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Kernel B–Splines and Interpolation

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Abstract

This paper applies difference operators to conditionally positive definite kernels in order to generate *kernel B–splines* that have fast decay towards infinity. Interpolation by these new kernels provides better condition of the linear system, while the kernel B–spline inherits the approximation orders from its native kernel. We proceed in two different ways: either the kernel B–spline is constructed adaptively on the data knot set X, or we use a fixed difference scheme and shift its associated kernel B–spline around. In the latter case, the kernel B–spline so obtained is strictly positive in general. Furthermore, special kernel B–splines obtained by hexagonal second finite differences of multiquadrics are studied in more detail. We give suggestions in order to get a consistent improvement of the condition of the interpolation matrix in applications.

AMS subject classification: 41A05, 65D10

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1 Introduction

The reconstruction of multivariate functions from discrete data by using the reproducing kernel of some semi–Hilbert space is an increasingly popular [9, 3, 4] technique. For computational efficiency, one should look for cheaply available kernels with good decay at infinity, and for other practical reasons certain unbounded kernels like the multiquadrics or the thin–plate spline are useful. To overcome this apparent contradiction, the situation on the infinite grid [3] and certain preconditioning techniques [6] suggest to take linear combinations of unbounded kernels in order to generate new kernels with strong decay properties.

Applying difference operators to linear systems arising from kernel interpolation problems is a well-known strategy for preconditioning (see [3] and references there). In particular, we recall the paper by Dyn et al. [5] in which the bidimensional discretization Δ_h of the Laplacian operator is used, and the paper by Powell [7] in which it is observed that the use of Δ_h provides a difference operator annihilating all linear polynomials. However, we do not apply differences to linear systems. Instead, we use new kernels defined by difference operators acting on the original kernels. These two strategies are closely related, but the latter allows direct application of all powerful tools for kernel analysis.

In §2 and in §3 we define kernel B-splines via differences of conditionally positive definite kernels. This reduces the order of a possible singularity of their Fourier transform at the origin and turns increasing kernels into decaying kernels, if the difference order is large enough. Using this technique, we prove sharp results on the decay of our kernel B–splines at infinity in the case of multiquadrics and of polyharmonic splines. The construction of the required differences proceeds via linear combinations of local point evaluations that annihilate all polynomials of some order k greater than the order of conditional positivity of the kernel.

In §5 we take a fixed local difference scheme and apply it twice to a translation–invariant kernel to define a new B–spline kernel which turns out to be (strictly) positive definite, if the order of the difference scheme is large enough. This leads to strongly decaying kernels with good approximation and stability properties. In contrast to [3] on the grid, we assume no regularity of the data here.

2 Notation

Let $\Phi : I\!R^d \to I\!R^d$ be a translation–invariant [10] kernel used on two arguments $x, y \in I\!R^d$ as $\Phi(x-y)$ conditionally positive of order m on $I\!R^d$ and let $X, Y, \ldots$ denote finite subsets of separated points of $I\!R^d$. We let the space dimension d be fixed and remove it from further notation. The space of d–variate polynomials up to order at most k will be denoted by $I\!P_k$, its dimension is $q(k) = \binom{k+d-1}{d}$, and a basis is denoted by $p_1, \ldots, p_{q(k)}$. If X is a point set in $I\!R^d$ consisting of $M := |X|$ elements, we define the $|X| \times q(k)$ matrix

$$P_{X,k} := (p_j(x_i))_{x_i \in X, 1 \le j \le q(k)}$$

and the space

$$V_{X,k} = \left\{ \alpha \in I\!R^{|X|} \; : \; P^T_{X,k}\alpha = 0 \right\}. \tag{1}$$

We generally assume $k \ge m$, $|X| \ge q(k) \ge q(m)$, and

$$\text{rank } P_{X,k} = q(k). \tag{2}$$

Note that for $k = m$ this is the standard additional condition on the point locations, ensuring solvability of the interpolation problem on X for a conditionally positive definite kernel of order m. For any two finite sets X and Y we use the notation

$$A_{X,Y} := (\Phi(x_j - y_l))_{x_j \in X, y_l \in Y}$$

for the $|X| \times |Y|$ matrix of values of the kernel Φ on X and Y.

Definition 2.1 *For each $|X| \subset I\!R^d$, each k with $q(k) \le |X|$ and each $\alpha \in V_{X,k}$ we call a function*

$$u_{X,k,\alpha}(x) := \sum_{x_j \in X} \alpha_j \Phi(x - x_j) \tag{3}$$

a kernel B–spline based on Φ with knot set X and annihilation order k.

Note that this generalizes the standard univariate B–spline definition, but it will in general not yield a function with compact support. However, it will in many cases provide a function with strong decay towards infinity, and we shall address this question in the next section.

3 Properties

First of all we observe that the decay of any kernel to infinity depends on the behavior of its Fourier transform at the origin. Therefore we take (3) and evaluate the Fourier transform as

$$\begin{aligned} \hat{u}_{X,k,\alpha}(\omega) &= \hat{\Phi}(\omega)\sigma_{X,k,\alpha}(\omega) \\ \sigma_{X,k,\alpha}(\omega) &= \sum_{x_j \in X} \alpha_j e^{-i\omega^t x_j}. \end{aligned} \tag{4}$$

The condition in (1) yields

$$\sigma_{X,k,\alpha}(\omega) = \mathcal{O}(\|\omega\|_2^k) \text{ near } \omega = 0,$$

following from Taylor expansion of the exponential around zero (see e.g. [11]). Thus the second factor in the Fourier transform of the kernel B–spline modifies the behavior at the origin. When going over from Φ to the B–spline kernel $u_{X,k,\alpha}$, the order of the possible singularity of the Fourier transform reduces by at least k.

