THE MESHLESS KERNEL-BASED METHOD OF LINES FOR PARABOLIC EQUATIONS

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Abstract. Using the heat equation as a simple example, we give a rigid theoretical analysis of the Method of Lines, implemented as a meshless method based on spatial trial spaces spanned by translates of positive definite kernels. The technique can be generalized to other parabolic problems, and some numerical illustrations are given.

Key words. Method of Lines, time integration, CFL condition, partial differential equations, parabolic equations

AMS subject classifications. 65M20, 65M12, 65M22, 35K05, 35K08,35Q79, 45H99

1. Introduction. There are plenty of application papers in which kernels or radial basis functions are successfully used for solving partial differential equations by meshless methods. The usage of kernels is typically based on spatial interpolation at scattered locations, writing the trial functions "*entirely in terms of nodes*"[2]. For *stationary* partial differential equations, the discretization can take pointwise analytic derivatives of the trial functions to end up with a linear system of equations. This started in [6] and was pursued in the following years, including a convergence theory in [12]. There are also variations that use weak data, like the Meshless Local Petrov–Galerkin method [1] with a convergence theory in [14]. For the potential equation, there are special kernels that allow the use of trial functions that satisfy the differential equation exactly [13, 5].

For *time–dependent* partial differential equations, meshless kernel–based methods were similarly based on a fixed spatial interpolation, but now the coefficients are time–dependent, and one obtains a system of ordinary differential equations for these. This is the well–known *Method of Lines*, and it turned to be experimentally useful in various cases (see e.g. [16, 7, 4, 15]). However, a rigid analysis of its behavior seems to be still missing.

For the simple case of the heat equation, this paper provides an analysis of the Method of Lines. To this end, we start with basics on kernels, then describe the Method of Lines and analyze it. Though the Method of Lines needs no explicit CFL condition, we show how a CFL condition acts behind ths scene. Some numerical examples are provided as well, and a short section showing how to generalize this to much more general parabolic equations.

2. Kernel-Based Space Discretization. A *kernel* is a symmetric function

$$K \ : \ \Omega \times \Omega \to \mathbb{R}$$

on some spatial domain $\Omega \subset \mathbb{R}^d$. The kernel usually is assumed to be *positive definite*, i.e. for all selections of finite point sets $X = \{x_1, \ldots, x_n\} \subset \Omega$, the $n \times n$ *kernel matrices* $A = A(X)$ with entries $K(x_j, x_k)$, $1 \le j, k \le n$ are symmetric and positive definite. Standard examples are *radial basis functions* like the *Gaussian*

$$K(x, y) = \exp(-\|x - y\|_2^2) \text{ for all } x,\, y \in \mathbb{R}^d$$

or the compactly supported *Wendland function*

$$K(x, y) = \begin{cases} (1 - \|x - y\|_2)^4 (1 + 4\|x - y\|_2) & \|x - y\|_1 \le 1 \\ 0 & \|x - y\|_1 \ge 1 \end{cases}$$

for all $x,\, y \in \mathbb{R}^d$ with $d \leq 3$.

The standard way to use kernels for solving time–dependent partial differential equations is to introduce a fixed space discretization via finite spatial point sets $X = \{x_1, \ldots, x_n\} \subset \Omega$ and to generate spatial trial functions via translates of a kernel K in the form

$$s(x) := \sum_{j=1}^{n} \alpha_j K(x, x_j),\ x \in \Omega. \tag{2.1}$$

Interpolation of a spatial function $f\ :\ \Omega \to \mathbb{R}$ on the given point set X is done by solving the system

$$s(x_k) = \sum_{j=1}^{n} \alpha_j K(x_k, x_j) = f(x_k),\ 1 \leq k \leq n$$

involving the $n \times n$ *kernel matrix* A with entries $K(x_j, x_k),\ 1 \leq j, k \leq n$ which is positive definite for all positive definite kernels.

For further use we note that one can construct a Lagrange basis $u_1(x), \ldots, u_n(x)$ of the span of the functions $K(\cdot, x_j),\ 1 \leq j \leq n$ via solving the system

$$u(x) = K(x)A^{-1} \tag{2.2}$$

where we use the notation

$$u(x) := (u_1(x), \ldots, u_n(x)),\ K(x) = (K(x, x_1), \ldots, K(x, x_n)).$$

Here and in what follows, indices running over functions will be column indices, while indices running over points will be row indices. In particular, it is convenient to introduce the column-valued *evaluation operator* defined as

$$E(f) := (f(x_1), \ldots, f(x_n))^T \text{ for all } f\ :\ \Omega \to \mathbb{R},$$

and application of this operator to a row of m functions $v_1, \ldots, v_m$ should generate the $n \times m$ matrix with entries $v_j(x_k)$ with $1 \leq j \leq m$ for the columns and $1 \leq k \leq n$ for the rows. In particular, the kernel matrix then is $A = E(K(x))$ and the Lagrange property simply follows from

$$E(u(x)) = E(K(x)A^{-1}) = E(K(x))A^{-1} = AA^{-1} = I_{n \times n}.$$

Using the Lagrange basis, the representation (2.1) of an interpolant to a function f turns into

$$s(x) = \sum_{j=1}^{n} u_j(x) f(x_j) = u(x)E(f),\ x \in \Omega,$$

which is "entirely in terms of nodes" as required for *meshless methods* [2].

