d, \alpha)$ -kissing configuration* is a packing of unit balls touching  $\alpha$  pairwise tangent unit balls. The  *$(d, \alpha)$ -kissing number*  $k(d, \alpha)$  is the maximum number of balls in a  $(d, \alpha)$ -kissing configuration.

So the  $d$ -kissing configuration discussed before is actually the  $(d, 1)$ -kissing configuration. It is easy to see that if  $G$  is the tangency graph of a  $(d, \alpha)$ -kissing configuration,  $G \star K_1$  must be the graph of a  $(d, \alpha - 1)$ -kissing configuration, and  $G \star K_{\alpha-1}$  must be the graph of a  $d$ -kissing configuration.

With a similar argument as before, we have

**Theorem 3.4.**  *$K_{2+\alpha} \star G$  is  $d$ -ball packable if and only if  $G$  is the tangency graph of a  $(d - 1, \alpha)$ -kissing configuration.*

For the proof, just represent  $K_{2+\alpha}$  by two half-spaces distance 2 apart and  $\alpha$  pairwise tangent unit balls, then the other balls must form a  $(d - 1, \alpha)$ -kissing configuration.

For example,  $K_{2+\alpha} \star G_m$  is not  $d$ -ball packable if  $m > k(d - 1, \alpha)$ . The following corollary is from the fact that  $k(d, d) = 2$  for all  $d > 0$

**Corollary 3.5.**  *$K_{d+1} \star G_3$  is not  $d$ -ball packable.*

From Theorem 2.8, we see that a  $(d+1)$ -tree is  $d$ -ball packable only if it is stacked  $(d + 1)$ -polytopal.

A  *$(d, \cos \theta)$ -spherical code* [8] of minimal angle  $\theta$  is a set of points on the unit  $(d - 1)$ -sphere such that the spherical distance between any two points in the set is at least  $\theta$ . We denote by  $A(d, \cos \theta)$  the maximal number of points in such a spherical code. This is in fact a generalization of kissing configurations: the minimal angle corresponds to the tangency relations, and  $A(d, \cos \theta) = k(1, d)$  if  $\theta = \pi/3$ . Corresponding to the tangency graph, the *minimal-angle graph* of a spherical code takes the points as vertices and connects two vertices if the corresponding points achieve the minimal spherical distance.

As noticed by Bannai and Sloane [2, Theorem 1], the centers of unit balls in a  $(d, \alpha)$ -kissing configuration correspond to a  $(d - \alpha + 1, \frac{1}{\alpha+1})$ -spherical code after rescaling. Therefore:

**Corollary 3.6.**  $K_{2+\alpha} \star G$  is  $(d + \alpha)$ -ball packable if and only if  $G$  is the minimal-angle graph of a  $(d, \frac{1}{\alpha+1})$ -spherical code.

We give in Table 1 an incomplete list of  $(d, \frac{1}{\alpha+1})$ -spherical codes for integer values of  $\alpha$ . They are therefore  $(d + \alpha - 1, \alpha)$ -kissing configurations for the  $\alpha$  and  $d$  given in the table.

The first column is the name of the polytope whose vertices form the spherical code. Some of them are from Klitzing's list of segmentochora [18], which can be viewed as a special type of spherical codes. Some others are inspired from Sloane's collection of optimal spherical codes [29]. For those polytopes with no conventional name, we keep Klitzing's notation, or give a name following Klitzing's method.

The second column is the corresponding minimal-angle graph, if it is possible to write out. Here are some notations used in the table: For a graph  $G$ , its *line graph*  $L(G)$  takes the edges of  $G$  as vertices, and two vertices are adjacent iff the corresponding edges share a vertex in  $G$ . The *Johnson graph*  $J_{n,k}$  takes the  $k$ -element subsets of an  $n$ -element set as vertices, and two vertices are adjacent whenever their intersection contains  $k - 1$  elements. Especially,  $J_{n,2} = L(K_n)$ . For two graph  $G$  and  $H$ ,  $G \square H$  denotes the *Cartesian product*.

We would like to point out that for  $1 \leq \alpha \leq 6$ , vertices of the uniform  $(5 - \alpha)_{21}$  polytope form an  $(8, \alpha)$ -kissing configuration. These codes are derived from the  $E_8$

TABLE 1. Some known  $(d, \frac{1}{\alpha+1})$ -spherical codes for integer  $\alpha$

| spherical code                                     | minimal distance graph               | $\alpha$ | $d$     |
|--|--------------------------------------|----------|---------|
| $k$ -orthoplicial prism                            | $\diamond_k \square K_2$             | 2        | $k + 1$ |
| $k$ -orthoplicial-pyramidal prism                  | $(\diamond_k \star K_1) \square K_2$ | 2        | $k + 2$ |
| rectified $k$ -orthoplex                           | $L(\diamond_k)$                      | 1        | $k$     |
| augmented $k$ -simplicial prism                    |                                      | $k$      | $k + 1$ |
| 2-simplicial prism( $-1_{21}$ ) [18, 3.4.1]        | $K_3 \square K_2$                    | 6        | 3       |
| 3-simplicial prism( $-1_{31}$ ) [18, 4.9.2]        | $K_4 \square K_2$                    | 4        | 4       |
| 5-simplicial prism                                 | $K_6 \square K_2$                    | 3        | 6       |
| triangle-triangle duoprism( $-1_{22}$ ) [18, 4.10] | $K_3 \square K_3$                    | 3        | 4       |
| tetrahedron-tetrahedron duoprism                   | $K_4 \square K_4$                    | 2        | 6       |
| triangle-hexahedron duoprism                       | $K_3 \square K_6$                    | 2        | 7       |
| rectified 4-simplex( $0_{21}$ ) [1]                | $J_{5,2}$                            | 5        | 4       |
| rectified 5-simplex( $0_{31}$ )                    | $J_{6,2}$                            | 3        | 5       |
| rectified 7-simplex                                | $J_{8,2}$                            | 2        | 7       |
| birectified 5-simplex( $0_{22}$ )                  | $J_{6,3}$                            | 2        | 5       |
| birectified 8-simplex                              | $J_{9,3}$                            | 1        | 8       |
| trirectified 7-simplex                             | $J_{8,4}$                            | 1        | 7       |
| 5-demicube( $1_{21}$ ) [29, pack.5.16]             |                                      | 4        | 5       |
| 6-demicube( $1_{31}$ )                             |                                      | 2        | 6       |
| 8-demicube   |                                      | 1        | 8       |
| $1_{22}$   |                                      | 1        | 6       |
| $2_{31}$   |                                      | 1        | 7       |
| $2_{21}$ [7, Appendix A]                           |                                      | 3        | 6       |
| $3_{21}$ [2]                                       |                                      | 2        | 7       |
| $4_{21}$ [2]                                       |                                      | 1        | 8       |
| $3p \parallel \text{refl ortho } 3p$ [18, 4.13]    |                                      | 2        | 4       |
| $3g \parallel \text{gyro } 3p$ [18, 4.6.2]         |                                      | 5        | 4       |
| $3g \parallel \text{ortho } 4g$ [18, 4.7.3]        |                                      | 5        | 4       |
| $3p \parallel \text{ortho line}$ [18, 4.8.2]       |                                      | 5        | 4       |
| $\text{oct} \parallel \text{hex}$ [29, pack.5.14]  |                                      | 4        | 5       |