To prove somewhat more precise results, consider the identity

$$\|x-y\|_2^2 = \|x-z\|_2^2 \cdot (1+F(x,y,z))$$

with

$$F(x,y,z) = \frac{\|y\|_2^2 - \|z\|_2^2 + 2x^T(z-y)}{\|x-z\|_2^2}$$

for sufficiently large x and bounded y, z. We shall use the fact that F is a quadratic polynomial in y and decays like $\|x\|_2^{-1}$ for $x \to \infty$.

Consider classical multiquadrics first. Then, by expansion of $-\sqrt{1+t}$ around zero, we get

$$\begin{array}{rl} & -(1+\|x-x_j\|_2^2)^{1/2} \\ = & -\|x-z\|_2\Big(\|x-z\|_2^{-2} + 1 + F(x,x_j,z)\Big)^{1/2} \\ = & -\|x-z\|_2 \sum\limits_{\ell=0}^{\infty} c_\ell \left((\|x-z\|_2^{-2} + F(x,x_j,z)\right)^\ell \\ = & -\sum\limits_{\ell=0}^{\infty} c_\ell \|x-z\|_2^{1-2\ell}\left(1-\|z\|_2^2+\|x_j\|_2^2+2x^T(z-x_j)\right)^\ell \\ = & -\sum\limits_{\ell=0}^{\infty} c_\ell \|x-z\|_2^{1-2\ell} \sum\limits_{i_0+i_1+i_2=\ell} \binom{\ell}{i}\left(1-\|z\|_2^2\right)^{i_0} \|x_j\|_2^{2i_1}\left(2x^T(z-x_j)\right)^{i_2}. \end{array}$$

Now let α be a vector that annihilates polynomials on X up to order k as in (1), and form the multiquadric kernel B–spline with these coefficients. Then only terms with $2i_1 + i_2 \geq k$ are left in the sum, and we get a decay order of at least

$$1-2\ell+i_2 = 1-2i_0-2i_1-2i_2+i_2 = 1-2i_0-2i_1-i_2 \leq 1-2i_0-k \leq 1-k.$$

Let us now do the same trick with the polyharmonic spline. First we consider the case of d even, $\Phi(t) = \|t\|^{2s-d} log\|t\|^2$ with $s \in \mathcal{Z}_+$, and we put $2r = 2s-d$. We find

$$\begin{array}{rl} & \|x-x_j\|_2^{2r}\log\|x-x_j\|_2^2 \\ = & \|x-x_j\|_2^{2r}\log\left(\|x-z\|_2^2\,(1+F(x,x_j,z))\right) \\ = & \|x-x_j\|_2^{2r}\left(\log\|x-z\|_2^2+\log(1+F(x,x_j,z))\right) \\ = & \left(\|x-x_j\|_2^{2r}\log\|x-z\|_2^2\right)+\left(\|x-x_j\|_2^{2r}\log(1+F(x,x_j,z))\right). \end{array}$$

Again, let α be a vector that annihilates polynomials on X up to order k as in (1), and form the kernel B–spline with these coefficients. We now assume

$k \geq (2r+1)$ to get rid of the first summand of the above equation, because it is a quadratic polynomial in x_j. We are left with the second one, rewrite it as

$$\begin{aligned} & \|x - x_j\|_2^{2r} \log(1 + F(x, x_j, z)) \\ = & \|x - z\|_2^{2r} (1 + F(x, x_j, z))^r \log(1 + F(x, x_j, z)) \end{aligned}$$

and expand $(1+t)^r \log(1+t)$ around zero to get

$$\begin{aligned} & \|x - z\|_2^{2r} (1 + F(x, x_j, z))^r \log(1 + F(x, x_j, z)) \\ = & \|x - z\|_2^{2r} \sum_{\ell=1}^{\infty} c_\ell F(x, x_j, z)^\ell \\ = & \sum_{\ell=1}^{\infty} c_\ell \|x - z\|_2^{2r-2\ell} \left(\|x_j\|_2^2 - \|z\|_2^2 + 2x^T(z - x_j) \right)^\ell \\ = & \sum_{\ell=1}^{\infty} c_\ell \|x - z\|_2^{2r-2\ell} \sum_{i_0+i_1+i_2=\ell} \binom{\ell}{i} \left(-\|z\|_2^2 \right)^{i_0} \|x_j\|_2^{2i_1} \left(2x^T(z - x_j) \right)^{i_2} . \end{aligned}$$

With the same argument as for the multiquadric, we now get a decay order of at least $2r - k$ for $k \geq 2r + 1$.

In the case of d odd, the polyharmonic spline is defined as $\|t\|^{2s-d}$ with $s \in \mathcal{Z}_+$; by proceeding as before, we obtain a decay order of at least $2s-d-k$ for $k \geq 2s - d + 1$.

Theorem 3.1 *If k is the annihilation order of its coefficient vector, a kernel B–spline based on a classical multiquadric has decay order $1 - k$ at least at infinity, for $k \geq 2$. A kernel B–spline based on a classical polyharmonic spline has decay order $2s - d - k$ at least, for $k \geq 2s - d + 1$.*

Note that the above technique does not make specific use of radiality, and it gives decay results for any point configuration.

