On the fractional derivatives of radial basis functions: Theories and applications

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Abstract

The paper provides the fractional integrals and derivatives of the Riemann-Liouville and Caputo type for the five kinds of radial basis functions, including the Powers, Gaussian, Multiquadric, Matérn and Thin-plate splines, in one dimension. It allows to use high order numerical methods for solving fractional differential equations. The results are tested by solving two test problems. The first test case focuses on the discretization of the fractional differential operator while the second considers the solution of a fractional order differential equation.

Keywords: Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, Radial basis functions

1. Introduction

The basic idea behind fractional calculus has a history similar to that of more classical calculus and the topic has attracted the interests of mathematicians who contributed fundamentally to the development of classical calculus [7]. However, the development and analysis of fractional calculus and fractional differential equations is less mature than that associated with classical calculus. In spite of this, during the last decade fractional calculus emerges increasingly as a tool for the description of a broad range of non-classical phenomena in the applied sciences and engineering [10, 22]. A striking example of this is a model for anomalous transport processes and diffusion, leading to fractional partial differential equations [29, 30], but other examples are readily available for the modeling of frequency dependent damping behavior of many viscoelastic materials [1, 2], continuum and statistical mechanics [28], solid mechanics [40], and economics [3]. Models involving fractional derivatives can be divided into two major types: time fractional differential equations, typically associated with phenomena with memory or non-Markovian processes and spatial fractional partial differential equations (FPDEs), often used to model anomalous diffusion or dispersion with enhanced diffusion speed [32]. With an expanding range of applications and models based on fractional calculus comes a need for the development of robust and accurate computational methods for solving these equations. For the time fractional problems, there is a substantial number of publications on a variety of numerical schemes [9, 24, 25, 50, 51, 52]. For the spatial FPDEs, publications on the numerical schemes are relatively sparse, and the majority of the publications are based on finite difference methods of order one or at most two [8, 27, 31, 32, 44, 45, 46, 47, 48]. Some other numerical schemes using low-order finite elements [11, 15, 39], modified homotopy analysis method [41], and spectral method [20] have also been proposed. One of the ongoing issues with fractional models is the design of efficient high-order numerical discretizations. One approach is to discretize these non-local differential operators with non-local numerical methods. Following that approach, Hanert has proposed a pseudo-spectral method based on Chebyshev basis functions in space and Mittag-Leffler basis functions in time to discretize the space-time fractional diffusion equation [18]. A similar approach has been followed by

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Li and Xu to discretize the time-fractional diffusion equation with a Jacobi pseudo-spectral method [26]. Recently, Xu and Hesthaven proposed stable multi-domain spectral penalty methods for solving fractional partial differential equations [49].

Unlike traditional numerical methods for solving differential equations, meshless methods need no mesh generation, which is the major problem in finite difference, finite element and spectral methods [33, 35, 38, 43]. Radial basis function (RBF) methods are truly meshless and simple enough to allow modelling of rather high dimensional problems [16, 17, 19, 21, 33]. These methods can be very efficient numerical schemes to discretize non-local operators like fractional differential operators. Piret and Hanert [36] recently proposed a Guassian RBF discretization for the one-dimensional space-fractional diffusion equations. In this paper, we provide the required formulas for the fractional integrals and derivatives of the Riemann-Liouville and Caputo types for RBFs in one dimension. It allows to use high order numerical methods for solving fractional differential equations.

The rest of the paper is organized as follows. In section 2 we give some important definitions and theorems which are needed throughout the remaining sections of the paper. In section 3 we get a recursive relation for fractional integrals and derivatives of $x^k (k \in \mathbb{N}_0)$. The corresponding formulas of the fractional integrals and derivatives of Riemann-Liouville and Caputo type for the five kinds of RBFs are given in section 4. The results are applied to solve two test problems in section 5. The last section is devoted to a brief conclusion.

2. Preliminaries

2.1. Radial Basis Functions

In this subsection, we give an account of the RBFs that are in our focus [42].

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary nonempty set. A function $K : \Omega \times \Omega \to \mathbb{R}$ is called (real) kernel on Ω.*

Definition 2.2. *A kernel K is symmetric, if $K(x, y) = K(y, x)$ holds for all $x, y \in \Omega$.*

For scattered nodes $x_1, \ldots, x_n \in \mathbb{R}^d$, the *translates* $K_j(x) = K(x_j, x)$ are the trial functions one can start with. If the kernel is translation-invariant on $\mathbb{R}^d$, it is of the form

$$K(x, y) = \phi(x - y), \quad \forall x, y \in \mathbb{R}^d.$$

An important class of kernels are radial kernels, with significant properties [6]. Radial kernels can be defined as

$$K(x, y) = \phi(r),\ r = \|x - y\|,\ x, y \in \mathbb{R}^d,$$

for a scalar function

$$\phi : [0, \infty) \to \mathbb{R},$$

the function ϕ is called a radial basis function. Kernels on $\mathbb{R}^d$ can be scaled by a positive factor c which is called shape parameter and can be found numerically for getting accurate numerical solutions and good conditioning of the collocation matrix [5, 14]. The new scaled kernel is given by:

$$K_c(x, y) = K\left(\frac{x}{c}, \frac{y}{c}\right) \quad \forall x, y \in \mathbb{R}^d.$$

Therefore, the scaled radial kernels on $\mathbb{R}^d$ can be defined as:

$$K_c(x, y) = \phi\left(\frac{r}{c}\right),\ r = \|x - y\|,\ x, y \in \mathbb{R}^d.$$

The most commonly used global RBFs $\phi(r)$ are listed in Table 1, where n, β and ν are RBF parameters.

Table 1: Global RBFs.

Name	$\phi(r)$	condition
Gaussian	$\exp(-\frac{r^2}{2})$	
Multiquadric	$(1+r^2)^{\beta/2}$	$\beta \in \mathbb{R}_{\neq 0} \setminus 2\mathbb{N}$
Matérn/Sobolev	$r^{\nu} K_{\nu}(r)$	$\nu > 0$
Powers	r^{β}	$0 < \beta \notin 2\mathbb{N}$
Thin-plate splines	$r^{2n} \ln(r)$	$n \in \mathbb{N}$

Remark 2.1. *One of the most important examples of scaled radial kernels are the Whittle-Matérn kernels*

$$\left(\frac{r}{c}\right)^{\nu} K_{\nu}\left(\frac{r}{c}\right), \ \nu = m - \frac{d}{2}, \ r = \|x-y\|, \ x, y \in \mathbb{R}^d,$$

reproducing in the Sobolev space $W_2^m(\mathbb{R}^d)$ for $m > d/2$, where K_{ν} is the modified Bessel function of the second kind defined as follows [4]:

$$K_{\nu}(x) = \frac{\pi}{2\sin(\pi\nu)} \left(J_{-\nu}(x) - J_{\nu}(x)\right),$$

where

$$J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^n}{n!\Gamma(\nu+n+1)}.$$

Remark 2.2. *Throughout this paper, we work with the scaled RBFs in 1 dimension, that is,*

$$\phi\left(\frac{r}{c}\right), \ r = |x-y|, \ x, y \in \mathbb{R}.$$

2.2. Fractional Calculus

Here we give some basic definitions and properties of the fractional calculus theory which are used further in this paper [12, 23]. In all cases, a and b are arbitrary real numbers and α is a non-integer positive number.

Definition 2.3. *The left and the right-sided Riemann-Liouville fractional integrals of order α of a function f are defined in the following forms:*

$$\left(I_{a^+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x f(t)\,(x-t)^{\alpha-1}\,dt, \qquad (x > a),$$

and

$$\left(I_{b^-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b f(t)\,(t-x)^{\alpha-1}\,dt, \qquad (x < b).$$

Definition 2.4. *The left and the right-sided Riemann-Liouville fractional derivatives of order α of a function f are defined in the following forms:*

$$\left(D_{a^+}^{\alpha} f\right)(x) := \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x f(t)\,(x-t)^{n-\alpha-1}\,dt, \qquad (x > a),$$

and

$$\left(D_{b^-}^{\alpha} f\right)(x) := \left(-\frac{d}{dx}\right)^n \left(I_{b^-}^{n-\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b f(t)\,(t-x)^{n-\alpha-1}\,dt, \qquad (x < b),$$

where $n = [\alpha] + 1$.

Definition 2.5. *The left and the right-sided Caputo fractional derivatives of order α of a function f are defined in the following forms:*

$$\left({}^{C}D_{a+}^{\alpha}f\right)(x) := \left(I_{a+}^{n-\alpha}D^{n}f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x} f^{(n)}(t)\,(x-t)^{n-\alpha-1}\,dt, \qquad (n=[\alpha]+1,\ x>a)\,,$$

and

$$\left({}^{C}D_{b-}^{\alpha}f\right)(x) := (-1)^{n}\left(I_{b-}^{n-\alpha}D^{n}f\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b} f^{(n)}(t)\,(t-x)^{n-\alpha-1}\,dt, \qquad (n=[\alpha]+1,\ x<b)\,,$$

where $D=\dfrac{d}{dx}$.

The following results are easy to prove [12, 23].