root lattice [2, Example 2]. They are optimal and unique except for the trigonal prism( $(-1)_{21}$  polytope) [1; 7, Appendix A]. There are also spherical codes similarly derived from the *Leech lattice* [2, Example 3; 6].

As another example, since

$$k(d, \alpha) = A\left(d - \alpha + 1, \frac{1}{\alpha + 1}\right).$$

the following fact provides another way for proving Corollary 3.2:

$$k(d, d-1) = A(2, 1/d) = \begin{cases} 4 & \text{if } d \geq 4 \\ 5 & \text{if } d = 3 \\ 6 & \text{if } d = 2(\text{optimal}) \end{cases}$$

To end this part, the following theorem is trivial but more general.

**Theorem 3.7.**  $K_2 \star G$  is  $d$ -ball packable if and only if  $G$  is  $(d-1)$ -unit-ball packable.

For the proof, just use disjoint half-spaces to represent  $K_2$ , then  $G$  has to be representable by a packing of unit balls.

### 3.3. Graphs in form of $\diamond_d \star G_m$ .

**Theorem 3.8.**  $\diamond_{d-1} \star P_4$  is not  $d$ -ball packable, but  $\diamond_{d+1} = \diamond_{d-1} \star C_4$  is.

*Proof.*  $\diamond_{d-1}$  is the graph of the  $(d-1)$ -dimensional orthoplex. The vertices of a regular orthoplex of edge length  $\sqrt{2}$  forms an optimal spherical code of minimal angle  $\pi/2$ .

As in the proof of Theorem 3.1, we first construct the ball packing of  $\diamond_{d-1} \star P_2$ , where  $P_2$  is represented by two disjoint half-spaces, and  $\diamond_{d-1}$  is represented by  $2(d-1)$  unit balls, whose centers are the vertices of a regular  $(d-1)$ -dimensional orthoplex of edge length 2. Therefore the unit balls are centered on a  $(d-2)$ -sphere of radius  $\sqrt{2}$ .

We now construct  $\diamond_{d-1} \star P_3$  by adding the unique ball that is tangent to all the unit balls and also to one half-space. After an elementary calculation, the radius of this ball is  $1/2$ . By symmetry, a ball touching the other half-space has the same radius. These two balls must be tangent since their diameters sum up to 2.

Therefore, an attempt for constructing a ball packing of  $\diamond_{d-1} \star P_4$  results in a ball packing of  $\diamond_{d+1} = \diamond_{d-1} \star C_4$ .  $\square$

For example,  $C_4 \star C_4$  is 3-ball packable.

By the same argument as in the proof of Corollary 3.2, we have

**Corollary 3.9.**  $\diamond_{d-1} \star G_4$  is not  $d$ -ball packable, with the exception of  $\diamond_{d+1} = \diamond_{d-1} \star C_4$ .

**3.4. Graphs in form of  $G_n \star G_m$ .** The following is a corollary of Corollary 3.2.

**Corollary 3.10.**  $G_6 \star G_3$  is not 3-ball packable, with the exception of  $C_6 \star C_3$ .

*Proof.* As in the proof of Theorem 3.1, we first construct a 3-ball packing of  $P_2 \star G_3$ , where  $P_2$  is represented by two disjoint half-spaces distance 2 apart, and  $G_3$  by three unit balls touching both hyperplanes, whose tangency graph is  $G_d$ .

If the centers of these unit balls are collinear, further construction is not possible. Otherwise, they must be on a circle in a 2-dimensional hyperplane parallel to the half-spaces. A ball touching all the three unit balls must center on the straight line passing through the center of this circle and perpendicular to this hyperplane.

We then continue the construction as in the proof of Theorem 3.1. However, since the three unit balls are not pairwise tangent, a ball must have a larger radius in order to touch all of them. Again, this makes it more difficult to avoid

self-intersection, which is already a mission impossible when the unit balls are pairwise tangent.

And as in the proof of Corollary 3.2, gaps on the path make the situation even worse. So the construction must fail.  $\square$

Therefore, if a graph is 3-ball packable, any induced subgraph in form of  $G_6 \star G_3$  must be in form of  $C_6 \star K_3$ .

By the same argument, we derive the following corollary from the fact that  $C_4 \star P_4$  is not 3-ball packable

**Corollary 3.11.**  $G_4 \star G_4$  is not 3-ball packable, with the exception of  $C_4 \star C_4$ .

Therefore, if a graph is 3-ball packable, any induced subgraph in form of  $G_4 \star G_4$  must be in form of  $C_4 \star C_4$ .

From the fact that  $\diamond_3 \star P_4$  is not 4-ball packable, we derive the following corollary, but the argument is slightly different:

**Corollary 3.12.**  $G_4 \star G_6$  is not 4-ball packable, with the exception of  $C_4 \star \diamond_3$ .

*Proof.* The proof is basically the same as Corollary 3.10.

