Multivariate approximation

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1 Synonyms

Approximation by functions of several variables

2 Mathematics Subject Classification

41-00,41A63, 65Mxx

3 Short Definition

Approximations of functions are *multivariate*, if they replace functions of $n \geq 2$ variables defined on a domain $\Omega \subseteq I\!R^n$ by simpler or explicitly known or computable functions from a *trial space* of n–variate functions.

4 Overview

Multivariate approximation is an extension of ⇒ *Approximation Theory* and ⇒ *Approximation Algorithms*. In general, approximations can be provided via ⇒ *Interpolation*, but this works in the multivariate case only if trial spaces are *data–dependent*. Consequently, multivariate approximation splits into subfields depending on the chosen trial spaces which in turn are tailored to meet the demands of applications. In all cases, there is a strong dependence on the domain Ω. If Ω is a Cartesian product of univariate domains, e.g. an n–dimensional cube or rectangle, one can use *tensor products*, i.e. sums of products of univariate functions [Light and Cheney(1985)]. In the periodic case, i.e. on a multivariate torus, there are multivariate *Fourier series*, a special case of tensor products. On the sphere, expansions into *spherical harmonics* yield useful multivariate approximations with plenty of applications in geophysics. Other special applications in Physics and Engineering may require special multivariate trial functions like *plane waves* for approximation. In general, ⇒ *spectral methods* [Canuto et al(2006)Canuto, Hussaini, Quarteroni, and Zang, Canuto et al(2007)Canuto, Hussaini, Quarteroni, and *pseudo–spectral* methods [Fornberg and Sloan(1994), Fornberg(1996)] use application–adapted multivariate trial functions for solving ordinary or partial differential equations via some form of multivariate approximation.

But there is also a number of multi–purpose trial spaces. They often require a *triangulation* or *mesh* of the domain $\Omega \subseteq \mathbb{R}^n$. If the triangulation is *regular* in the sense of a net or grid, *box splines* [de Boor et al(1993)de Boor, Höllig, and Riemenschneider], living on a *multi–direction mesh*, generalize the well–known univariate ⇒ *spline functions* [de Boor(2001), Schumaker(2007)] which are piecewise polynomial functions. General splines on triangulations are treated in [Lai and Schumaker(2007)]. On grids, and with special applications to imaging, multivariate ⇒ *wavelets* are particularly useful, with a huge literature, e.g. [Cohen(2003), Mallat(2009)].

On general triangulations, and with a vast range of applications in ⇒ *computational partial differential equations*, the ⇒ *finite element method* (FEM) [Babuška et al(2011)Babuška, Whiteman, and Strouboulis, Brenner and Scott(2008)] is the most popular multivariate approximation technique. Via *Cea's Lemma*, the error analysis of FEM techniques for solving elliptic PDEs boils directly down to the error of multivariate approximation to the solution. Various extensions (XFEM, GFEM) enrich the finite element trial spaces by special functions to model phenomena like boundary singularities or crack discontinuities.

Non–Uniform Rational B–Splines, (*NURBS*, [Farin(1999)]) form vector–valued multivariate trial spaces related to finite elements. They dominate the applications of *Computer–Aided Design* (CAD, ⇒ *Geometrical Design*) of curves and surfaces in Engineering [Dassault(2012)]. It is a generalization of the *Bernstein–Bézier* technique (⇒ *Bezier Curves and Surfaces*) for parametrizing spaces of multivariate polynomials over triangles, rectangles, tetrahedra, or simplices. Here, vector–valued multivariate functions. e.g. complicated 3D surfaces, are approximated by smoothly patching simpler surfaces together.

If users want to work without triangulations, they have to resort to *mesh-free* or ⇒ *meshless methods* [Liu(2003)]. They come in various forms, based on either *particles* [Li and Liu(2004)] like in ⇒ *smooth particle hydrodynamics* [Liu and Liu(2003)], or on *shape functions* [Belytschko et al(1996)Belytschko, Krongauz, Organ, Fleming, and Liu(2003)] that generate meshless trial spaces and often form a *partition of unity*. The shape functions may be generated via *Moving Least Squares* [Levin(1998)] as a per–point calculation, but they can also be provided in explicit form by translates of *kernels* or ⇒ *radial basis functions*. These techniques provide general tools for handling multivariate scattered data [Wendland(2005), Fasshauer(2007), Schaback and Wendland(2006)] and are connected to pseudospectral and particle methods, since they furnish multivariate approximations from superpositions of smooth global or compactly supported functions (⇒ *Spectral collocation methods*,⇒ *Spectral methods*). They are instances of ⇒ *Reproducing kernel methods* and also allow ⇒ *Fast Multipole Methods* [Beatson and Greengard(1997)]. A particularly important application area for such techniques is ⇒ *Computational Mechanics* [Liu(2003)].