Theorem 2.1. *Let $\beta>-1$, then*

$$\begin{aligned}
\left(I_{a+}^{\alpha}(t-a)^{\beta}\right)(x) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(x-a)^{\beta+\alpha}, && (x>a),\\
\left(D_{a+}^{\alpha}(t-a)^{\beta}\right)(x) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}, && (x>a),\\
\left(I_{b-}^{\alpha}(b-t)^{\beta}\right)(x) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(b-x)^{\beta+\alpha}, && (x<b),\\
\left(D_{b-}^{\alpha}(b-t)^{\beta}\right)(x) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha}, && (x<b).
\end{aligned}$$

Specifically, if $\alpha-\beta\in\{1,2,\ldots,[\alpha]+1\}$ then we have

$$\begin{aligned}
\left(D_{a+}^{\alpha}(t-a)^{\beta}\right)(x) &= 0, && (x>a),\\
\left(D_{b-}^{\alpha}(b-t)^{\beta}\right)(x) &= 0, && (x<b).
\end{aligned}$$

□

Theorem 2.2. *Let $n=[\alpha]+1$ and $\beta>n-1$ then the following relations hold*

$$\begin{aligned}
\left({}^{C}D_{a+}^{\alpha}(t-a)^{\beta}\right)(x) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}, && (x>a)\,,\\
\left({}^{C}D_{b-}^{\alpha}(b-t)^{\beta}\right)(x) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha}, && (x<b)\,.
\end{aligned}$$

Specifically, if $k\in\{0,1,\ldots n-1\}$ then we have

$$\begin{aligned}
\left({}^{C}D_{a+}^{\alpha}(t-a)^{k}\right)(x) &= 0, && (x>a),\\
\left({}^{C}D_{b-}^{\alpha}(b-t)^{k}\right)(x) &= 0, && (x<b).
\end{aligned}$$

□

Theorem 2.3. *Assume that f is such that ${}^{C}D_{a+}^{\alpha}f$, ${}^{C}D_{b-}^{\alpha}f$, $D_{a+}^{\alpha}f$ and $D_{b-}^{\alpha}f$ exist. Then*

$$\begin{aligned}
\left({}^{C}D_{a+}^{\alpha}f\right)(x) &= \left(D_{a+}^{\alpha}f\right)(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}, && (n=[\alpha]+1,\ x>a)\,,\\
\left({}^{C}D_{b-}^{\alpha}f\right)(x) &= \left(D_{b-}^{\alpha}f\right)(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}, && (n=[\alpha]+1,\ x<b)\,.
\end{aligned}$$

□

3. Fractional derivatives of x^k

In this section we get a recursive relation for fractional integrals and derivatives of x^k, $k \in \mathbb{N}_0$, which are needed in the sequel. In all cases, a and b are arbitrary real numbers and α is a non-integer positive number.

Theorem 3.1. *For all $k \in \mathbb{N}_0$, we have*

$$\left(I_{a^+}^{\alpha} t^k\right)(x) = \frac{k!\,(x-a)^{\alpha}\,\Gamma(-\alpha+1)}{\Gamma(k+\alpha+1)} \sum_{i=0}^{k} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(-\alpha-i+1)}.$$

Proof. By Definition 2.3 we have

$$\left(I_{a^+}^{\alpha} t^k\right)(x) \;=\; \frac{1}{\Gamma(\alpha)} \int_a^x t^k\,(x-t)^{\alpha-1}\,dt.$$

Since

$$\left(I_{a^+}^{\alpha} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\,dt = \frac{1}{\Gamma(\alpha+1)}\,(x-a)^{\alpha},$$

the theorem is correct in the case $k=0$. Now let $k>0$, by change of variable $t = y(x-a)+a$, we have

$$\begin{aligned}
\left(I_{a^+}^{\alpha} t^k\right)(x) &= \frac{(x-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 (yx + a(1-y))^k\,(1-y)^{\alpha-1}\,dy \\
&= \frac{(x-a)^{\alpha}}{\Gamma(\alpha)} \sum_{i=0}^{k} \binom{k}{i} x^{k-i} a^i \int_0^1 y^{k-i}\,(1-y)^{\alpha-1+i}\,dy \\
&= \frac{(x-a)^{\alpha}}{\Gamma(\alpha)} \sum_{i=0}^{k} \binom{k}{i} x^{k-i} a^i \frac{\Gamma(k-i+1)\Gamma(\alpha+i)}{\Gamma(k+\alpha+1)} \\
&= \frac{k!\,(x-a)^{\alpha}\,\Gamma(-\alpha+1)}{\Gamma(k+\alpha+1)} \sum_{i=0}^{k} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(-\alpha-i+1)}.
\end{aligned}$$

□

Theorem 3.2. *Let $n = [\alpha]+1$. Then for all $k \in \mathbb{N}_0$,*

$$\left({}^{C}D_{a^+}^{\alpha} t^k\right)(x) = \begin{cases} 0, & k<n, \\ \dfrac{k!\,(x-a)^{n-\alpha}\,\Gamma(\alpha-n+1)}{\Gamma(k-\alpha+1)} \displaystyle\sum_{i=0}^{k-n} \frac{x^{k-n-i}\,(-a)^i}{i!\;\Gamma(\alpha-n-i+1)}, & k \ge n. \end{cases}$$

Proof. By Definition 2.5 we have

$$\left({}^{C}D_{a^+}^{\alpha} t^k\right)(x) \;=\; \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(t^k\right)^{(n)} (x-t)^{n-\alpha-1}\,dt.$$

If $k<n$ it is clear that $\left({}^{C}D_{a^+}^{\alpha} t^k\right)(x)$=0. Since

$$\begin{aligned}
\left({}^{C}D_{a^+}^{\alpha} t^n\right)(x) &= \frac{n!}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1}\,dt \\
&= \frac{n!}{\Gamma(n-\alpha+1)}\,(x-a)^{n-\alpha},
\end{aligned}$$

the theorem is also correct for the case $k=n$. Now let $k>n$, by change of variable $t=y(x-a)+a$ we have

$$
\begin{aligned}
&\left({}^C D^\alpha_{a^+} t^k\right)(x) \\
&= \frac{k(k-1)\dots(k-n+1)(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (yx+a(1-y))^{k-n} (1-y)^{n-\alpha-1}\, dy \\
&= \frac{k(k-1)\dots(k-n+1)(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \sum_{i=0}^{k-n} \binom{k-n}{i} x^{k-n-i} a^i \int_0^1 y^{k-n-i} (1-y)^{n-\alpha-1+i}\, dy \\
&= \frac{k(k-1)\dots(k-n+1)(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \sum_{i=0}^{k-n} \binom{k-n}{i} x^{k-n-i} a^i \frac{\Gamma(k-n-i+1)\Gamma(n-\alpha+i)}{\Gamma(k-\alpha+1)} \\
&= \frac{k!\,(x-a)^{n-\alpha}\,\Gamma(\alpha-n+1)}{\Gamma(k-\alpha+1)} \sum_{i=0}^{k-n} \frac{x^{k-n-i}\,(-a)^i}{i!\,\Gamma(\alpha-n-i+1)}.
\end{aligned}
$$

□

Lemma 3.1. *Let $\alpha>1$, $\alpha\notin\mathbb{N}$, and $n=[\alpha]+1$. Then for all $k\in\mathbb{N}$,*

$$\left({}^C D^\alpha_{a^+} t^k\right)(x) = k\left({}^C D^{\alpha-1}_{a^+} t^{k-1}\right)(x).$$

Proof. If $k<n$ it is clear from Definition 2.5 that

$$\left({}^C D^\alpha_{a^+} t^k\right)(x) = \left({}^C D^{\alpha-1}_{a^+} t^{k-1}\right)(x) = 0.$$

Now let $k\geq n$ then we have

$$
\begin{aligned}
\left({}^C D^\alpha_{a^+} t^k\right)(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(t^k\right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\
&= \frac{k}{\Gamma(n-\alpha)} \int_a^x \left(t^{k-1}\right)^{(n-1)} (x-t)^{n-\alpha-1}\, dt \\
&= k\left({}^C D^{\alpha-1}_{a^+} t^{k-1}\right)(x).
\end{aligned}
$$

□

Lemma 3.2. *Let $\alpha>1$, $\alpha\notin\mathbb{N}$, and $n=[\alpha]+1$. Then for all $k\in\mathbb{N}$,*

$$\left(D^\alpha_{a^+} t^k\right)(x) = k\left(D^{\alpha-1}_{a^+} t^{k-1}\right)(x) + \frac{a^k\,(x-a)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Proof. By Theorem 2.3 and Lemma 3.1 we have

$$
\begin{aligned}
\left(D^\alpha_{a^+} t^k\right)(x) &= \left({}^C D^\alpha_{a^+} t^k\right)(x) + \sum_{i=0}^{n-1} \frac{\left(x^k\right)^{(i)}|_a}{\Gamma(i-\alpha+1)} (x-a)^{i-\alpha} \\
&= k\left({}^C D^{\alpha-1}_{a^+} t^{k-1}\right)(x) + k\sum_{i=1}^{n-1} \frac{\left(x^{k-1}\right)^{(i-1)}|_a}{\Gamma(i-\alpha+1)} (x-a)^{i-\alpha} + \frac{a^k\,(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= k\left(\left({}^C D^{\alpha-1}_{a^+} t^{k-1}\right)(x) + \sum_{i=0}^{n-2} \frac{\left(x^{k-1}\right)^{(i)}|_a}{\Gamma(i-\alpha+2)} (x-a)^{i+1-\alpha}\right) + \frac{a^k\,(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= k\left(D^{\alpha-1}_{a^+} t^{k-1}\right)(x) + \frac{a^k\,(x-a)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