Two vertices of  $G_4$  are represented by half-spaces, and  $G_6$  are represented by unit balls. If the centers of these unit balls are collinear, further construction is not possible. If the centers are on a 2-sphere, its diameter reaches its minimum *only* when  $G_6 = \diamond_3$ . If  $G_6$  is in any other form, a ball touching all the unit balls must have a larger radius, and the construction must fail.

We should be careful that it is possible to have the six unit balls centered on a circle. In this case, the radius of a ball touching all of them is at least 1 (luckily), which rules out the possibility of further construction.  $\square$

*Remark.* The argument in the proof of Corollaries 3.10 and 3.12 should be used with caution. As mentioned in the proof of Corollary 3.12, one must check carefully the non-generic cases, and make sure that nothing goes wrong.

For example, Corollary 3.12 can *not* be derived from the fact that  $K_4 \star P_6$  is not 4-ball packable. If we use the same argument, unit balls representing  $G_4$  must be centered on a 2-sphere, whose radius is minimum when  $G_4 = K_4$ . However, it is possible to have the centers on a circle, for example when  $G_4 = C_4$ . In this case, a ball touching all the four unit balls can have a radius as small as  $1/2$ , and its center is not restricted on a line. Indeed, we have the counterexample  $\diamond_3 \star C_4$ .

#### 4. BALL PACKABLE STACKED-POLYTOPAL GRAPHS

**4.1. More on stacked polytopes.** Since  $K_d \star P_m$  is the graph of a stacked  $(d+1)$ -polytope, Theorem 3.1 provides some examples of stacked  $(d+1)$ -polytope whose graph is not  $d$ -ball packable, and  $C_3 \star C_6$  provides an example of Apollonian 3-ball packing whose tangency graph is not stacked 4-polytopal. Therefore, in higher dimensions, the relation between Apollonian ball packings and stacked polytopes becomes much weaker.

The following remains true:

**Theorem 4.1.** *If the graph of a stacked  $(d+1)$ -polytope is  $d$ -ball packable, its ball packing is Apollonian and unique up to Möbius transformations and reflections.*

*Proof.* The Apollonianity can be easily seen by comparing the construction processes.

The uniqueness can be proved by an induction on the construction process. While a stacked polytope is built from a simplex, we construct its ball packing from a Descartes configuration, which is unique up to Möbius transformations and

reflections. For every stacking operation, a new ball was added into the packing to form a new Descartes configuration. We have an unique choice for every newly added ball, so the uniqueness is preserved at every step of construction.  $\square$

Given a ball packing  $\mathcal{S} = \{S_1, \dots, S_n\}$ , let  $\mathbf{c}_i$  be the center of  $S_i$ , a *stress* of  $\mathcal{S}$  is a real function  $T$  on the edge set of  $G(\mathcal{S})$  such that for all  $S_i \in \mathcal{S}$

$$\sum_{S_i S_j \text{ edge of } G(\mathcal{S})} T(S_i S_j)(\mathbf{c}_j - \mathbf{c}_i) = 0$$

We can view stress as forces between tangent balls when all the balls are in equilibrium. We say that  $\mathcal{S}$  is *stress-free* if it has no non-zero stress.

**Theorem 4.2** (Stress free). *If the graph of a stacked  $(d+1)$ -polytope is  $d$ -ball packable, its ball packing is stress-free.*

*Proof.* We construct the ball packing as we did in the proof of Theorem 4.1, and assume a non-zero stress.

The last ball  $S$  that is added into the packing has  $d+1$  “neighbor” balls tangent to it. If the stress is not zero on all the  $d+1$  edges incident to  $S$ , they can not be of the same sign, so there must be a hyperplane separating positive edges and negative edges. This contradicts the assumption that  $S$  is in equilibrium. So the stress must vanish on the edges incident to  $S$ . We then remove  $S$  and repeat the same argument on the second last ball, and so on, and finally conclude that the stress has to be zero on all the edges of  $G(\mathcal{S})$ .  $\square$

The theorem and the proof above was informally discussed in Section 8 in Kotlov, Lovász and Vempala’s famous paper on Colin de Verdière number [20]. In that paper, they defined another graph invariant  $\nu(G)$  using the notion of stress-freeness (a slightly different version), which turns out to be strongly related to Colin de Verdière number. Applying their results on stress-freeness, they concluded that (in our formulation) if the graph  $G$  of a stacked  $(d+1)$ -polytope with  $n$  vertices is  $d$ -ball packable, then  $\nu(G) \leq d+2$ , and the upper bound is achieved if  $n \geq d+4$ . However, they didn’t pay much attention to the existence of the ball packing in question. Theorem 3.1 shows that this kind of ball packing is not always constructible.

For a  $d$ -polytope  $\mathcal{P}$ , the *link* of a  $k$ -face  $F$  is the subgraph of  $G(\mathcal{P})$  induced by the common neighbors of the vertices of  $F$ . The following observation will be useful:

**Lemma 4.3.** *If  $\mathcal{P}$  is a stacked  $d$ -polytope, then the link of every  $k$ -face is stacked  $(d-k-1)$ -polytopal.*

**4.2. Weighted mass of a word.** The following theorem was proved in [16]

**Theorem 4.4.** *The 3-dimensional Apollonian group is a hyperbolic Coxeter group generated by the relations  $\mathbf{R}_i \mathbf{R}_i = \mathbf{I}$  and  $(\mathbf{R}_i \mathbf{R}_j)^3 = \mathbf{I}$  for  $1 \leq i \neq j \leq 5$ .*

As a sketch, their proof was based on the study of reduced words.