Moreover, we point out that the stated decay rates can be numerically confirmed, but there is no mathematical proof yet for optimality of these rates.

4 Construction and Experiments

This section deals with the numerical construction of kernel B–splines. Omitting indices k and X from (1), we have to construct vectors α with $P^T\alpha = 0$. These will not be unique, and there may be additional conditions that we can impose. In what follows, we shall ignore permutations of points (or, equivalently, columns of P and elements of α). A standard way to handle the condition $P^T\alpha = 0$ in view of the rank property (2) is to find an orthogonal basis of the nullspace of P^T, as provided by the MATLAB command $B = null(P')$. This yields a matrix B of size $|X| \times (|X| - q(k))$ with $B^T B = I$ and $P^T B = 0$. Any $\alpha = B\gamma$ with some $\gamma \in I\!R^{|X|-q(k)}$ will do, and we can impose other conditions to restrict γ.

If the standard RBF system for solving an interpolation problem with $k \geq m$ on X is written as

$$\begin{array}{rcrcl} A_{X,X}\alpha & + & P_{X,k}\beta & = & f_X \\ P_{X,k}^T\alpha & + & 0 & = & 0 \end{array}$$

we can use the matrix B for solving

$$\begin{array}{rcl} B^T A_{X,X} B\gamma & = & B^T f_X \\ \alpha & = & B\gamma \\ P_{X,k}\beta & = & f_X - A_{X,X}\alpha \end{array} \tag{5}$$

instead.

We did many experiments in 2D with the multiquadric and the thin plate spline, respectively, using the matrix B as above and looking at the condition of $B^T A_{X,X} B$. The experiments show that when using a radial basis function $\Phi(r)$ in the usual way, we have a better condition for $B^T A_{X,X} B$ than for $A_{X,X}$. About the dependence on k, we found that as k increases, the condition $\mathcal{K}_2(B^T A_{X,X} B)$ decreases. Here are some examples in $2D$, for $M = 100$ scattered data points.

- For thin–plate splines, we found $\mathcal{K}_2(A_{X,X}) = 8.29 \times 10^6$, while for $k = 2$ we have $\mathcal{K}_2(B^T A_{X,X} B) = 2.42 \times 10^6$ and for $k = 6$ we get $\mathcal{K}_2(B^T A_{X,X} B) = 8.84 \times 10^4$; for $k = 13$ the largest value such that $q(k) \leq M$, we have $\mathcal{K}_2(B^T A_{X,X} B) = 8.16$.

- On the same X and using the scaled multiquadric $-\sqrt{1 + (x^2 + y^2)/\delta^2}$ with $\delta = 0.01$, we get $\mathcal{K}_2(A_{X,X}) = 4.22 \times 10^6$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.20 \times 10^6$ for $k = 1$; we get $\mathcal{K}_2(B^T A_{X,X} B) = 3.52 \times 10^4$ for $k = 6$, and for $k = 13$ we get $\mathcal{K}_2(B^T A_{X,X} B) = 7.72$. For $\delta = 1$: we get $\mathcal{K}_2(A_{X,X}) = 3.83 \times 10^{18}$, $\mathcal{K}_2(B^T A_{X,X} B) = 8.23 \times 10^{16}$ with $k = 1$; we get $\mathcal{K}_2(B^T A_{X,X} B) = 3.27 \times 10^{12}$ with $k = 6$, and we get $\mathcal{K}_2(B^T A_{X,X} B) = 2.62 \times 10^4$ with $k = 13$.

We note also that when choosing the polynomial order less or equal than the order m of the conditionally positive definite radial basis, the l^2-conditioning is not modified essentially.

The construction along (5) fits the data (X, f_X) by a combination of radial basis functions constrained to have a decay that is function of k, let us say $F_\phi(k)$, and then it calculates the polynomial of order k that fits the residual in the least squares sense. In general, as k increases, features of f are shifted from the combination of the radial basis function part constrained to decay as $F_\phi(k)$, and they are captured by the polynomial instead. It is known that the combination of the radial part, plus the polynomial of minimal order that guarantees strict positivity, can fit f with full accuracy, but the polynomial of large order might present undue oscillations in regions with scarce data and in particular at the boundary, so in general it is not recommended to take the largest k such that $q(k) \le M$.

Our experience shows that k should not be larger than six for smooth bivariate functions, when M is of the order of one hundred. However, values of k larger than six can provide accurate results when f is well fitted by a polynomial (with the multiquadric and with δ large enough, we usually have an improvement of many orders of $\mathcal{K}_2(B^T A_{X,X} B)$ with respect to $\mathcal{K}_2(A_{X,X})$ in this case) or, when using uniformly scattered data, we have in addition a good information at the border. Here we provide examples. We use the multiquadric with suitable choices of δ and k such that the results are accurate both in terms of error and of graphical appearance. The discrete root mean squared error e_2 and the discrete maximum error e_∞, both computed at the points of a uniform grid 61×61, are provided.

Example 1: $M = 121$ mildly scattered data from $f(x, y) = (\sqrt{x^2 + y^2} - 0.6)_+^4$ within $[0, 1] \times [0, 1]$. The results for $k = 10$ and $\delta = 0.35$ are $e_2 = 3.1 \times 10^{-5}$, $e_\infty = 1 \times 10^{-3}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.4 \times 10^5$ while $\mathcal{K}_2(A_{X,X}) = 1.7 \times 10^{10}$. The graphical output is shown in Fig. 1.