□

Theorem 3.3. *Let $n = [\alpha] + 1$. Then for all $k \in \mathbb{N}_0$,*

$$\left(D_{a^+}^{\alpha} t^k\right)(x) = \frac{k!\ \Gamma(\alpha+1)\,(x-a)^{-\alpha}}{\Gamma(k-\alpha+1)} \sum_{i=0}^{k} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)}.$$

Proof. By Definition 2.4 we have

$$\begin{aligned}
\left(D_{a^+}^{\alpha} 1\right)(x) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1}\,dt \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left(\frac{d}{dx}\right)^n (x-a)^{n-\alpha} \\
&= \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

Therefore the theorem is correct in the case $k = 0$. For $k \in \mathbb{N}$, the proof is based on induction over n. Beginning the induction at $n = 1$ ($0 < \alpha < 1$), we use the Theorems 2.3 and 3.2 as follows:

$$\begin{aligned}
&\left(D_{a^+}^{\alpha} t^k\right)(x) \\
&= \left({}^C D_{a^+}^{\alpha} t^k\right)(x) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{1-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \sum_{i=0}^{k-1} \frac{x^{k-1-i}\,(-a)^i}{i!\ \Gamma(\alpha-i)} + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i)} + \sum_{i=0}^{k-1} \frac{x^{k-1-i}\,(-a)^{i+1}}{i!\ \Gamma(\alpha-i)}\right) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i)} + \sum_{i=1}^{k} \frac{x^{k-i}\,(-a)^i}{(i-1)!\ \Gamma(\alpha-i+1)}\right) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\alpha \sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)} + \frac{(-a)^k}{(k-1)!\Gamma(\alpha-k+1)}\right) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\alpha \sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)} + \frac{\alpha\,(-a)^k}{k!\ \Gamma(\alpha-k+1)} \left(\frac{k}{\alpha} + \frac{(-1)^k\,\Gamma(k-\alpha+1)\Gamma(\alpha-k+1)}{\alpha\ \Gamma(\alpha)\ \Gamma(1-\alpha)}\right)\right) \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\alpha \sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)} + \frac{\alpha\,(-a)^k}{k!\ \Gamma(\alpha-k+1)} \left(\frac{k}{\alpha} + \frac{\alpha-k}{\alpha}\right)\right) \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha+1)}{\Gamma(k-\alpha+1)} \sum_{i=0}^{k} \frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)}.
\end{aligned}$$

Now assume that the theorem is correct for $n - 1 < \alpha < n$ where $n > 1$ (induction hypothesis). We must then prove that the theorem also holds for $n < \alpha < n + 1$. To do this, we use Lemma 3.2 and the induction hypothesis, as follows:

$$
\begin{aligned}
&\left(D_{a^+}^{\alpha} t^k\right)(x) \\
&= k\left(D_{a^+}^{\alpha-1} t^{k-1}\right)(x) + \frac{a^k\,(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= \frac{k!\,(x-a)^{1-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \sum_{i=0}^{k-1} \frac{x^{k-1-i}\,(-a)^i}{i!\;\Gamma(\alpha-i)} + \frac{a^k\,(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(\alpha-i)} + \sum_{i=0}^{k-1} \frac{x^{k-1-i}\,(-a)^{i+1}}{i!\;\Gamma(\alpha-i)}\right) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(\alpha-i)} + \sum_{i=1}^{k} \frac{x^{k-i}\,(-a)^i}{(i-1)!\;\Gamma(\alpha-i+1)}\right) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\alpha\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(\alpha-i+1)} + \frac{(-a)^k}{(k-1)!\Gamma(\alpha-k+1)}\right) + \frac{a^k}{\Gamma(1-\alpha)}\,(x-a)^{-\alpha} \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\alpha\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(\alpha-i+1)} + \frac{\alpha\,(-a)^k}{k!\;\Gamma(\alpha-k+1)}\left(\frac{k}{\alpha} + \frac{(-1)^k\,\Gamma(k-\alpha+1)\Gamma(\alpha-k+1)}{\alpha\;\Gamma(\alpha)\;\Gamma(1-\alpha)}\right)\right) \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha)}{\Gamma(k-\alpha+1)} \left(\alpha\sum_{i=0}^{k-1} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(\alpha-i+1)} + \frac{\alpha\,(-a)^k}{k!\;\Gamma(\alpha-k+1)}\left(\frac{k}{\alpha} + \frac{\alpha-k}{\alpha}\right)\right) \\
&= \frac{k!\,(x-a)^{-\alpha}\,\Gamma(\alpha+1)}{\Gamma(k-\alpha+1)} \sum_{i=0}^{k} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(\alpha-i+1)}.
\end{aligned}
$$

□

At the end of this section we use Theorems 3.1-3.3 to get a recursive relation between fractional integrals and derivatives of t^k and t^{k+1}.

Theorem 3.4. *For all $k \in \mathbb{N}_0$, we have*

$$\left(I_{a^+}^{\alpha} t^{k+1}\right)(x) = \frac{(k+1)\,x}{k+\alpha+1}\left(I_{a^+}^{\alpha} t^k\right)(x) + \frac{\alpha\;a^{k+1}}{k+\alpha+1}\left(I_{a^+}^{\alpha} 1\right)(x).$$

Proof. By theorem 3.1 we have

$$
\begin{aligned}
\left(I_{a^+}^{\alpha} t^{k+1}\right)(x) &= \frac{(k+1)!\,(x-a)^{\alpha}\,\Gamma(-\alpha+1)}{\Gamma(k+\alpha+2)} \sum_{i=0}^{k+1} \frac{x^{k-i+1}\,(-a)^i}{i!\;\Gamma(-\alpha-i+1)} \\
&= \frac{(k+1)!\;(x-a)^{\alpha}\,\Gamma(-\alpha+1)}{\Gamma(k+\alpha+2)} \sum_{i=0}^{k} \frac{x^{k-i+1}\,(-a)^i}{i!\;\Gamma(-\alpha-i+1)} \\
&\quad + \frac{(-a)^{k+1}\,\Gamma(-\alpha+1)\,(x-a)^{\alpha}}{\Gamma(k+\alpha+2)\Gamma(-\alpha-k)} \\
&= \frac{(k+1)\,x}{k+\alpha+1}\left(\frac{k!\;(x-a)^{\alpha}\,\Gamma(-\alpha+1)}{\Gamma(k+\alpha+1)} \sum_{i=0}^{k} \frac{x^{k-i}\,(-a)^i}{i!\;\Gamma(-\alpha-i+1)}\right) \\
&\quad + \frac{(-a)^{k+1}\,\Gamma(-\alpha+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+2)\Gamma(-\alpha-k)}\left(I_{a^+}^{\alpha} 1\right)(x) \\
&= \frac{(k+1)\,x}{k+\alpha+1}\left(I_{a^+}^{\alpha} t^k\right)(x) + \frac{\alpha\;a^{k+1}}{k+\alpha+1}\left(I_{a^+}^{\alpha} 1\right)(x).
\end{aligned}
$$

□

Theorem 3.5. *Let $n = [\alpha] + 1$. Then for all $k \in \mathbb{N}$ which $k \geq n$,*

$$\left({}^C D^{\alpha}_{a^+} t^{k+1}\right)(x) = \frac{(k+1)\,x}{k-\alpha+1}\left({}^C D^{\alpha}_{a^+} t^{k}\right)(x) + \binom{k+1}{n}\frac{(n-\alpha)\ a^{k-n+1}}{k-\alpha+1}\left({}^C D^{\alpha}_{a^+} t^{n}\right)(x).$$

Proof. By theorem 3.2 we have

$$\begin{aligned}
\left({}^C D^{\alpha}_{a^+} t^{k+1}\right)(x) &= \frac{(k+1)!\,(x-a)^{n-\alpha}\,\Gamma(\alpha-n+1)}{\Gamma(k-\alpha+2)}\sum_{i=0}^{k-n+1}\frac{x^{k-n-i+1}\,(-a)^i}{i!\ \Gamma(\alpha-n-i+1)} \\
&= \frac{(k+1)!\ \ (x-a)^{n-\alpha}\,\Gamma(\alpha-n+1)}{\Gamma(k-\alpha+2)}\sum_{i=0}^{k-n}\frac{x^{k-n-i+1}\,(-a)^i}{i!\ \Gamma(\alpha-n-i+1)} \\
&\quad + \frac{(-a)^{k-n+1}\,(k+1)!\ \Gamma(\alpha-n+1)\,(x-a)^{n-\alpha}}{(k-n+1)!\ \Gamma(k-\alpha+2)\Gamma(\alpha-k)} \\
&= \frac{(k+1)\,x}{k-\alpha+1}\left(\frac{k!\ \ (x-a)^{n-\alpha}\,\Gamma(\alpha-n+1)}{\Gamma(k-\alpha+1)}\sum_{i=0}^{k-n}\frac{x^{k-n-i}\,(-a)^i}{i!\ \Gamma(\alpha-n-i+1)}\right) \\
&\quad + \frac{(-a)^{k-n+1}\,(k+1)!\ \Gamma(\alpha-n+1)\Gamma(n-\alpha+1)}{n!\ \ (k-n+1)!\ \Gamma(k-\alpha+2)\Gamma(\alpha-k)}\left({}^C D^{\alpha}_{a^+} t^{n}\right)(x) \\
&= \frac{(k+1)\,x}{k-\alpha+1}\left({}^C D^{\alpha}_{a^+} t^{k}\right)(x) + \binom{k+1}{n}\frac{(n-\alpha)\,a^{k-n+1}}{k-\alpha+1}\left({}^C D^{\alpha}_{a^+} t^{n}\right)(x).
\end{aligned}$$