**Definition 4.5.** A word  $\mathbf{U} = \mathbf{U}_1 \mathbf{U}_2 \dots \mathbf{U}_n$  over the generator of the 3-dimensional Apollonian group (i.e.  $\mathbf{U}_i \in \{\mathbf{R}_1, \dots, \mathbf{R}_5\}$ ) is *reduced* if it does not contain

- subword of form  $\mathbf{R}_i \mathbf{R}_i$  for  $1 \leq i \leq 5$ ; or
- subword of form  $\mathbf{V}_1 \mathbf{V}_2 \dots \mathbf{V}_{2m}$  in which  $\mathbf{V}_1 = \mathbf{V}_3$ ,  $\mathbf{V}_{2m-2} = \mathbf{V}_{2m}$  and  $\mathbf{V}_{2j} = \mathbf{V}_{2j+3}$  for  $1 \leq j \leq 2m-2$ .

Notice that  $m = 2$  excludes the subwords of form  $(\mathbf{R}_i \mathbf{R}_j)^2$ . One can verify that non-reduced words can be simplified to reduced words using the generating relations. Then it suffices to prove that no nonempty reduced word, treated as product of matrices, is identity.

For proving this, they studied the sum of entries in the  $i$ -th row of  $\mathbf{U}$ , i.e.  $\sigma_i(\mathbf{U}) := \mathbf{e}_i^\top \mathbf{U} \mathbf{e}$ , and the sum of all the entries in  $\mathbf{U}$ , i.e.  $\Sigma(\mathbf{U}) := \mathbf{e}^\top \mathbf{U} \mathbf{e}$ . The latter is called the *mass* of  $\mathbf{U}$ . The quantities  $\Sigma(\mathbf{U}), \Sigma(\mathbf{R}_j \mathbf{U}), \sigma_i(\mathbf{U}), \sigma_i(\mathbf{R}_j \mathbf{U})$  satisfy a series of linear equations, which was used to inductively prove that  $\Sigma(\mathbf{U}) > \Sigma(\mathbf{U}')$  for a reduced word  $\mathbf{U} = \mathbf{R}_i \mathbf{U}'$ . Therefore  $\mathbf{U}$  is not an identity since  $\Sigma(\mathbf{U}) \geq \Sigma(\mathbf{R}_i) = 7 > \Sigma(\mathbf{I}) = 5$ .

We propose the following adaption: Given a weight vector  $\mathbf{w}$ , we define  $\sigma_i^w(\mathbf{U}) = \mathbf{e}_i^\top \mathbf{U} \mathbf{w}$  the weighted sum of entries in the  $i$ -th row of  $\mathbf{U}$ , and  $\Sigma^w(\mathbf{U}) = \mathbf{e}^\top \mathbf{U} \mathbf{w}$  the *weighted mass* of  $\mathbf{U}$ . We find that the following lemma can be proved with an argument similar as in [16]:

**Lemma 4.6.** *For dimension 3, if  $\Sigma^w(\mathbf{R}_i) \geq \Sigma^w(\mathbf{I})$  for any  $1 \leq i \leq 5$ , then for a reduced word  $\mathbf{U} = \mathbf{R}_i \mathbf{U}'$ , we have  $\Sigma^w(\mathbf{U}) \geq \Sigma^w(\mathbf{U}')$ .*

*Sketch of proof.* It suffices to replace “sum” by “weighted sum”, “mass” by “weighted mass”, and “ $>$ ” by “ $\geq$ ” in the proof of [16, Theorem 5.1].

It turns out that the following relations hold for  $1 \leq i, j \leq 5$ .

$$(7) \quad \begin{aligned} \sigma_i^w(\mathbf{R}_j \mathbf{U}) &= \begin{cases} \sigma_i^w(\mathbf{U}) & \text{if } i \neq j \\ \Sigma^w(\mathbf{U}) - 2\sigma_i^w(\mathbf{U}) & \text{if } i = j \end{cases} \\ \Sigma^w(\mathbf{R}_i \mathbf{U}) &= 2\Sigma^w(\mathbf{U}) - 3\sigma_i^w(\mathbf{U}) \end{aligned}$$

Then, if we define  $\delta_i^w(\mathbf{U}) := \Sigma^w(\mathbf{R}_i \mathbf{U}) - \Sigma^w(\mathbf{U})$ , the following relations hold:

$$\begin{aligned} \delta_i^w(\mathbf{R}_j \mathbf{U}) &= \begin{cases} \delta_i^w(\mathbf{U}) + \delta_j^w(\mathbf{U}) & \text{if } i \neq j \\ -\delta_i^w(\mathbf{U}) & \text{if } i = j \end{cases} \\ \delta_i^w(\mathbf{R}_j \mathbf{U}) &= \delta_j^w(\mathbf{R}_i \mathbf{U}) \text{ if } i \neq j \\ \delta_i^w(\mathbf{R}_j \mathbf{R}_i \mathbf{U}) &= \delta_j^w(\mathbf{U}) \end{aligned}$$

These are all the relations that are useful for the induction. The base case is already assumed in the condition of the theorem, which reads  $\delta_i^w(\mathbf{I}) \geq 0$  for  $1 \leq i \leq 5$ .

So the rest of the proof is exactly the same as in the proof of [16, Theorem 5.1]. For details of the induction, please refer to the original proof.

The conclusion is  $\delta_i^w(\mathbf{U}') \geq 0$ , i.e.  $\Sigma^w(\mathbf{U}) \geq \Sigma^w(\mathbf{U}')$ .  $\square$

**4.3. A generalization of Coxeter’s sequence.** Let  $\mathbf{U} = \mathbf{U}_n \cdots \mathbf{U}_2 \mathbf{U}_1$  be a word over the generators of the 3-dimensional Apollonian group (we have a good reason for inverting the order of the index). Let  $\mathbf{M}_0$  be the curvature-center matrix of an initial Descartes configuration, consisting of the first five balls in the sequence  $S_1, \dots, S_5$ .

The curvature-center matrices recursively defined by  $\mathbf{M}_i = \mathbf{U}_i \mathbf{M}_{i-1}$  ( $1 \leq i \leq n$ ), define a sequence of Descartes configurations. We take  $S_{4+i}$  to be the single ball that is in the configuration at step  $i$  but not in the configuration at step  $i-1$ . This generates a sequence of  $4+n$  balls.