Example 2: $M = 161$ data of which 121 are mildly scattered within $[0, 1] \times [0, 1]$ and 40 on the boundary from Franke's "humps and dips" function. The results for $k = 10$ and $\delta = 0.35$ are $e_2 = 8.6 \times 10^{-4}$, $e_\infty = 5.6 \times 10^{-3}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 10^6$. The graphical output is shown in Fig. 2. We found $\mathcal{K}_2(A_{X,X}) = 6.3 \times 10^{10}$.

5 Shifted B–Spline Kernels

Theorem 5.1 *Let Φ be translation–invariant and conditionally positive definite of order m on $\mathbb{R}^d$ with m minimal. Furthermore, Φ must have a generalized Fourier transform which is positive almost everywhere on $\mathbb{R}^d$. Let Y be a discrete set of points around the origin, and let $\alpha \in \mathbb{R}^{|Y|}$ be a nonzero vector with annihilation order $k \ge m$ on Y. Then the* shifted kernel *B*–spline

$$\Psi(x) := \sum_{y_j \in Y, y_l \in Y} \alpha_j \alpha_\ell \Phi(x - (y_j - y_\ell))$$

is positive definite on $\mathbb{R}^d$.

The proof is a simple consequence of the relation (4) and of the condition (1). In fact, the Fourier transform of Ψ is

$$\begin{aligned}\hat{\Psi}(\omega) &= \hat{\Phi}(\omega) \sum_{y_j \in Y, y_l \in Y} \alpha_j \alpha_\ell e^{-i\omega^T (y_j - y_\ell)} \\ &= \hat{\Phi}(\omega) \left| \sum_{y_j \in Y} \alpha_j e^{-i\omega^T y_j} \right|^2\end{aligned}$$

and since the singularity of the transform at zero is canceled, the assertion follows. Here, we made use of the fact [11] that m is the smallest nonnegative integer such that $\hat{\Phi}(\omega)\|\omega\|_2^{2m}$ is integrable around the origin.

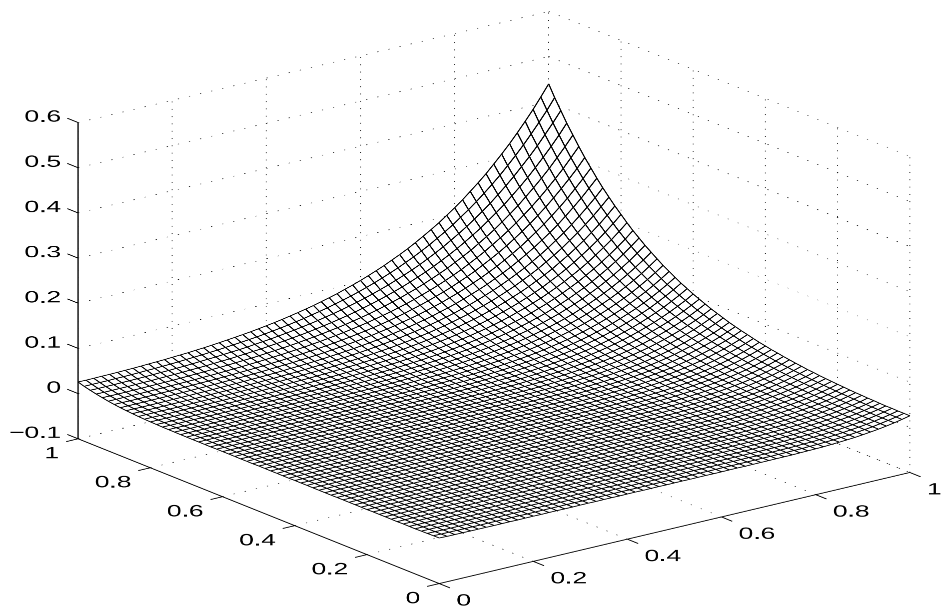

Figure 1: Example 1: reconstruction.

Remark. If $k \geq m$ is not satisfied, one gets a kernel that still is conditionally positive definite of order $m - k$.

The above result allows to use shifted kernel B–splines within the standard setting of interpolation by kernel functions. The decay of the shifted kernel B–splines is useful for system solving, but it may be necessary to add polynomials to cope with global trends.

Since the error analysis and stability properties of kernels are dominated by smoothness properties, as far as orders are concerned, the shifted kernel B–spline Ψ inherits the properties of Φ. Improvements in error and stability behavior can therefore only be effected via multiplicative constants that differ from those obtained for the pure kernels. We provide a specific example later.

The intrinsically bad condition of the matrix A of an interpolation system is mainly due to the part $\|A^{-1}\|$ contributed by the smallest eigenvalue of A. This part is a function of the separation distance of points, and its order is not affected by taking linear combinations. Even for quickly decaying kernels, the norm $\|A\|$ will be increasing linearly with the cardinality $|X|$ of the set X of centers, if the domain is kept fixed. But if the kernel itself is increasing, one has to expect an additional factor which grows like a power of the diameter of the domain. This is why an improvement of condition can be expected when going over from an increasing kernel to a decaying B–spline kernel.