□

Theorem 3.6. *For all $k \in \mathbb{N}_0$, we have*

$$\left(D^{\alpha}_{a^+} t^{k+1}\right)(x) = \frac{(k+1)\,x}{k-\alpha+1}\left(D^{\alpha}_{a^+} t^{k}\right)(x) - \frac{\alpha\ a^{k+1}}{k-\alpha+1}\left(D^{\alpha}_{a^+} 1\right)(x).$$

Proof. By Theorem 3.3 we have

$$\begin{aligned}
\left(D^{\alpha}_{a^+} t^{k+1}\right)(x) &= \frac{(k+1)!\,\Gamma(\alpha+1)\,(x-a)^{-\alpha}}{\Gamma(k-\alpha+2)}\sum_{i=0}^{k+1}\frac{x^{k-i+1}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)} \\
&= \frac{(k+1)!\,\Gamma(\alpha+1)\,(x-a)^{-\alpha}}{\Gamma(k-\alpha+2)}\sum_{i=0}^{k}\frac{x^{k-i+1}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)} \\
&\quad + \frac{(-a)^{k+1}\,\Gamma(\alpha+1)\,(x-a)^{-\alpha}}{\Gamma(k-\alpha+2)\Gamma(\alpha-k)} \\
&= \frac{(k+1)\,x}{k-\alpha+1}\left(\frac{k!\ \Gamma(\alpha+1)\,(x-a)^{-\alpha}}{\Gamma(k-\alpha+1)}\sum_{i=0}^{k}\frac{x^{k-i}\,(-a)^i}{i!\ \Gamma(\alpha-i+1)}\right) \\
&\quad + \frac{(-a)^{k+1}\,\Gamma(\alpha+1)\Gamma(1-\alpha)}{\Gamma(k-\alpha+2)\Gamma(\alpha-k)}\left(D^{\alpha}_{a^+} 1\right)(x) \\
&= \frac{(k+1)\,x}{k-\alpha+1}\left(D^{\alpha}_{a^+} t^{k}\right)(x) - \frac{\alpha\ a^{k+1}}{k-\alpha+1}\left(D^{\alpha}_{a^+} 1\right)(x).
\end{aligned}$$

□

4. RBFs to discretize fractional operators

Given a set of centers $\bar{x}_j$, $j = 1, \dots, n$, the RBF interpolant in one dimension takes the form

$$s(x) = \sum_{j=1}^{n} \lambda_j \phi\left(\frac{|x - \bar{x}_j|}{c}\right),$$

where ϕ can be one of the RBFs listed in Table 1 and $c \in \mathbb{R}^+$ is the shape parameter. The coefficients λ_j are chosen by enforcing the interpolation condition

$$s(x_i) = f(x_i), \quad i = 1, \dots, n$$

at a set of collocation points x_i, $i = 1, \dots, n$. This leads to a $n \times n$ linear system

$$A\boldsymbol{\lambda} = \boldsymbol{F},$$

where

$$\begin{aligned} A &= \left(\phi\left(\frac{|x_i - \bar{x}_j|}{c}\right)\right)_{1 \le i,j \le n}, \\ \boldsymbol{\lambda} &= (\lambda_j,\ j = 1, \dots, n)^T, \\ \boldsymbol{F} &= (f(t_i),\ i = 1, \dots, n)^T. \end{aligned}$$

We wish to find a matrix D that discretizes the fractional differential operator $\mathcal{D}^\alpha$ with an RBF expansion, where $\mathcal{D}^\alpha$ can be one of the fractional operators defined in subsection 2.2. Applying the fractional differential operator $\mathcal{D}^\alpha$ to the RBF gives

$$\sum_{j=1}^{n} \lambda_j \left(\mathcal{D}^\alpha \phi\left(\frac{|x - \bar{x}_j|}{c}\right)\right)(x_i) = g_i, \qquad i = 1, \dots, n$$

where g_i is the value of the underlying function's fractional operator at each x_i. It leads to the matrix equation

$$B\boldsymbol{\lambda} = \boldsymbol{G},$$

where

$$\begin{aligned} B &= \left(\left(\mathcal{D}^\alpha \phi\left(\frac{|x - \bar{x}_j|}{c}\right)\right)(x_i)\right)_{1 \le i,j \le n}, \\ \boldsymbol{G} &= (g_i,\ i = 1, \dots, n)^T. \end{aligned}$$

The collocation matrix A being unconditionally nonsingular [13], and we can eliminate the expansion coefficient vector $\boldsymbol{\lambda}$ and obtain $\boldsymbol{G} = BA^{-1}\boldsymbol{F}$. The matrix $D = BA^{-1}$ thus gives an RBF discretization of $\mathcal{D}^\alpha$. In order to find

$$\left(\mathcal{D}^\alpha \phi\left(\frac{|\cdot - y|}{c}\right)\right)(x), \tag{1}$$

we define the function

$$\begin{aligned} &\phi_y : \mathbb{R} \to \mathbb{R}, \\ &\phi_y(x) := \phi\left(\frac{|x - y|}{c}\right), \end{aligned}$$

for arbitrary $y \in \mathbb{R}$. Therefore, evaluating

$$(\mathcal{D}^\alpha \phi_y)(x),$$

results in evaluating (1). The following theorems show that finding the fractional integrals and derivatives of the single real variable RBF ϕ can lead to fractional integrals and derivatives of ϕ_y.

Theorem 4.1. *If ϕ is an even function then for all $x > a$ we have*

$$\left(I^{\alpha}_{a+}\ \phi_y\right)(x) = c^{\alpha}\left(I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right).$$

Proof. Since ϕ is an even function, by Definition 2.3 we have

$$\begin{aligned}\left(I^{\alpha}_{a+}\ \phi_y\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{|t-y|}{c}\right)(x-t)^{\alpha-1}\,dt\\ &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt\\ &= \frac{c^{\alpha}}{\Gamma(\alpha)}\int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi(u)\left(\frac{x-y}{c}-u\right)^{\alpha-1}du,\end{aligned}$$

where $u = \frac{t-y}{c}$. Then

$$\left(I^{\alpha}_{a+}\ \phi_y\right)(x) = c^{\alpha}\left(I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right).$$

□

Remark 4.1. *Similarly, one can show that if ϕ is an even function then for all $x < b$ we have*

$$\left(I^{\alpha}_{b-}\ \phi_y\right)(x) = c^{\alpha}\left(I^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\right)\left(\frac{x-y}{c}\right).$$

Theorem 4.2. *If ϕ is an odd function then for all $x > a$ we have*

$$\left(I^{\alpha}_{a+}\ \phi_y\right)(x) = \begin{cases} c^{\alpha}\left(I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right), & y \le a,\\ c^{\alpha}\left(-I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2I^{\alpha}_{0+}\ \phi\right)\left(\frac{x-y}{c}\right), & a < y < x,\\ -c^{\alpha}\left(I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right), & y \ge x.\end{cases}$$

Proof. In case $y \le a$ by Definition 2.3 we have

$$\begin{aligned}\left(I^{\alpha}_{a+}\ \phi_y\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{|t-y|}{c}\right)(x-t)^{\alpha-1}\,dt\\ &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt\\ &= \frac{c^{\alpha}}{\Gamma(\alpha)}\int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi(u)\left(\frac{x-y}{c}-u\right)^{\alpha-1}du,\end{aligned}$$

where $u = \frac{t-y}{c}$. Then

$$\left(I^{\alpha}_{a+}\ \phi_y\right)(x) = c^{\alpha}\left(I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right).$$

In case $a < y < x$, by Definition 2.3 we have

$$\begin{aligned}
\left(I^{\alpha}_{a^+}\ \phi_y\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{|t-y|}{c}\right)(x-t)^{\alpha-1}\,dt \\
&= \frac{-1}{\Gamma(\alpha)}\int_a^y \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt + \frac{1}{\Gamma(\alpha)}\int_y^x \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt \\
&= \frac{-1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt + \frac{2}{\Gamma(\alpha)}\int_y^x \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt \\
&= \frac{-c^{\alpha}}{\Gamma(\alpha)}\int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi(u)\left(\frac{x-y}{c}-u\right)^{\alpha-1}du + \frac{2c^{\alpha}}{\Gamma(\alpha)}\int_0^{\frac{x-y}{c}} \phi(u)\left(\frac{x-y}{c}-u\right)^{\alpha-1}du,
\end{aligned}$$

where $u = \frac{t-y}{c}$. Then

$$\left(I^{\alpha}_{a^+}\ \phi_y\right)(x) = c^{\alpha}\left(-I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2I^{\alpha}_{0^+}\ \phi\ \right)\left(\frac{x-y}{c}\right).$$