Coxeter’s loxodromic sequence in dimension 3 is therefore generated by an infinite word of period 5, e.g.  $\mathbf{U} = \cdots \mathbf{R}_2 \mathbf{R}_1 \mathbf{R}_5 \mathbf{R}_4 \mathbf{R}_3 \mathbf{R}_2 \mathbf{R}_1$ , which can be viewed as a special case of our sequence.

The following is a corollary of Lemma 4.6:

**Corollary 4.7.** *If  $\mathbf{U}$  is reduced and  $\mathbf{U}_1 = \mathbf{R}_1$ , then in the sequence constructed as above,  $S_1$  is disjoint from all balls except the first five.*

*Proof.* We take the initial configuration to be the configuration used in the proof of Theorem 3.1. Assume  $S_1$  to be the lower half-space  $x_1 \leq 0$ , then the initial

curvature-center matrix is

$$\mathbf{M}_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & \sqrt{1/3} \\ 1 & 1 & -1 & \sqrt{1/3} \\ 1 & 1 & 0 & -2\sqrt{1/3} \end{pmatrix}$$

Every row corresponds to the curvature-center coordinates  $\mathbf{m}$  of a ball. The first coordinate  $m_1$  is the curvature  $\kappa$ . If the curvature is not zero, the second coordinate  $m_2$  is the ‘‘height’’ of the center times the curvature, i.e.  $x_1\kappa$ .

Now take the second column of  $\mathbf{M}_0$  to be the weight vector  $\mathbf{w}$ . That is,

$$\mathbf{w} = (-1, 1, 1, 1, 1)^\top.$$

We have  $\Sigma^w(\mathbf{R}_1) = 9 > \Sigma^w(\mathbf{I}) = 3$  and  $\Sigma^w(\mathbf{R}_j) = 3 = \Sigma^w(\mathbf{I})$  for  $j > 1$ . Applying Lemma 4.6 we have

$$\Sigma^w(\mathbf{U}_k \mathbf{U}_{k-1} \cdots \mathbf{U}_2 \mathbf{R}_1) \geq \Sigma^w(\mathbf{U}_{k-1} \cdots \mathbf{U}_2 \mathbf{R}_1)$$

By (7), this means that

$$\sigma_j^w(\mathbf{U}_k \cdots \mathbf{U}_2 \mathbf{R}_1) \geq \sigma_j^w(\mathbf{U}_{k-1} \cdots \mathbf{U}_2 \mathbf{R}_1)$$

if  $\mathbf{U}_k = \mathbf{R}_j$ , or equality if  $\mathbf{U}_k \neq \mathbf{R}_j$ .

The key observation is that  $\sigma_j^w(\mathbf{U}_k \cdots \mathbf{R}_1)$  is exactly the second curvature-center coordinate  $m_2$  of  $j$ -th ball in the  $k$ -th Descartes configuration. So at every step, a ball is replaced by another ball with a larger or same value for  $m_2$ . Especially, since  $\sigma_j^w(\mathbf{R}_1) \geq 1$  for  $1 \leq j \leq 5$ , we conclude that  $m_2 \geq 1$  for every ball.

Four balls in the initial configuration have  $m_2 = 1$ . Once they are replaced, the new ball must have a strictly larger value of  $m_2$ . This can be seen from (4) and notice that the r.h.s. of (4) is at least 4 since the very first step of the construction. We then conclude that  $m_2 > 1$  for all balls except the first five. This exclude the possibility of curvature zero, so  $x_1\kappa > 1$  for all balls except the first five.

For dimension 3, Equation (4) is integral. Therefore the curvature-center coordinates of all balls are integral (see [16] for more on integrality of Apollonian packings). Since the sequence is a packing (by the result of [4]), no ball in the sequence has a negative curvature. By the definition of the curvature-center coordinates, the fact that  $m_2 > 1$  exclude the possibility of curvature 0. Therefore all balls have a positive curvature  $\kappa \geq 1$  except the first two.

For conclusion,  $x_1\kappa > 1$  and  $\kappa \geq 1$  implies that  $x_1 > 1/\kappa$ , therefore disjoint from the half-space  $x_1 \leq 0$ .  $\square$

#### 4.4. Main result.

**Lemma 4.8.** *Let  $G$  be a stacked 4-polytopal graph. If  $G$  has an induced subgraph of form  $G_3 \star G_6$ ,  $G$  must have an induced subgraph of form  $K_3 \star P_6$ .*

Note that  $C_6 \star K_3$  is not an induced subgraph of any stacked polytopal graph.

*Proof.* Let  $H$  be an induced subgraph of  $G$  of form  $G_3 \star G_6$ . Let  $v \in V(H)$  be the last vertex in  $H$  that is added into the polytope during the construction. We have  $\deg_H v = d + 1$ , and the neighbors of  $v$  induce a complete graph.

Since  $\ell(d) = 6 > d + 1 = 4$ ,  $v$  must be a vertex in the part  $G_{\ell(d)} = G_6$ , so that the other part, being an induced subgraph of  $K_4$ , is the complete graph  $K_3$ . Therefore  $H$  is of the form  $K_3 \star G_6$ .

By Lemma 4.3, in the stacked 4-polytope with graph  $G$ , the link of every 2-face is stacked 1-polytopal. In other words, the common neighbors of  $K_3$  induce a path  $P_n$  where  $n \geq 6$ . Therefore  $G$  must have an induced subgraph of form  $P_6 \star K_3$ .  $\square$

*proof of Theorem 1.1.* The “only if” is by Theorem 3.1 and Lemma 4.8. We prove “if” by induction on number of vertices.

The complete graph on 5 vertices is of course 3-ball packable. Assume that every stacked 4-polytope with less than  $n$  vertices satisfies this theorem. We now study a stacked 4-polytope  $\mathcal{P}$  of  $n + 1$  vertices that do not have six 4-cliques in its graph with 3 vertices in common, and assume that  $G(\mathcal{P})$  is not ball packable.