6 Special B–Spline Kernels

Here we consider two examples in $2D$ related to the case $k = 2$. The first example is a generalization of polyharmonic splines of order $m = 2$. Let us consider a scaled multiquadric $\Phi_\delta(x) := \sqrt{\delta^2 + \|x\|_2^2}$. Then we define as a *Laplacian multiquadric B–spline* the function

$$u_0 := \Delta_\rho \Phi_\delta = \alpha_\rho * \Phi_\delta$$

centered at the origin, where Δ_ρ is the classical five–point discretization of the Laplacian with step ρ occurring in [5]. The parameter ρ should be chosen considerably less than δ so that the function $u_0(x)$ is close to radial. In Fig. 3 we show the function normalized to have maximum equal to one, while the parameters are $\rho = 0.05$ and $\delta = 1$. We recall that the classical polyharmonic spline is not radial; in fact the directions of the bisectrices of the axes are privileged.

For the second example, we take the seven points in $Y := \{y_0, y_1, \ldots, y_6\} \subset \mathbb{R}^2$ to be the origin $y_0 = 0$ and the six roots of unity $y_1, \ldots, y_6$ scaled by the factor $\rho > 0$. The coefficient vector is $\alpha = \frac{1}{6}(6, -1, -1, -1, -1, -1, -1)$, and it is easy to see that we have annihilation order 2. In fact the Fourier transform factor in (4) is

$$\begin{aligned} \sigma(\omega) &= \sum_{j=0}^{6} \alpha_j e^{-i\omega^T y_j} \\ &= 1 - \frac{4}{6}\cos(\frac{\rho\omega_1}{2})\cos(\frac{\rho\omega_2\sqrt{3}}{2}) - \frac{2}{6}\cos(\rho\omega_1) =: T(\rho\omega) \end{aligned} \tag{6}$$

by straightforward computations. It is nonnegative and vanishes only at the isolated points

$$\left(\frac{4k\pi}{\rho}, \frac{4\ell\pi}{\sqrt{3}\rho}\right)$$

$$\left(\frac{(4k+2)\pi}{\rho}, \frac{(4\ell+2)\pi}{\sqrt{3}\rho}\right)$$

for all integers k, ℓ, forming a hexagonal grid as an overlay of two standard grids. Clearly, the expansions around all of these points, including zero, vanish to second order.

When considering the multiquadric $\Phi_\delta(x) := \sqrt{\delta^2 + \|x\|_2^2}$, the resulting *hexagonal multiquadric B–spline* is not radial, but close to radial for $\rho < \delta$. Furthermore, the perturbation theory of [2] applies here, because it can be easily generalized to the translation–invariant setting.

In Fig. 4 we show the behavior of the close-to-radial hexagonal multiquadric B–spline. It was calculated with $\rho = 0.142$ and $\delta = 1$ and normalized to have maximum equal to one. The set of functions $u_i(x) = u_0(x - x_i)$ for $x_i \in X$ is our basis.

7 Stability

Theorem 7.1 *If q is the minimal separation distance of the data, and if we take $\rho = 0.142q$, the smallest eigenvalue of the interpolation matrix defined via the hexagonal multiquadric B–spline is at least by a factor of $4/3$ larger than the smallest eigenvalue λ_{A^Φ} of the matrix for unscaled multiquadric interpolation. With the interpolation matrix $A^u := \{a_{ij}\}_{i,j=1,\dots|X|}$ of the unscaled u_Y basis shifted around on X, and $A^\Phi = \{a^\Phi_{ij}\}_{i,j=1,\dots|X|}$, the following holds:*

$$1 \le \mathcal{K}_2(A^u) \le \mid X \mid \rho^2/2\ (3/4\lambda^{-1}_{A^\Phi}),$$

$$1 \le \mathcal{K}_2(A^\Phi) \le \mid X \mid \sqrt{1 + \mid X \mid_2^2}\ \lambda^{-1}_{A^\Phi}.$$

Proof. Following [11] we have the generalized Fourier transform of the 2D multiquadric as

$$\hat{\Phi}(\omega) = \frac{(1 + \|\omega\|_2)\exp(-\|\omega\|_2)}{\|\omega\|_2^3},$$

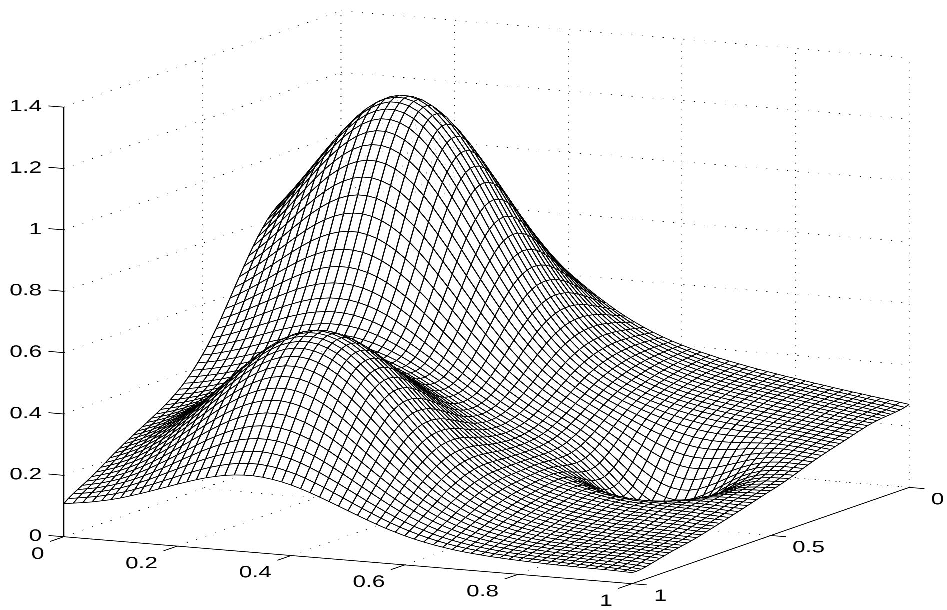

Figure 2: Example 2: reconstruction.