In case $y \geq x$, by Definition 2.3 we have

$$\begin{aligned}
\left(I^{\alpha}_{a^+}\ \phi_y\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{|t-y|}{c}\right)(x-t)^{\alpha-1}\,dt \\
&= \frac{-1}{\Gamma(\alpha)}\int_a^x \phi\left(\frac{t-y}{c}\right)(x-t)^{\alpha-1}\,dt \\
&= \frac{-c^{\alpha}}{\Gamma(\alpha)}\int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi(u)\left(\frac{x-y}{c}-u\right)^{\alpha-1}du,
\end{aligned}$$

where $u = \frac{t-y}{c}$. Then

$$\left(I^{\alpha}_{a^+}\ \phi_y\right)(x) = -c^{\alpha}\left(I^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ \right)\left(\frac{x-y}{c}\right).$$

$\square$

Remark 4.2. *Similarly, one can show that if ϕ is an odd function then for all $x < b$ we have*

$$\left(I^{\alpha}_{b^-}\ \phi_y\right)(x) = \begin{cases} c^{\alpha}\left(I^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\ \right)\left(\frac{x-y}{c}\right), & y \leq x, \\ c^{\alpha}\left(I^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\ - 2I^{\alpha}_{0^-}\ \phi\ \right)\left(\frac{x-y}{c}\right), & x < y < b, \\ -c^{\alpha}\left(I^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\ \right)\left(\frac{x-y}{c}\right), & y \geq b. \end{cases}$$

Theorem 4.3. *If ϕ is an even function then for all $x > a$ we have*

$$\left(D^{\alpha}_{a^+}\ \phi_y\right)(x) = c^{-\alpha}\left(D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ \right)\left(\frac{x-y}{c}\right).$$

Proof. Let $n = [\alpha]+1$, by Definition 2.4 and Theorem 4.1 we have

$$\begin{aligned}
\left(D^{\alpha}_{a^+}\phi_y\right)(x) &= \left(I^{n-\alpha}_{a^+}\phi_y\right)^{(n)}(x) \\
&= c^{n-\alpha}\left(\left(I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\phi\right)\left(\frac{x-y}{c}\right)\right)^{(n)} \\
&= c^{-\alpha}\left(I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\phi\right)^{(n)}\left(\frac{x-y}{c}\right) \\
&= c^{-\alpha}\left(D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ \right)\left(\frac{x-y}{c}\right).
\end{aligned}$$

$\square$

Remark 4.3. *Similarly, one can show that if ϕ is an even function then for all $x < b$ we have*

$$\left(D^{\alpha}_{b-}\ \phi_y\right)(x) = c^{-\alpha}\left(D^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\right)\left(\frac{x-y}{c}\right).$$

Theorem 4.4. *If ϕ is an odd function then for all $x > a$ we have*

$$\left(D^{\alpha}_{a+}\ \phi_y\right)(x) = \begin{cases} c^{-\alpha}\left(D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right), & y \leq a, \\ c^{-\alpha}\left(-D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2D^{\alpha}_{0+}\ \phi\right)\left(\frac{x-y}{c}\right), & a < y < x, \\ -c^{-\alpha}\left(D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right), & y \geq x. \end{cases}$$

Proof. Let $n = [\alpha] + 1$, by Definition 2.4 and Theorem 4.2 we have

$$\begin{aligned} \left(D^{\alpha}_{a+}\phi_y\right)(x) &= \left(I^{n-\alpha}_{a+}\phi_y\right)^{(n)}(x) \\ &= \begin{cases} c^{n-\alpha}\left(\left(I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right)\right)^{(n)}, & y \leq a, \\ c^{n-\alpha}\left(\left(-I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2I^{n-\alpha}_{0+}\ \phi\right)\left(\frac{x-y}{c}\right)\right)^{(n)}, & a < y < x, \\ -c^{n-\alpha}\left(\left(I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right)\right)^{(n)}, & y \geq x, \end{cases} \\ &= \begin{cases} c^{-\alpha}\left(I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)^{(n)}\left(\frac{x-y}{c}\right), & y \leq a, \\ c^{-\alpha}\left(-I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2I^{n-\alpha}_{0+}\ \phi\right)^{(n)}\left(\frac{x-y}{c}\right), & a < y < x, \\ -c^{-\alpha}\left(I^{n-\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)^{(n)}\left(\frac{x-y}{c}\right), & y \geq x, \end{cases} \\ &= \begin{cases} c^{-\alpha}\left(D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right), & y \leq a, \\ c^{-\alpha}\left(-D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2D^{\alpha}_{0+}\ \phi\right)\left(\frac{x-y}{c}\right), & a < y < x, \\ -c^{-\alpha}\left(D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right), & y \geq x. \end{cases} \end{aligned}$$

$\square$

Remark 4.4. *Similarly, one can show that if ϕ is an odd function then for all $x < b$ we have*

$$\left(D^{\alpha}_{b-}\ \phi_y\right)(x) = \begin{cases} c^{-\alpha}\left(D^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\right)\left(\frac{x-y}{c}\right), & y \leq x, \\ c^{-\alpha}\left(D^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\ - 2D^{\alpha}_{0-}\ \phi\right)\left(\frac{x-y}{c}\right), & x < y < b, \\ -c^{-\alpha}\left(D^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi\right)\left(\frac{x-y}{c}\right), & y \geq b. \end{cases}$$

Theorem 4.5. *If ϕ is an even function then for all $x > a$ we have*

$$\left({}^{C}D^{\alpha}_{a+}\ \phi_y\right)(x) = c^{-\alpha}\left({}^{C}D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\right)\left(\frac{x-y}{c}\right).$$

Proof. Let $n = [\alpha] + 1$, by Definition 2.5 we have

$$\begin{aligned}\left({}^C D^{\alpha}_{a^+}\ \phi_y \right)(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{|t-y|}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{t-y}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\ &= \frac{1}{c^n \Gamma(n-\alpha)} \int_a^x \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt \\ &= \frac{c^{-\alpha}}{\Gamma(n-\alpha)} \int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi^{(n)}(u) \left(\frac{x-y}{c} - u \right)^{n-\alpha-1} du,\end{aligned}$$

where $u = \frac{t-y}{c}$. Then

$$\left({}^C D^{\alpha}_{a^+}\ \phi_y \right)(x) = c^{-\alpha} \left({}^C D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi \right) \left(\frac{x-y}{c} \right).$$

□

Remark 4.5. *Similarly, one can show that if ϕ is an even function then for all $x < b$ we have*

$$\left({}^C D^{\alpha}_{b^-}\ \phi_y \right)(x) = c^{-\alpha} \left({}^C D^{\alpha}_{\left(\frac{b-y}{c}\right)^-}\ \phi \right) \left(\frac{x-y}{c} \right).$$

Theorem 4.6. *If ϕ is an odd function then for all $x > a$ we have*

$$\left({}^C D^{\alpha}_{a^+}\ \phi_y \right)(x) = \begin{cases} c^{-\alpha} \left({}^C D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi \right) \left(\frac{x-y}{c} \right), & y \le a, \\ c^{-\alpha} \left(- {}^C D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi\ + 2\, {}^C D^{\alpha}_{0^+}\ \phi \right) \left(\frac{x-y}{c} \right), & a < y < x, \\ -c^{-\alpha} \left({}^C D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi \right) \left(\frac{x-y}{c} \right), & y \ge x. \end{cases}$$

Proof. In case $y \le a$ by Definition 2.5 we have

$$\begin{aligned}\left({}^C D^{\alpha}_{a^+}\ \phi_y \right)(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{|t-y|}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{t-y}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\ &= \frac{1}{c^n \Gamma(n-\alpha)} \int_a^x \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt \\ &= \frac{c^{-\alpha}}{\Gamma(n-\alpha)} \int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi^{(n)}(u) \left(\frac{x-y}{c} - u \right)^{n-\alpha-1} du,\end{aligned}$$

where $u = \frac{t-y}{c}$. Then

$$\left({}^C D^{\alpha}_{a^+}\ \phi_y \right)(x) = c^{-\alpha} \left({}^C D^{\alpha}_{\left(\frac{a-y}{c}\right)^+}\ \phi \right) \left(\frac{x-y}{c} \right).$$

In case $a < y < x$, by Definition 2.5 we have

$$
\begin{aligned}
&\left({}^C D_{a+}^{\alpha}\ \phi_y \right)(x) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{|t-y|}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-1}{\Gamma(n-\alpha)} \int_a^y \left(\phi\left(\frac{t-y}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt + \frac{1}{\Gamma(n-\alpha)} \int_y^x \left(\phi\left(\frac{t-y}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-1}{c^n\Gamma(n-\alpha)} \int_a^y \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt + \frac{1}{c^n\Gamma(n-\alpha)} \int_y^x \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-1}{c^n\Gamma(n-\alpha)} \int_a^x \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt + \frac{2}{c^n\Gamma(n-\alpha)} \int_y^x \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-c^{-\alpha}}{\Gamma(n-\alpha)} \int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi^{(n)}(u) \left(\frac{x-y}{c} - u \right)^{n-\alpha-1} du + \frac{2c^{-\alpha}}{\Gamma(n-\alpha)} \int_0^{\frac{x-y}{c}} \phi^{(n)}(u) \left(\frac{x-y}{c} - u \right)^{n-\alpha-1} du,
\end{aligned}
$$

where $u = \frac{t-y}{c}$. Then

$$
\left({}^C D_{a+}^{\alpha}\ \phi_y \right)(x) = c^{-\alpha} \left(-\ {}^C D_{\left(\frac{a-y}{c}\right)^+}^{\alpha}\ \phi\ + 2\ {}^C D_{0+}^{\alpha}\ \phi \right) \left(\frac{x-y}{c} \right).
$$