5 Algorithms

Numerical methods (⇒ *Approximation Algorithms*) for multivariate approximation problems arise in many forms, in particular if solutions of partial differential equations are approximated. They range from the classical ⇒ *Galerkin methods* and the *Meshless Local Petrov Galerkin* approach (MLPG, [Atluri and Shen(2002)]) via all forms of pseudospectral techniques to ⇒ *finite volume methods* and ⇒ *smooth particle hydrodynamics* in fluid dynamics. In most cases, a multivariate function from a suitably parametrized *trial* space is required to satisfy certain *test* equations arising from *weak* formulations using *test functions* or *strong* formulations using ⇒ *Collocation Methods*. If there are enough test conditions to identify trial functions uniquely and with additional *stability* properties, numerical solutions will usually provide an accuracy that is roughly the error of the best approximation of the true solution by functions from the trial space [Schaback(2010)].

By the *curse of dimensionality*, the ⇒ *computational complexity* of algorithms for multivariate approximation usually grows exponentially with the number of variables, if the required accuracy is fixed. Such problems can only be handled by reducing the degrees of freedom using techniques based on ⇒ *sparsity*. Sparse *tensor product* methods are connected to *sparse grids* [Barthelmann et al(2000)Barthelmann, Novak, and Ritter, Bungartz and Griebel(2004)] and hyperbolic cross approximations [Sickel and Ullrich(2009), Döhler et al(2010)Döhler, Kunis, and Potts]. *N–term approximation* [DeVore(1998)], ⇒ *wavelets*, and ⇒ *compressive sensing* [Donoho(2006), Cohen et al(2009)Cohen, Dahmen, and DeVore] aim at ⇒ *sparse approximation* in general, even if there are only a few independent variables, e.g. when it comes to solve PDEs [Urban(2009), Cohen et al(2010)Cohen, DeVore, and Schwab] or dealing with images. These multivariate approximations are behind modern ⇒ *data compression algorithms* like JPEG 2000 and MPEG-4 for images and videos.

References

[Atluri and Shen(2002)] Atluri SN, Shen S (2002) The Meshless Local Petrov-Galerkin (MLPG) Method. Tech Science Press, Encino, CA

[Babuška et al(2011)Babuška, Whiteman, and Strouboulis] Babuška I, Whiteman JR, Strouboulis T (2011) Finite elements. Oxford University Press, Oxford, an introduction to the method and error estimation

[Barthelmann et al(2000)Barthelmann, Novak, and Ritter] Barthelmann V, Novak E, Ritter K (2000) High dimensional polynomial interpolation on sparse grids. Adv Comput Math 12(4):273–288, DOI 10.1023/A:1018977404843, URL http://dx.doi.org/10.1023/A:1018977404843, multivariate polynomial interpolation

[Beatson and Greengard(1997)] Beatson R, Greengard L (1997) A short course on fast multipole methods. In: Ainsworth M, Levesley J, Light W, Marletta M (eds) Wavelets, Multilevel Methods and Elliptic PDEs, Oxford University Press, pp 1–37

[Belytschko et al(1996)Belytschko, Krongauz, Organ, Fleming, and Krysl] Belytschko T, Krongauz Y, Organ D, Fleming M, Krysl P (1996) Meshless methods: an overview and recent developments. Computer Methods in Applied Mechanics and Engineering, special issue 139:3–47

[de Boor(2001)] de Boor C (2001) A practical guide to splines, Applied Mathematical Sciences, vol 27, revised edn. Springer-Verlag, New York

[de Boor et al(1993)de Boor, Höllig, and Riemenschneider] de Boor C, Höllig K, Riemenschneider S (1993) Box splines, Applied Mathematical Sciences, vol 98. Springer-Verlag, New York

[Brenner and Scott(2008)] Brenner SC, Scott LR (2008) The mathematical theory of finite element methods, Texts in Applied Mathematics, vol 15, 3rd edn. Springer, New York, DOI 10.1007/978-0-387-75934-0, URL http://dx.doi.org/10.1007/978-0-387-75934-0

[Bungartz and Griebel(2004)] Bungartz HJ, Griebel M (2004) Sparse grids. Acta Numer 13:147–269, DOI 10.1017/S0962492904000182, URL http://dx.doi.org/10.1017/S0962492904000182

[Canuto et al(2006)Canuto, Hussaini, Quarteroni, and Zang] Canuto C, Hussaini MY, Quarteroni A, Zang TA (2006) Spectral methods. Scientific Computation, Springer-Verlag, Berlin, fundamentals in single domains

[Canuto et al(2007)Canuto, Hussaini, Quarteroni, and Zang] Canuto C, Hussaini MY, Quarteroni A, Zang TA (2007) Spectral methods. Scientific Computation, Springer, Berlin, evolution to complex geometries and applications to fluid dynamics