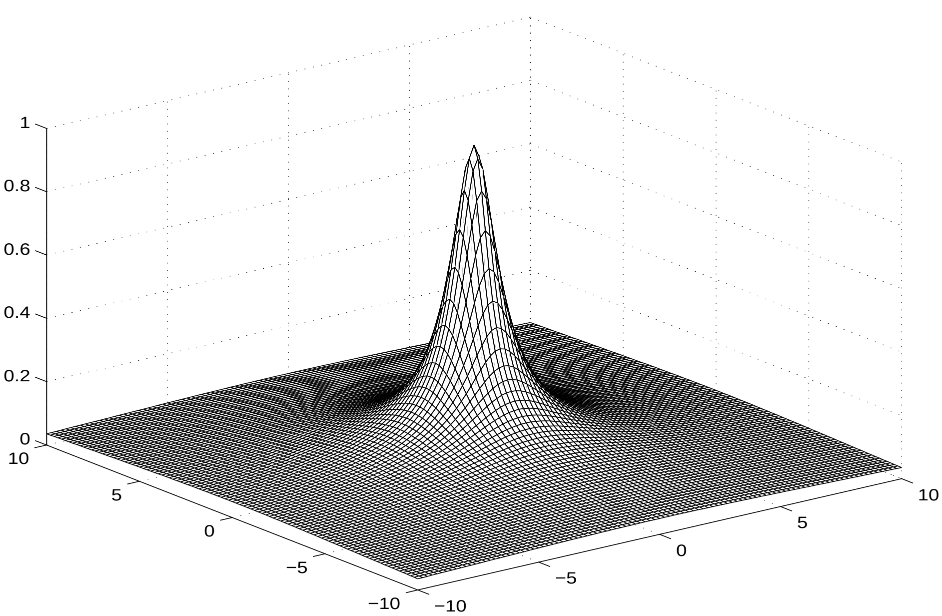

Figure 3: Laplacian multiquadric B–spline.

up to a constant, which is monotone decreasing and thus attains its minimum

$$\varphi_0(M) := \frac{(1+2M)\exp(-2M)}{8M^3}$$

on all $\|\omega\|_2 \le 2M$. Note that this function is central in Theorem 3.1 of [8] for proving stability bounds for the multiquadric. In order to apply this theorem, we have to evaluate

$$\psi_0(M,\rho) := \inf_{\|\omega\|_2 \le 2M} \hat{\Phi}(\omega) T(\rho\omega)$$

with T of (6). We used MAPLE to give a radial local lower bound for T by

$$T(\rho\omega) \ge 1 - \frac{1}{3}\cos(\frac{\sqrt{3}}{2}\rho\|\omega\|_2) - \frac{2}{3}\cos^2(\frac{\sqrt{3}}{4}\rho\|\omega\|_2) =: G(\rho\|\omega\|_2)$$

in the range $I_\omega := \rho\|\omega\|_2 \le \frac{2\pi}{\sqrt{3}}$. The quantity $\hat{\Phi}(\omega)G(\rho\|\omega\|_2)$ is monotone decreasing with respect to $\|\omega\|_2$ for what follows:

$$D(\Phi(\omega)G(\rho\|\omega\|_2)) = \frac{d\Phi(\omega)}{d\|\omega\|_2} G(\rho\|\omega\|_2) + \Phi(\omega)\frac{dG(\rho\|\omega\|_2)}{d\|\omega\|_2}$$

with the first summand negative and the second summand nonnegative within the range I_ω. We consider

$$R(\|\omega\|_2) = \mid \frac{d\Phi(\omega)}{d\|\omega\|_2} G(\rho\|\omega\|_2) \mid / \mid \Phi(\omega)\frac{dG(\rho\|\omega\|_2)}{d\|\omega\|_2} \mid;$$

and decompose it as

$$R(\|\omega\|_2) = R_1(\|\omega\|_2) \cdot R_2(\|\omega\|_2),$$

with

$$R_1(\|\omega\|_2) = (\|\omega\|_2^2 + 3\|\omega\|_2 + 3)/(6(1+\|\omega\|_2))$$

and

$$R_2(\|\omega\|_2) = (\tan\sqrt{3}/4\ \rho\|\omega\|_2)/(\sqrt{3}/4\ \rho\|\omega\|_2),$$

both increasing on I_ω. Because of $D(\Phi(\omega)G(\rho\|\omega\|_2))_{\|\omega\|_2=0} < 0$, it follows that for each $\|\omega\|_2 \in I_\omega$, the negative first summand is dominating, and so it follows that $D(\Phi(\omega)G(\rho\|\omega\|_2 < 0$ in I_ω. □

Thus we look at $\hat{\Phi}(2M)G(2M\rho)$ and get the value $\frac{4}{3}\hat{\Phi}(2M)$ for $\rho = \pi/(M\sqrt{3})$. This proves $\psi_0(M, \pi/(M\sqrt{3})) \ge \frac{4}{3}\varphi_0(M)$ for all M.

To provide a lower bound to the smallest eigenvalue λ of A, we follow the line of argument of section 3 in [8] with $M := \frac{12.76}{q}$, where q is the minimal separation distance of the data locations. This value of M is the optimal one for bounding the smallest eigenvalue of the multiquadric interpolation matrix from below. Thus for $\rho = 0.142q$ we have an improvement of the lowest eigenvalue by a factor of 4/3.