In case $y \geq x$, by Definition 2.5 we have

$$
\begin{aligned}
\left({}^C D_{a+}^{\alpha}\ \phi_y \right)(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{|t-y|}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-1}{\Gamma(n-\alpha)} \int_a^x \left(\phi\left(\frac{t-y}{c} \right) \right)^{(n)} (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-1}{c^n\Gamma(n-\alpha)} \int_a^x \phi^{(n)}\left(\frac{t-y}{c} \right) (x-t)^{n-\alpha-1}\, dt \\
&= \frac{-c^{-\alpha}}{\Gamma(n-\alpha)} \int_{\frac{a-y}{c}}^{\frac{x-y}{c}} \phi^{(n)}(u) \left(\frac{x-y}{c} - u \right)^{n-\alpha-1} du,
\end{aligned}
$$

where $u = \frac{t-y}{c}$. Then

$$
\left({}^C D_{a+}^{\alpha}\ \phi_y \right)(x) = -c^{-\alpha} \left({}^C D_{\left(\frac{a-y}{c}\right)^+}^{\alpha}\ \phi \right) \left(\frac{x-y}{c} \right).
$$

□

Remark 4.6. *Similarly, one can show that if ϕ is an odd function then for all $x < b$ we have*

$$
\left({}^C D_{b-}^{\alpha}\ \phi_y \right)(x) = \begin{cases} c^{-\alpha} \left({}^C D_{\left(\frac{b-y}{c}\right)^-}^{\alpha}\ \phi \right) \left(\frac{x-y}{c} \right), & y \leq x, \\ c^{-\alpha} \left({}^C D_{\left(\frac{b-y}{c}\right)^-}^{\alpha}\ \phi\ - 2\ {}^C D_{0-}^{\alpha}\ \phi \right) \left(\frac{x-y}{c} \right), & x < y < b, \\ -c^{-\alpha} \left({}^C D_{\left(\frac{b-y}{c}\right)^-}^{\alpha}\ \phi \right) \left(\frac{x-y}{c} \right), & y \geq b. \end{cases}
$$

In the sequel, we see that all five kinds of RBFs listed in subsection 2.1 are even functions, odd functions or linear combinations of them. Therefore, we can use results of above theorems and only find the analytic expression of fractional integrals and derivatives of a single real variable RBF $\phi(r)$ for evaluating (1). For simplicity in notation, we work with the function $\phi(x)$ instead of $\phi(r)$. In order to find an analytic expression for the fractional integral or derivative $\mathcal{D}^{\alpha}\phi(x)$, we represent each $\phi(x)$ as Taylor series expansions and then apply the fractional operator term by term. This can lead to fractional integrals or derivatives of x^k, $k \in \mathbb{N}$, which can be calculated by the following recursive relations derived from Theorems 3.4-3.6.

$$\begin{aligned} \left(I_{a^+}^{\alpha} t^{k+1}\right)(x) &= \frac{(k+1)\,x}{k+\alpha+1}\left(I_{a^+}^{\alpha} t^{k}\right)(x) + \frac{\alpha\ a^{k+1}}{k+\alpha+1}\left(I_{a^+}^{\alpha} 1\right)(x), \\ \left(I_{a^+}^{\alpha} 1\right)(x) &= \tfrac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}. \end{aligned} \tag{2}$$

$$\begin{aligned} \left({}^{C}D_{a^+}^{\alpha} t^{k+1}\right)(x) &= \frac{(k+1)\,x}{k-\alpha+1}\left({}^{C}D_{a^+}^{\alpha} t^{k}\right)(x) + \binom{k+1}{n}\frac{(n-\alpha)\ a^{k-n+1}}{k-\alpha+1}\left({}^{C}D_{a^+}^{\alpha} t^{n}\right)(x), \\ \left({}^{C}D_{a^+}^{\alpha} t^{n}\right)(x) &= \tfrac{n!}{\Gamma(n-\alpha+1)}(x-a)^{n-\alpha}. \end{aligned} \tag{3}$$

$$\begin{aligned} \left(D_{a^+}^{\alpha} t^{k+1}\right)(x) &= \frac{(k+1)\,x}{k-\alpha+1}\left(D_{a^+}^{\alpha} t^{k}\right)(x) - \frac{\alpha\ a^{k+1}}{k-\alpha+1}\left(D_{a^+}^{\alpha} 1\right)(x), \\ \left(D_{a^+}^{\alpha} 1\right)(x) &= \tfrac{1}{\Gamma(1-\alpha)}(x-a)^{-\alpha}. \end{aligned} \tag{4}$$

- **Powers:** $\phi(x) = x^n$, $n \in \mathbb{N}$.
  It is clear that if n is an even number then the Power function $\phi(x) = x^n$ is even, otherwise, it is odd. Also $I_{a^+}^{\alpha} x^n$, ${}^{C}D_{a^+}^{\alpha} x^n$ and $D_{a^+}^{\alpha} x^n$ can be easily evaluated by (2)-(4), recursively.

- **Gaussian:** $\phi(x) = \exp(-\frac{x^2}{2})$.
  It is clear that the Gaussian function is even. Furthermore, we represent $\phi(x)$ as a MacLaurin series expansion and then apply the fractional operators term by term as follows:

$$\mathrm{e}^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} \Longrightarrow \begin{cases} I_{a^+}^{\alpha} \mathrm{e}^{-\frac{x^2}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}\ I_{a^+}^{\alpha} x^{2n}, \\ D_{a^+}^{\alpha} \mathrm{e}^{-\frac{x^2}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}\ D_{a^+}^{\alpha} x^{2n}, \\ {}^{C}D_{a^+}^{\alpha} \mathrm{e}^{-\frac{x^2}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}\ {}^{C}D_{a^+}^{\alpha} x^{2n}. \end{cases}$$

  Since the radius of convergence of above series is ∞, we can truncate the infinite sum once the terms are smaller in magnitude than machine precision. Also $I_{a^+}^{\alpha} x^{2n}$, ${}^{C}D_{a^+}^{\alpha} x^{2n}$ and $D_{a^+}^{\alpha} x^{2n}$ can be easily evaluated by (2)-(4), recursively.

- **Multiquadric:** $\phi(x) = (1+x^2)^{\beta/2}$, $\beta \in \mathbb{R}_{\neq 0} \setminus 2\mathbb{N}$.
  It is evident that the Multiquadric function is even. Here we represent the Multiquadric function as a MacLaurin series expansion and then apply the fractional operators term by term as follows:

$$\left(1+x^2\right)^{\frac{\beta}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{(n)}}{n!} x^{2n} \Longrightarrow \begin{cases} I_{a^+}^{\alpha}\left(1+x^2\right)^{\frac{\beta}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{(n)}}{n!} I_{a^+}^{\alpha} x^{2n}, \\ D_{a^+}^{\alpha}\left(1+x^2\right)^{\frac{\beta}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{(n)}}{n!} D_{a^+}^{\alpha} x^{2n}, \\ {}^{C}D_{a^+}^{\alpha}\left(1+x^2\right)^{\frac{\beta}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{(n)}}{n!}\ {}^{C}D_{a^+} x^{2n}, \end{cases}$$

  where

$$\left(\frac{\beta}{2}\right)^{(n)} = \begin{cases} \dfrac{\beta}{2}\left(\dfrac{\beta}{2}-1\right)\cdots\left(\dfrac{\beta}{2}-n+1\right), & n \in \mathbb{N}, \\ 0, & n = 0. \end{cases}$$

Since the radius of convergence of above series is 1, we can truncate the infinite sum once the terms are smaller in magnitude than machine precision for all $x \in (-1,1)$. Also $I^{\alpha}_{a+}x^{2n}$, ${}^{C}D^{\alpha}_{a+}x^{2n}$ and $D^{\alpha}_{a+}x^{2n}$ can be easily evaluated by (2)-(4), recursively. Note that according to the results of section 4, in practice, we evaluate fractional integrals and derivatives at $\frac{x-y}{c}$, and so we have to choose proper shape parameters c such that $\frac{x-y}{c} \in (-1,1)$.