[Cohen(2003)] Cohen A (2003) Numerical analysis of wavelet methods, Studies in Mathematics and its Applications, vol 32. North-Holland Publishing Co., Amsterdam

[Cohen et al(2009)Cohen, Dahmen, and DeVore] Cohen A, Dahmen W, DeVore R (2009) Compressed sensing and best k-term approximation. J Amer Math Soc 22(1):211–231, DOI 10.1090/S0894-0347-08-00610-3, URL http://dx.doi.org/10.1090/S0894-0347-08-00610-3

[Cohen et al(2010)Cohen, DeVore, and Schwab] Cohen A, DeVore R, Schwab C (2010) Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. Found Comput Math 10(6):615–646, DOI 10.1007/s10208-010-9072-2, URL http://dx.doi.org/10.1007/s10208-010-9072-2

[Dassault(2012)] Dassault (2012) CATIA. URL http://www.3ds.com/products/catia, [Online; accessed 6-February-2012]

[DeVore(1998)] DeVore RA (1998) Nonlinear approximation. In: Acta numerica, 1998, Acta Numer., vol 7, Cambridge Univ. Press, Cambridge, pp 51–150

[Döhler et al(2010)Döhler, Kunis, and Potts] Döhler M, Kunis S, Potts D (2010) Nonequispaced hyperbolic cross fast Fourier transform. SIAM J Numer Anal 47(6):4415–4428, DOI 10.1137/090754947, URL http://dx.doi.org/10.1137/090754947

[Donoho(2006)] Donoho D (2006) Compressed sensing. IEEE Transactions on Information Theory 52:1289–1306

[Farin(1999)] Farin GE (1999) NURBS, 2nd edn. A K Peters Ltd., Natick, MA, from projective geometry to practical use

[Fasshauer(2007)] Fasshauer GF (2007) Meshfree Approximation Methods with MATLAB, Interdisciplinary Mathematical Sciences, vol 6. World Scientific Publishers, Singapore

[Fornberg(1996)] Fornberg B (1996) A practical guide to pseudospectral methods, Cambridge Monographs on Applied and Computational Mathematics, vol 1. Cambridge University Press, Cambridge, DOI 10.1017/CBO9780511626357, URL http://dx.doi.org/10.1017/CBO9780511626357

[Fornberg and Sloan(1994)] Fornberg B, Sloan D (1994) A review of pseudospectral methods for solving partial differential equations. Acta Numerica pp 203–267

[Lai and Schumaker(2007)] Lai MJ, Schumaker LL (2007) Spline functions on triangulations, Encyclopedia of Mathematics and its Applications, vol 110. Cambridge University Press, Cambridge, DOI 10.1017/CBO9780511721588, URL http://dx.doi.org/10.1017/CBO9780511721588

[Levin(1998)] Levin D (1998) The approximation power of moving least-squares. Mathematics of Computation 67:1517–1531

[Li and Liu(2004)] Li S, Liu WK (2004) Meshfree particle methods. Springer-Verlag, Berlin

[Light and Cheney(1985)] Light WA, Cheney EW (1985) Approximation theory in tensor product spaces, Lecture Notes in Mathematics, vol 1169. Springer-Verlag, Berlin

[Liu(2003)] Liu GR (2003) Mesh free methods. CRC Press, Boca Raton, FL, moving beyond the finite element method

[Liu and Liu(2003)] Liu GR, Liu MB (2003) Smoothed particle hydrodynamics: a meshfree particle method. World Scientific

[Mallat(2009)] Mallat S (2009) A wavelet tour of signal processing, 3rd edn. Elsevier/Academic Press, Amsterdam, the sparse way, With contributions from Gabriel Peyré

[Schaback(2010)] Schaback R (2010) Unsymmetric meshless methods for operator equations. Numer Math 114:629–651

[Schaback and Wendland(2006)] Schaback R, Wendland H (2006) Kernel techniques: from machine learning to meshless methods. Acta Numerica 15:543–639

[Schumaker(2007)] Schumaker LL (2007) Spline functions: basic theory, 3rd edn. Cambridge Mathematical Library, Cambridge University Press, Cambridge, DOI 10.1017/CBO9780511618994, URL http://dx.doi.org/10.1017/CBO9780511618994

[Sickel and Ullrich(2009)] Sickel W, Ullrich T (2009) Tensor products of Sobolev-Besov spaces and applications to approximation from the hyperbolic cross. J Approx Theory 161(2):748–786, DOI 10.1016/j.jat.2009.01.001, URL http://dx.doi.org/10.1016/j.jat.2009.01.001

[Urban(2009)] Urban K (2009) Wavelet methods for elliptic partial differential equations. Numerical Mathematics and Scientific Computation, Oxford University Press

[Wendland(2005)] Wendland H (2005) Scattered Data Approximation. Cambridge University Press