To get a comparison for the norms $\|A^u\|$ and $\|A^\Phi\|$, we remark that $\|A^u\|$ behaves like $\mathcal{O}(|X|)$ where $|X|$ is the number of data centers, while $\|A^\Phi\|$ will increase like $|X| \cdot \sqrt{1+\rho^2(\Omega)}$ where $\rho(\Omega)$ is the diameter of the domain Ω. □

In the case of the Laplacian multiquadric B–spline, there is no improvement of the lowest eigenvalue respect to the one of the multiquadric, but because of ρ smaller than in the case of the hexagonal multiquadric B–spline, the two kernel B–splines can get equivalent stability.

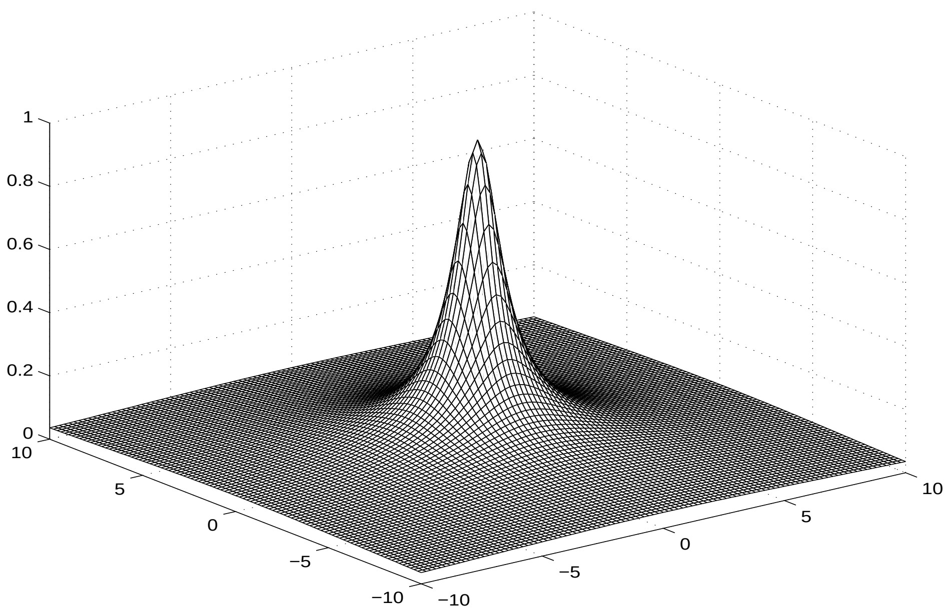

Figure 4: Hexagonal multiquadric B–spline.

8 Numerical Experiments with B–Spline Kernels

In this section we show that in numerical applications actually we have a good stability together with a good recovery of the interpolated function. In all cases we start with the scaled multiquadric $\Phi_\delta(x) := \sqrt{\delta^2 + \|x\|_2^2}$ and then form the hexagonal multiquadric B–spline $u = u_Y$, which gave good results when we approximate a function by few significant points obtained from a large sample [1]. First of all we consider the condition number of the interpolation matrix $A^u(\delta)$ in comparison with the condition number of the interpolation matrix $A^\Phi(\delta)$ for the classical multiquadric.

We provide the values for the case of 100 scattered data on $[0,1] \times [0,1]$.

- For $\delta = 0.01$ we have $\mathcal{K}_2(A^u(\delta)) = 840$ and $\mathcal{K}_2(A^\Phi(\delta)) = 4.2 \times 10^6$,

- For $\delta = 0.5$ we have $\mathcal{K}_2(A^u(\delta)) = 6.9 \times 10^5$ and $\mathcal{K}_2(A^\Phi(\delta)) = 1.1 \times 10^{14}$,
- For $\delta = 1$ we have $\mathcal{K}_2(A^u(\delta)) = 6.8 \times 10^{12}$ and $\mathcal{K}_2(A^\Phi(\delta)) = 3.8 \times 10^{18}$.

Now we provide two examples for the recovery of a function. As before we consider the errors e_2 and e_∞ computed on the uniform grid 61×61.

Example 1: $M = 101$ scattered data (mildly scattered data except for a cluster of two data) from the function defined as

$$\begin{cases} \Gamma := (x_\Gamma = x, y_\Gamma = 0.6 \cdot sin(\pi x/1.2)) \quad x \in [0.3, 0.7] \\ d(x,y) := \min_\Gamma((x - x_\Gamma)^2 + (y - y_\Gamma)^2) \\ f(x,y) := 0.1 \; exp(-d(x,y)) \end{cases}$$

within $[0,1] \times [0,1]$. The results for $\delta = 0.5$ are $e_2 = 6.0 \times 10^{-5}$, $e_\infty = 4.4 \times 10^{-4}$. We have $\mathcal{K}_2(A^u(\delta)) = 4.6 \times 10^{13}$ while $\mathcal{K}_2(A^\Phi(\delta)) = 5.6 \times 10^{17}$. The reconstruction is shown in the figure 5.

We get a similar value of the condition when using the kernel B–spline adapted to X and based on the multiquadric with $\delta = 0.5$, but with a slight loss of accuracy. In fact with $k = 5$ we get $e_2 = 7.4 \times 10^{-5}$, $e_\infty = 7.0 \times 10^{-4}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.3 \times 10^{14}$; with $k = 6$ we get $e_2 = 9.7 \times 10^{-5}$ and $e_\infty = 1.5 \times 10^{-3}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.9 \times 10^{13}$.