- **Matérn:** $\phi(x) = x^{\nu}K_{\nu}(x)$, $\nu = m - \frac{1}{2}$, $m \in \mathbb{N}$.
We know that

$$x^{\nu}K_{\nu}(x) = \frac{\pi}{2\sin(\pi\nu)}\left(\frac{1}{2^{-\nu}}\sum_{n=0}^{\infty}\frac{x^{2n}}{4^n\ n!\ \Gamma(-\nu+n+1)} - \frac{1}{2^{\nu}}\sum_{n=0}^{\infty}\frac{x^{2n+2\nu}}{4^n\ n!\ \Gamma(\nu+n+1)}\right). \tag{5}$$

If 2ν is an even number then the Matérn function is even otherwise it is sum of the two odd and even functions. Moreover, according to (5), the fractional integral and derivatives of the Matérn function are given as follows:

$$\begin{aligned}
I^{\alpha}_{a+}x^{\nu}K_{\nu}(x) &= \frac{\pi}{2\sin(\pi\nu)}\left(\frac{1}{2^{-\nu}}\sum_{n=0}^{\infty}\frac{I^{\alpha}_{a+}x^{2n}}{4^n\ n!\ \Gamma(-\nu+n+1)} - \frac{1}{2^{\nu}}\sum_{n=0}^{\infty}\frac{I^{\alpha}_{a+}x^{2n+2\nu}}{4^n\ n!\ \Gamma(\nu+n+1)}\right),\\
D^{\alpha}_{a+}x^{\nu}K_{\nu}(x) &= \frac{\pi}{2\sin(\pi\nu)}\left(\frac{1}{2^{-\nu}}\sum_{n=0}^{\infty}\frac{D^{\alpha}_{a+}x^{2n}}{4^n\ n!\ \Gamma(-\nu+n+1)} - \frac{1}{2^{\nu}}\sum_{n=0}^{\infty}\frac{D^{\alpha}_{a+}x^{2n+2\nu}}{4^n\ n!\ \Gamma(\nu+n+1)}\right),\\
{}^{C}D^{\alpha}_{a+}x^{\nu}K_{\nu}(x) &= \frac{\pi}{2\sin(\pi\nu)}\left(\frac{1}{2^{-\nu}}\sum_{n=0}^{\infty}\frac{{}^{C}D^{\alpha}_{a+}x^{2n}}{4^n\ n!\ \Gamma(-\nu+n+1)} - \frac{1}{2^{\nu}}\sum_{n=0}^{\infty}\frac{{}^{C}D^{\alpha}_{a+}x^{2n+2\nu}}{4^n\ n!\ \Gamma(\nu+n+1)}\right).
\end{aligned}$$

Since the radius of convergence of above series is ∞, we can truncate the infinite sum once the terms are smaller in magnitude than machine precision. Also fractional integrals and derivatives of x^{2n} and $x^{2n+2\nu}$ can be easily evaluated by (2)-(4), recursively.

- **Thin-plate spline:** $\phi(x) = x^{2n}\ln(x)$, $n \in \mathbb{N}$.
We know that $x^{2n}\ln(x) = \dfrac{1}{2}x^{2n}\ln(x^2)$. It is clear that $\phi(x) = \dfrac{1}{2}x^{2n}\ln(x^2)$ is an even function. Now, we represent $\phi(x)$ as a Taylor series expansion about the point $a = 1$ as follows:

$$\begin{aligned}
\frac{1}{2}x^{2n}\ln(x^2) &= \frac{1}{2}x^{2n}\sum_{k=0}^{\infty}\frac{(-1)^k(x^2-1)^{k+1}}{k+1} = \frac{1}{2}x^{2n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\sum_{i=0}^{k+1}\frac{(k+1)!\,(-1)^{k+1-i}}{i!\,(k+1-i)!}x^{2i}\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{i=0}^{k+1}\frac{k!\,(-1)^{1+i}}{i!\,(k+1-i)!}x^{2i+2n}.
\end{aligned}$$

So the fractionl integrals and derivatives of the Thin-plate spline function are given as follows:

$$\begin{cases}
I^{\alpha}_{a+}\dfrac{1}{2}x^{2n}\ln(x^2) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\sum_{i=0}^{k+1}\frac{k!\,(-1)^{1+i}}{i!\,(k+1-i)!}I^{\alpha}_{a+}x^{2i+2n},\\[2ex]
D^{\alpha}_{a+}\dfrac{1}{2}x^{2n}\ln(x^2) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\sum_{i=0}^{k+1}\frac{k!\,(-1)^{1+i}}{i!\,(k+1-i)!}D^{\alpha}_{a+}x^{2i+2n},\\[2ex]
{}^{C}D^{\alpha}_{a+}\dfrac{1}{2}x^{2n}\ln(x^2) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\sum_{i=0}^{k+1}\frac{k!\,(-1)^{1+i}}{i!\,(k+1-i)!}\ {}^{C}D^{\alpha}_{a+}x^{2i+2n}.
\end{cases}$$

Since the radius of convergence of above series is $\sqrt{2}$, we can truncate the infinite sum once the terms are smaller in magnitude than machine precision for all $x \in \left(-\sqrt{2},\sqrt{2}\right)$. Also $I^{\alpha}_{a+}x^{2i+2n}$, ${}^{C}D^{\alpha}_{a+}x^{2i+2n}$ and $D^{\alpha}_{a+}x^{2i+2n}$ can be easily evaluated by (2)-(4), recursively. Note that according to the results of section 4, in practice, we evaluate fractional integrals and derivatives at $\frac{x-y}{c}$, and so we have to choose proper shape parameters c such that $\frac{x-y}{c} \in \left(-\sqrt{2},\sqrt{2}\right)$.

Remark 4.7. *With the same way we can get* I^{α}_{b-}, D^{α}_{b-} *and* ${}^{C}D^{\alpha}_{b-}$ *for all five kinds of RBFs listed above.*

As an example, the algorithm of calculating $\left(D^{\alpha}_{a+}\phi_y\right)(x)$ in case $\phi(r) = r^n$ where n is an odd number is given as follows:

Algorithm

```
set $a_1 = \dfrac{a-y}{c}$, $b_1 = \dfrac{b-y}{c}$ and $x_1 = \dfrac{x-y}{c}$.
Obtain $\left(D^{\alpha}_{a_1^+}\phi\right)(x_1)$ by recursive relation 4.
if $a_1 \geq 0$ then
    $\left(D^{\alpha}_{a+}\phi_y\right)(x) = c^{-\alpha}\left(D^{\alpha}_{a_1^+}\phi\right)(x_1)$.
else if $x_1 \leq 0$ then
    $\left(D^{\alpha}_{a+}\phi_y\right)(x) = -c^{-\alpha}\left(D^{\alpha}_{a_1^+}\phi\right)(x_1)$.
else
    Obtain $\left(D^{\alpha}_{0+}\phi\right)(x_1)$ by recursive relation 4.
    $\left(D^{\alpha}_{a+}\phi_y\right)(x) = c^{-\alpha}\left(-\left(D^{\alpha}_{a_1^+}\phi\right)(x_1) + 2\left(D^{\alpha}_{0+}\phi\right)(x_1)\right)$.
end if
```

5. Numerical results

In this section we apply the results of the previous sections to solve two test problems. The first test case focuses only on the discretization of the fractional differential operator while the second considers the solution of a fractional differential equation.

5.1. Test problem 1

Consider the function $f(x) = \mathrm{e}^{kx}$ with $x \in [0,1]$, for some $k > 0$. The exact left-sided Riemann-Liouville fractional derivative of f is given by [4]:

$$D^{\alpha}_{a+}\mathrm{e}^{kx} = \frac{\mathrm{e}^{ka}}{\Gamma(1-\alpha)}(x-a)^{-\alpha}\;{}_1F_1(1, 1-\alpha; k(x-a)),$$

where a is an arbitrary real number, α is a non-integer positive number, and ${}_1F_1$ denotes Kummer's confluent hypergeometric function. In our computational work in this case, we choose n equispaced center points

$$\bar{x}_j = \frac{j}{n-1}, \quad j = 0, \dots, n-1,$$

in $[0,1]$ and put $a = 0$, $\alpha = 1.5$, and $\alpha = 0.8$. Then according to Definitions 2.3-2.5, we have to choose n collocation points x_i in $(0,1]$. We work with

$$x_0 = \frac{1}{2(n-1)}, \quad x_i = \frac{i}{n-1}, \quad i = 1, \dots, n-1.$$

The RBF approximations to $D^{\alpha}_{a+}\mathrm{e}^{x}$ are evaluated by five kinds of RBFs listed in subsection 2.1 with $n = 50$ and the L_∞ error norms are reported in Table 2. It must be noted that we use Not-a-Knot technique to improve the accuracy near the ends [34].