Let  $u, v$  be two vertices of  $G(\mathcal{P})$  of degree 4. Deleting  $v$  from  $\mathcal{P}$  leaves a stacked polytope  $\mathcal{P}'$  of  $n$  vertices that satisfies the condition of the theorem, so  $G(\mathcal{P}')$  is ball packable by the assumption of induction. In the ball packing of  $\mathcal{P}'$ , the four balls corresponding to the neighbors of  $v$  are pairwise tangent. We then construct the ball packing of  $\mathcal{P}$  by adding a ball  $S_v$  that is tangent to these four balls. We have only one choice (the other choice will coincide with another ball), but since  $G(\mathcal{P})$  is not ball packable,  $S_v$  must intersect some other balls.

However, deleting  $u$  also leaves a stacked polytope whose graph is ball packable. Therefore  $S_v$  must intersect  $S_u$  and only  $S_u$ . Now if there is another vertex  $w$  of degree 4 different from  $u$  and  $v$ , deleting  $w$  leaves a stacked polytope whose graph is ball packable, which produces a contradiction. Therefore  $u$  and  $v$  are the only vertices of degree 4.

Let  $\mathcal{T}$  be the dual tree of  $\mathcal{P}$ , its leaves correspond to vertices of degree 4. So  $\mathcal{T}$  must be a path, whose two ends correspond to  $u$  and  $v$ .

We can therefore construct the ball packing of  $\mathcal{P}$  as a generalised Coxeter’s sequence that we just studied. The first ball is  $S_u$ . The construction word does not contain any subword of form  $(\mathbf{R}_i \mathbf{R}_j)^2$  (which produces  $C_6 \star K_3$  and violates the condition) or  $\mathbf{R}_i \mathbf{R}_i$ , one can therefore always simplify the word into a *non-empty* reduced word. This does not change the corresponding matrix, so the last ball in the sequence,  $S_v$ , remains the same.

Then Corollary 4.7 implies that  $S_u$  and  $S_v$  are disjoint which contradicts our previous discussion. Therefore  $G(\mathcal{P})$  is ball packable.  $\square$

Therefore, the relation between 4-trees, stacked 4-polytopes and Apollonian 3-ball packings can be illustrated as follows:

$$4\text{-tree} \xrightarrow[\text{sharing a 4-clique}]{\text{no three 5-cliques}} \text{stacked 4-polytope} \xrightarrow[\text{sharing a 3-clique}]{\text{no six 4-cliques}} \text{Apollonian 3-ball packing}$$

However, there exist Apollonian 3-ball packings whose tangency graphs are not 4-trees.

**4.5. Higher dimensions.** In this part, we would like to show that the following theorem is not valid only in dimension 3.

**Theorem 4.9.** *For  $d > 3$ , if a  $d$ -ball packing is Apollonian, then its tangency graph is stacked  $(d + 1)$ -polytopal.*

We will need the following lemma:

**Lemma 4.10.** *Let  $d \neq 3$  and  $\mathbf{w}$  is a  $(d + 2)$  dimensional vector  $(-1, 1, \dots, 1)^\top$ . Let  $\mathbf{U} = \mathbf{U}_1 \mathbf{U}_2 \cdots \mathbf{U}_n$  be a word over the generator of the  $d$ -dimensional Apollonian group (i.e.  $\mathbf{U}_i \in \{\mathbf{R}_1, \dots, \mathbf{R}_{d+2}\}$ ). If  $\mathbf{U}$  does not contain any subword in form of  $\mathbf{R}_i \mathbf{R}_i$ , then  $\sigma_i^{\mathbf{w}}(\mathbf{U}) \neq 1$  for  $1 \leq i \leq d + 2$ .*

*Proof.* It is shown in [16, Theorem 5.2] that the  $j$ -th row of  $\mathbf{U} - \mathbf{I}$  is a linear combination of rows of a matrix  $\mathbf{A} = \frac{1}{d-1} \mathbf{e} \mathbf{e}^\top - d \mathbf{I}$ . However, the weighted row sum  $\sigma_i^{\mathbf{w}}(\mathbf{A})$  of the  $i$ -th row of  $\mathbf{A}$  is 0 except for  $i = 1$ , whose weighted row sum is  $\frac{2}{d-1}$ . So  $\sigma_i^{\mathbf{w}}(\mathbf{U} - \mathbf{I}) = \frac{2C_i}{d-1}$ , where  $C_i$  is the coefficient in the linear combination.

According to the calculation in [16],  $C_i$  is a polynomial in the variable  $x_d = \frac{1}{d-1}$  in form of

$$C_i(x_d) = \sum_{k=0}^{n_i-1} c_k 2^{k+1} x_d^k$$

where  $n_i$  is the length of the longest subword that starts with  $\mathbf{U}_i$  and ends with  $\mathbf{U}_1$ , and  $c_k$  are integer coefficients. The leading term is  $2^{n_i} x_d^{n_i-1}$  (i.e.  $c_{n_i-1} = 1$ ). Then, by the same argument as in [16], we can show that  $C_i(x_d)$  is not zero. Therefore, for  $i \neq 1$ ,

$$\sigma_i^w(\mathbf{U}) = \frac{2C_i}{d-1} + \sigma_i^w(\mathbf{I}) = \frac{2C_i}{d-1} + 1 \neq 1.$$

For  $i = 1$ , since  $\sigma_1^w(\mathbf{I}) = -1$ , we need to prove that  $C_1 \neq d-1$ . So the calculation is slightly different.

If  $C_1 = d-1$ , then  $x_d$  is a root of the polynomial  $x_d C_1(x_d) - 1$ , whose leading term is  $(2x_d)^{n_1}$ . By the rational root theorem,  $d-1$  divides  $2^{n_1}$ . So we must have  $d-1 = 2^p$  for some  $p > 1$ , that is,  $x_d = 2^{-p}$ . We then have

$$\sum_{k=1}^{n_1} c_{k-1} 2^{k(1-p)} = 1.$$

Multiply both side by  $2^{(p-1)n_1}$ , we got

$$\sum_{k=1}^{n_1} c_{k-1} 2^{(p-1)(n_1-k)} = 2^{(p-1)n_1}.$$

The right hand side is even since  $(p-1)n_1 > 0$ . The terms in the summation are even except for the last one since  $(p-1)(n_1-k) > 0$ . The last term in the summation is  $c_{n_1-1} 2^0 = 1$ , so the left hand side is odd, which is the desired contradiction. Therefore

$$\sigma_1^w(\mathbf{U}) = \frac{2C_1}{d-1} + \sigma_1^w(\mathbf{I}) \neq 1.$$

□

*proof of Theorem 4.9.* Consider a construction process of the Apollonian ball packing. The theorem is true at the first step. Assume that it remains true before the introduction of a ball  $S$ , we are going to prove that, when added,  $S$  touches exactly  $d+1$  pairwise tangent balls in the packing.