Example 2: $M = 1024$ mildly scattered data from the "peaks" function in MATLAB. The results for $\delta = 1$ are $e_2 = 3.8 \times 10^{-5}$, $e_\infty = 1.1 \times 10^{-3}$. We have $\mathcal{K}_2(A^u(\delta)) = 4.5 \times 10^{10}$ while $\mathcal{K}_2(A^\Phi(\delta)) = 2.0 \times 10^{14}$. The reconstruction is shown in the figure 6; by the kernel B–spline adapted to X and based on Φ multiquadric with $\delta = 1$ and $k = 6$, we get $e_2 = 2.3 \times 10^{-4}$ and $e_\infty = 7.1 \times 10^{-3}$, and we get $B^T A_{X,X} B = 2.3 \times 10^{11}$.

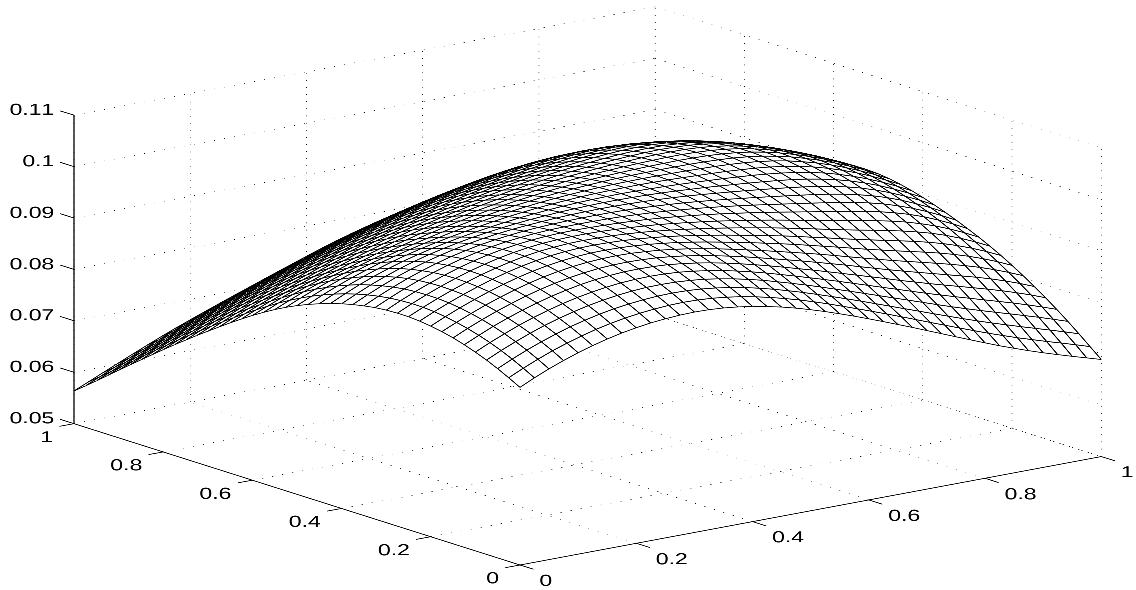

Figure 5: Example 1: reconstruction.

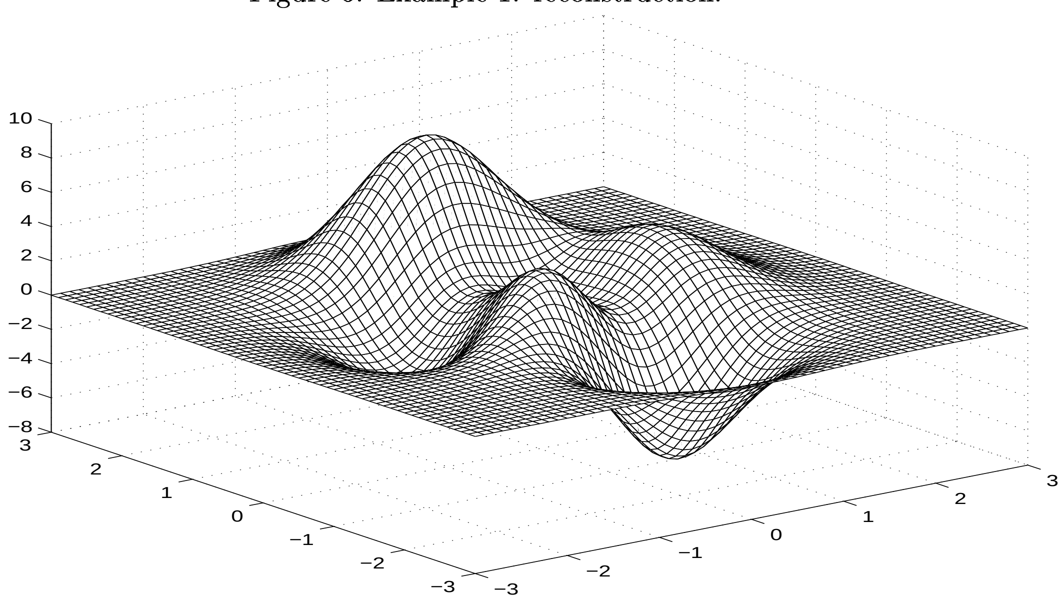

Figure 6: Example 2: reconstruction.

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