5.2. Test problem 2

Consider the following fractional differential equation [37]:

$$\begin{aligned} &D^{3/2}_{0+}u(t) + u(t) = f(t), \quad t \in (0,T], \\ &u(0) = u'(0) = 0. \end{aligned}$$

We choose n equispaced center points

$$\bar{t}_i = \frac{T}{n-1} i, \quad i = 0, \dots, n-1,$$

in $[0,T]$. Then according to Definitions 2.3-2.5, we have to choose n collocation points t_i in $(0,T]$. We work with

$$t_0 = \frac{T}{2(n-3)}, \quad t_i = \frac{T}{n-3} i, \quad i = 1, \dots, n-3.$$

Then the approximate solution can be written as

$$u(t) = \sum_{j=0}^{n-1} \lambda_j \phi \left(\frac{|t - \bar{t}_j|}{c} \right).$$

The unknown parameters λ_j are to be determined by the collocation method. Therefore, we get the following equations:

$$\sum_{j=0}^{n-1} \lambda_j \ D_{0+}^{3/2} \phi \left(\frac{|t_i - \bar{t}_j|}{c} \right) + \sum_{j=0}^{n-1} \lambda_j \phi \left(\frac{|t_i - \bar{t}_j|}{c} \right) = f(t_i) \tag{6}$$

for $i = 0, \dots n-3$, and the following equations for the initial conditions

$$\sum_{j=0}^{n-1} \lambda_j \ \phi \left(\frac{|0 - \bar{t}_j|}{c} \right) = 0, \tag{7}$$

$$\sum_{j=0}^{n-1} \lambda_j \ \phi' \left(\frac{|0 - \bar{t}_j|}{c} \right) = 0. \tag{8}$$

Then (6)-(8) lead to the following system of equations:

$$\begin{bmatrix} D_{0+}^{3/2} \boldsymbol{\phi} + \boldsymbol{\phi} \\ \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_1' \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{F} \\ 0 \\ 0 \end{bmatrix}.$$

The necessary matrices and vectors are

$$\begin{aligned}
\boldsymbol{\phi} &= \left(\phi \left(\frac{|t_i - \bar{t}_j|}{c} \right) \right)_{0 \le i \le n-3, 0 \le j \le n-1}, \\
D_{0+}^{3/2} \boldsymbol{\phi} &= \left(D_{0+}^{3/2} \phi (\frac{|t_i - \bar{t}_j|}{c}) \right)_{0 \le i \le n-3, 0 \le j \le n-1}, \\
\boldsymbol{\phi}_1 &= \left(\phi \left(\frac{|0 - \bar{t}_j|}{c} \right) \right)_{0 \le j \le n-1}, \\
\boldsymbol{\phi}_1' &= \left(\phi' \left(\frac{|0 - \bar{t}_j|}{c} \right) \right)_{0 \le j \le n-1}, \\
\boldsymbol{\lambda} &= (\lambda_j, \ j = 0, \dots, n-1)^T, \\
\boldsymbol{F} &= (f(t_i), \ i = 0, \dots, n-3)^T.
\end{aligned}$$

Now, we take 1001 points in the interval $0 \le t \le 50$ and work with three kinds of RBFs, Powers ($\beta = 5$), Gaussian and Matérn ($\nu = \frac{3}{2}$), with the shape parameter $c = 10$. As previously mentioned, the radius of convergence for series $D_{a+}^{\alpha} (1 + x^2)^{\frac{\beta}{2}}$ and $D_{a+}^{\alpha} \frac{1}{2} x^{2n} \ln(x^2)$ is 1 and $\sqrt{2}$ respectively, on the other hand in this example the length of interval is 50 thus if we work with Multiquadric or Thin-plate spline RBF we must get $c > 50$ that is a cause of decrease in accuracy. The numerical solutions with different right-hand side functions $f(t) = t\mathrm{e}^{-t}$ and $f(t) = \mathrm{e}^{-t} \sin(0.2t)$ are plotted in Figures 1, 2, and 3. The results are in agreement with the results of [37]. It must be noted that in this example, Not-a-Knot technique has also been used.

6. Conclusion

The fractional integrals and derivatives of Riemann-Liouville and Caputo type for five kinds of RBFs including the Powers, Gaussian, Multiquadric, Matérn and Thin-plate splines, in one dimension, are obtained. This allows to use high order numerical methods for solving fractional differential equations. Two test problems are given in order to validate formulas. The first test problem focuses only on the discretization of the fractional differential operator while the second one is a fractional differential equation which is solved by the RBF collocation method.

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Table 2: L_∞ error norms for $D^{\alpha}_{a+}\mathrm{e}^x$ by five kinds of RBFs. (Test Problem 1)

RBF	$\phi(r)$	shape parameter	L_∞ ($\alpha = 0.8$)	L_∞ ($\alpha = 1.5$)
Power	r^5	1	$2.25E-09$	$1.35E-07$
Gaussian	$\exp(-\frac{r^2}{2})$	1	$6.53E-10$	$1.12E-08$
Multiquadric	$\left(1+r^2\right)^{1/2}$	10	$1.88E-07$	$1.04E-05$
Matérn	$r^{3/2}K_{3/2}\left(r\right)$	1	$9.70E-14$	$2.96E-11$
Thin-plate spline	$r^6 ln\left(r\right)$	14	$2.50E-03$	$5.51E-03$

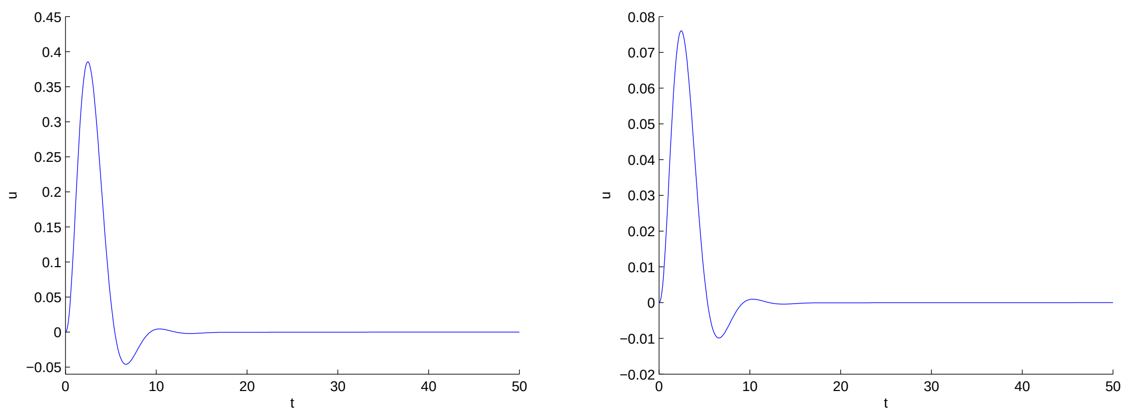

Figure 1: Numerical results of u with Powers RBF ($\beta = 5$), for $f(t) = t\mathrm{e}^{-t}$ (left), and $f(t) = \mathrm{e}^{-t}\sin(0.2t)$ (right). (Test problem 2)

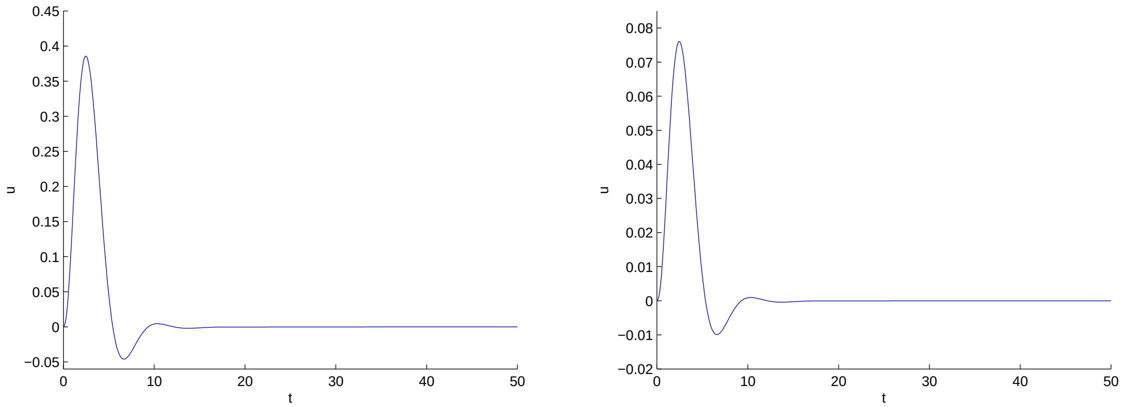

Figure 2: Numerical results of u with Matérn RBF ($\nu = \frac{3}{2}$), for $f(t) = t\mathrm{e}^{-t}$ (left), and $f(t) = \mathrm{e}^{-t}\sin(0.2t)$ (right). (Test problem 2)

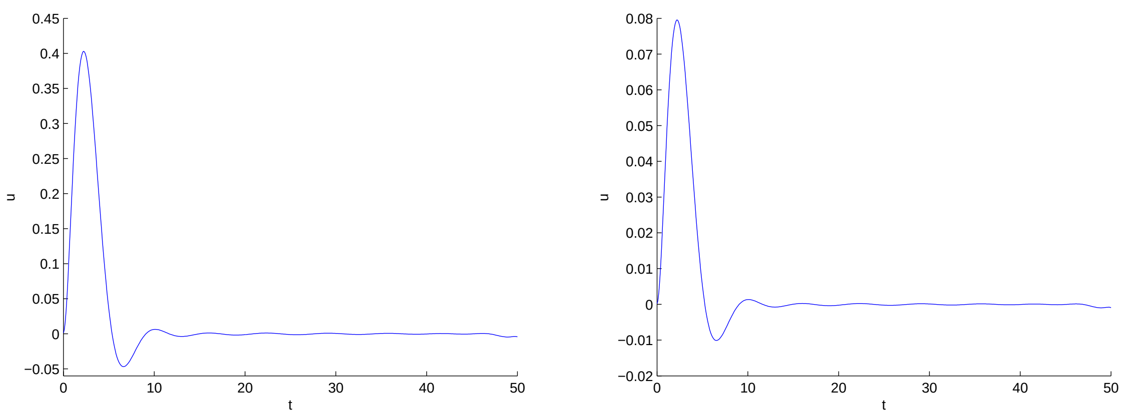

Figure 3: Numerical results of u with Gaussian RBF for $f(t) = t\mathrm{e}^{-t}$ (left), and $f(t) = \mathrm{e}^{-t}\sin(0.2t)$ (right). (Test problem 2)