If this is not the case, assume a  $(d+2)$ -th ball  $S'$  touching  $S$ , then we can find a sequence of Descartes configurations, with  $S'$  in the first configuration and  $S$  in the last one, that is generated by a word over the generators of the  $d$ -dimensional Apollonian group with distinct adjacent terms.

Without loss of generality, we arrange the first Descartes configuration in the sequence as in the proof of the Corollary 4.7, and let  $S'$  be the lower half-space  $x_1 \leq 0$ . Then Lemma 4.10 says that no ball (except for the first  $d+2$  balls) in this sequence is tangent to  $S'$ , contradicting our assumption.

By induction, every newly added ball touches exactly  $d+1$  pairwise tangent balls, so the tangency graph is a  $(d+1)$ -tree, and therefore  $(d+1)$ -polytopal. □

Now the remaining problem is to characterise stacked  $(d+1)$ -polytopal graphs that are  $d$ -ball packable. From Corollary 3.5, we know that if a  $(d+1)$ -tree is  $d$ -ball packable, the number of  $(\alpha+3)$ -cliques sharing a  $(\alpha+2)$ -clique is at most  $k(d-1, \alpha)$  for all  $1 \leq \alpha \leq d-1$ . Following the patterns in Theorems 1.1 and 2.8, we propose the following conjecture:

**Conjecture 4.11.** *For an integer  $d \geq 2$ , there is  $d-1$  integers  $n_1, \dots, n_{d-1}$  such that a  $(d+1)$ -tree is  $d$ -ball packable if and only if the number of  $(\alpha+3)$ -cliques sharing an  $(\alpha+2)$ -clique is at most  $n_\alpha$  for all  $1 \leq \alpha \leq d-1$ .*

## 5. A DISCUSSION ON EDGE-TANGENT POLYTOPES

A convex  $(d + 1)$ -polytope is *edge-tangent* if all of its edges are tangent to a  $d$ -sphere called *midsphere*.

One can derive from the disk packing theorem that<sup>1</sup>:

**Theorem 5.1.** *Every convex 3-polytope has an edge-tangent realization.*

Eppstein, Kuperberg and Ziegler have proved in [12] that no stacked 4-polytopes with more than six vertices has an edge-tangent realization. Comparing to Theorem 1.1, we see that ball packings and edge-tangent polytopes are not so closely related in higher dimensions: a polytope with ball packable graph does not, in general, have an edge-tangent realization.

In this last section, we would like to discuss about this difference in detail.

Let  $\mathbb{S}^d \subset \mathbb{R}^{d+1}$  be the unit sphere  $\{\mathbf{x} \mid x_0^2 + \cdots + x_d^2 = 1\}$ . For a spherical cap  $C \subset \mathbb{S}^d$  of radius smaller than  $\pi/2$ , its boundary can be viewed as the intersection of  $\mathbb{S}^d$  with a  $d$ -dimensional hyperplane  $H$ , which can be uniquely written in form of  $H = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{v} \rangle = 1\}$ .

Explicitly, if  $\mathbf{c} \in \mathbb{S}^d$  is the center of  $C$ , and  $\theta < \pi/2$  is its spherical radius, then  $\mathbf{v} = \mathbf{c}/\cos\theta$ . We can interpret  $\mathbf{v}$  as the center of the unique sphere that intersects  $\mathbb{S}^d$  *orthogonally* along the boundary of  $C$ , or as the apex of the unique cone whose boundary is tangent to  $\mathbb{S}^d$  along the boundary of  $C$ . We call  $\mathbf{v}$  the *polar vertex* of  $C$ , and  $H$  the *hyperplane* of  $C$ .

We see that  $\langle \mathbf{v}, \mathbf{v} \rangle > 1$ . If the boundary of two caps  $C$  and  $C'$  intersect orthogonally, their polar vertices  $\mathbf{v}$  and  $\mathbf{v}'$  satisfy  $\langle \mathbf{v}, \mathbf{v}' \rangle = 1$ , i.e. the polar vertex of one is on the hyperplane of the other. If  $C$  and  $C'$  have disjoint interiors,  $\langle \mathbf{v}, \mathbf{v}' \rangle < 1$ . If  $C$  and  $C'$  are tangent at  $\mathbf{t} \in \mathbb{S}^d$ , the segment  $\mathbf{v}\mathbf{v}'$  is tangent to  $\mathbb{S}^d$  at  $\mathbf{t}$ .

Now, given a  $d$ -ball packing  $\mathcal{S} = \{S_0, \dots, S_n\}$  in  $\hat{\mathbb{R}}^d$ , we can construct a  $(d + 1)$ -polytope  $\mathcal{P}$  as follows:

View  $\hat{\mathbb{R}}^d$  as the hyperplane  $x_0 = 0$  in  $\hat{\mathbb{R}}^{d+1}$ . Then a stereographic projection maps  $\hat{\mathbb{R}}^d$  to  $\mathbb{S}^d$ , and  $\mathcal{S}$  is mapped to a packing of spherical caps on  $\mathbb{S}^d$ . We may assume that the radii of all caps are smaller than  $\pi/2$ . If it is not the case, we apply a Möbius transformation. Then  $\mathcal{P}$  is obtained by taking the convex hull of the polar vertices of all the spherical caps.

**Theorem 5.2.** *If a  $(d + 1)$ -polytope  $\mathcal{P}$  is constructed as described above from a  $d$ -sphere packing  $\mathcal{S}$ , then  $G(\mathcal{S})$  is isomorphic to a spanning subgraph of  $G(\mathcal{P})$ .*

*Proof.* For every  $S_i \in \mathcal{S}$ , the polar vertex  $\mathbf{v}_i$  of the corresponding cap is a vertex of  $\mathcal{P}$ , since the hyperplane  $\{\mathbf{x} \mid \langle \mathbf{v}_i, \mathbf{x} \rangle = 1\}$  divides  $\mathbf{v}_i$  from other vertices.

For every edge  $S_i S_j$  of  $G(\mathcal{S})$ ,  $\mathbf{v}_i \mathbf{v}_j$  is an edge of  $\mathcal{P}$ . Since  $\mathbf{v}_i \mathbf{v}_j$  is tangent to the unit sphere,  $\langle \mathbf{v}, \mathbf{v} \rangle \geq 1$  for all points  $\mathbf{v}$  on the segment  $\mathbf{v}_i \mathbf{v}_j$ . If  $\mathbf{v}_i \mathbf{v}_j$  is not an edge of  $\mathcal{P}$ , some point  $\mathbf{v} = \lambda \mathbf{v}_i + (1 - \lambda) \mathbf{v}_j$  ( $0 \leq \lambda \leq 1$ ) can be written as a convex combination of other vertices  $\mathbf{v} = \sum_{k \neq i, j} \lambda_k \mathbf{v}_k$ , where  $\lambda_k \geq 0$  and  $\sum \lambda_k \leq 1$ . Then we have:

$$1 \leq \langle \mathbf{v}, \mathbf{v} \rangle = \left\langle \lambda \mathbf{v}_i + (1 - \lambda) \mathbf{v}_j, \sum_{k \neq i, j} \lambda_k \mathbf{v}_k \right\rangle < 1$$

since  $\langle \mathbf{v}_i, \mathbf{v}_j \rangle < 1$  if  $i \neq j$ . This is a contradiction.  $\square$

Now consider a  $d$ -ball packing  $\mathcal{S}$ . If a polytope  $\mathcal{P}$  is constructed from  $\mathcal{S}$  as described above, it is possible that  $G(\mathcal{P})$  is not isomorphic to  $G(\mathcal{S})$ . That is, there

<sup>1</sup>Schramm [28] said that the theorem is first claimed by Koebe [19], who only proved the simplicial and simple cases. He credits the full proof to Thurston [32], but the online version of Thurston's lecture notes only gave a proof for simplicial cases.



may be an edge of  $\mathcal{P}$  that does not correspond to any edge of  $G(\mathcal{S})$ . This edge will intersect  $\mathbb{S}^d$ , and  $\mathcal{P}$  is therefore not edge-tangent.

On the other hand, if the graph of a polytope  $\mathcal{P}$  is isomorphic to  $G(\mathcal{S})$ , since the graph does not determine the combinatorial type of a polytope,  $\mathcal{P}$  may be different from the one constructed from  $\mathcal{S}$ . So a polytope whose graph is ball packable may not be edge-tangent.

A polytope is edge-tangent if it's constructed from a ball packing, and its graph is isomorphic to the tangency relation of this ball packing. Neither condition can be removed. For the other direction, given an edge-tangent polytope  $\mathcal{P}$ , one can always obtain a ball packing of  $G(\mathcal{P})$  by reversing the construction above.

Disk packings are excepted from these problems. In fact, it is easier [27] to derive Theorem 5.1 from the following version of the disk packing theorem, which is equivalent but contains more information:

**Theorem 5.3** (Brightwell and Scheinerman [5]). *For every 3-polytope  $\mathcal{P}$ , there is a pair of disk packings, one consists of vertex-disks representing  $G(\mathcal{P})$ , the other consists of face-disks representing the dual  $G(\mathcal{P}^*)$ , such that:*

- *For each edge  $e$  of  $\mathcal{P}$ , the vertex-disks corresponding to the two endpoints of  $e$  and the face-disks corresponding to the two faces bounded by  $e$  meet at a same point;*
- *A vertex-disk and a face-disk intersect iff the corresponding vertex is on the boundary of the corresponding face, in which case their boundaries intersect orthogonally.*

*This representation is unique up to Möbius transformations.*

The presence of the face-disks and the orthogonal intersections guarantee the incidence relations between vertices and faces, and therefore fix the combinatorial type of the polytope.

We can generalize this into higher dimensions:

**Theorem 5.4.** *Given a  $(d + 1)$ -polytope  $\mathcal{P}$ , if there is a packing of  $d$ -dimensional vertex-balls representing  $G(\mathcal{P})$ , together with a collection of  $(d - 1)$ -dimensional facet-balls indexed by the facets of  $\mathcal{P}$ , such that:*

- *For each edge  $e$  of  $\mathcal{P}$ , the vertex-balls corresponding to the two endpoints of  $e$  and the boundaries of the facet-balls corresponding to the facets bounded by  $e$  meet at a same point;*
- *Either a vertex-ball and a facet-ball are disjoint, or their boundaries intersect at a nonobtuse angle;*
- *The boundary of a vertex-ball and the boundary of a facet-ball intersect orthogonally iff the corresponding vertex is on the boundary of the corresponding facet.*

*Then  $\mathcal{P}$  has an edge-tangent realization.*

Again, the convexity is guaranteed by the disjointness and nonobtuse intersections, and the incidence relations are guaranteed by the orthogonal intersections.

For an edge-tangent polytope, the facet-balls can be obtained by intersecting the midsphere with the facets. However, they do not form a  $d$ -ball packing for  $d > 2$ .

For an arbitrary polytope of dimension 4 or higher, even if its graph is ball packable, the facet-balls satisfying the conditions of Theorem 5.4 do not in general exist.

For example, consider the stacked 4-polytope with 7 vertices. The packing of its graph  $K_3 \star P_4$  is constructed in the proof of Theorem 3.1. We notice that a ball whose boundary orthogonally intersects the boundary of the three unit balls and the boundary of ball  $C$ , have to intersect the boundary of ball  $D$  orthogonally

(see Figure 1), thus violates the last condition in Theorem 5.4. One can check the polytope constructed from this packing, and find that it's not stacked and not simplicial.

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