If L is a linear spatial differential operator, and if the kernel K is sufficiently smooth to allow application of L, an advantage of kernel–based spatial discretizations is that

$$(Ls)(x) = \sum_{j=1}^{n} (Lu_j)(x) f(x_j) = Lu(x)E(f),\ x \in \Omega,$$

is explicitly available and again "entirely in terms of nodes". The required derivatives Lu_j of the Lagrange basis functions u_j come from (2.2) via solving

$$(Lu)(x) = (LK)(x)A^{-1}$$

provided that one can explicitly evaluate the action of L on K.

3. Method of Lines. With these notations concerning spatial functions and their derivatives, we now turn to modeling time–dependent functions $v(x,t)$ where the spatial argument x varies in Ω. One can always interpolate values $v(x_k,t)$ of $u(x,t)$ at all times t to get an interpolant

$$s(x,t) = \sum_{j=1}^{n} v(x_j,t)u_j(x) = u(x)E(u(\cdot,t)) \tag{3.1}$$

in terms of the Lagrange basis. This can be seen as a superposition of a separation of variables. The action of a spatial linear operator L then is

$$(Ls)(x,t) = \sum_{j=1}^{n} v(x_j,t)(Lu_j)(x) = (Lu)(x)E(v(\cdot,t)),$$

again separating the time variation from the space variation.

A linear evolution equation

$$u_t(x,t) = (Lu)(x,t),\ x \in \Omega,\ t \geq 0$$

with a starting function g on Ω with

$$u(x,0) = g(x) \text{ for all } x \in \Omega$$

can then be modeled by substituting (3.1) and discretizing the spatial variable to the points $x_1, \ldots, x_n$. This leads to

$$\begin{array}{rcll} s_t(x_k,t) & = & (Ls)(x_k,t) & 1 \leq k \leq n,\ t \geq 0 \\ s(x_j,0) & = & g(x_j), & 1 \leq j \leq n, \end{array}$$

i.e. the whole problem is posed discretely in space, but continuously in time. The seond part is just interpolation of the initial function, while the first takes the form

$$\frac{\partial}{\partial t} v(x_k,t) = \sum_{j=1}^{n} v(x_j,t)(Lu_j)(x_k),\ 1 \leq k \leq n,\ t \geq 0$$

which is a linear system of ordinary differential equations

$$y_k'(t) = \sum_{j=1}^{n} y_j(t)(Lu_j)(x_k),\ 1 \leq k \leq n,\ t \geq 0$$

for unknowns $y_k(t)$ having the meaning $y_k(t) = v(x_k,t)$ for an approximate solution $v(x,t)$ of the problem. The initial values are

$$y_j(0) = g(x_j),\ 1 \leq j \leq n.$$

This is the classical *Method of Lines* in the simplest linear case without additional boundary conditions. It is easy to generalize to nonlinear problems of the form

$$u_t(x,t) = F(t, u(x,t), (Lu)(x,t)),$$

leading to a nonlinear system

$$y_k'(t) = F\left(t, \sum_{j=1}^n y_j(t) u_j(x_k), \sum_{j=1}^n y_j(t)(Lu_j)(x_k)\right)$$

of ODEs, and it is also easy to incorporate multiple spatial differential operators. Additional time–dependent boundary conditions of the form

$$u(z,t) = u_B(z,t) \text{ for all } z \in \Gamma := \partial\Omega,\ t \geq 0$$

can also be handled. One discretizes them to

$$v(x_{n+i},t) = u_B(x_{n+i},t),\ 1 \leq i \leq m$$

for a choice of boundary points $x_{n+1}, \ldots, x_{n+m}$. These are added to the spatial interpolation problem, avoiding coalescence with the points $x_1, \ldots, x_n$. The trial functions are again specified in the form (3.1), but they split into

$$\begin{aligned} s(x,t) &= \sum_{j=1}^n v(x_j,t) u_j(x) + \sum_{i=1}^m v(x_{n+i},t) u_{n+i}(x) \\ &= \sum_{j=1}^n v(x_j,t) u_j(x) + \sum_{i=1}^m u_B(x_{n+i},t) u_{n+i}(x) \end{aligned}$$

being still entirely in terms of values at the nodes. Since we form the Lagrange basis with respect to all points $x_1, \ldots, x_{n+m}$, the first sum vanishes on the boundary points $x_{n+1}, \ldots, x_{n+m}$, while the second attains the correct boundary values there. The resulting ODE system then is the inhomogeneous system

$$y_k'(t) = \sum_{j=1}^n y_j(t)(Lu_j)(x_k) + \sum_{i=1}^m u_B(x_{n+i},t) L(u_{n+i})(x)$$

in the linear case, for $1 \leq k \leq n$ and all $t \geq 0$.

In numerical experiments, this technique was reported to work well (see e.g. [16, 7, 4, 15]), but a thorough mathematical analysis of its behavior is still missing, since one has to fight stability properties [10]. We shall supply a thorough analysis for a simple special case, the heat equation.

4. Method of Lines for Heat Equation. Consider

$$u_t = u_{xx}$$

on $(x,t) \in [0,1] \times [0,\infty)$ under boundary conditions

$$\begin{aligned} u(x,0) &= g(x),\ x \in [0,1], \\ u(0,t) &= g(0) = 0,\ t \geq 0, \\ u(1,t) &= g(1) = 0,\ t \geq 0 \end{aligned}$$

defined by a smooth function g on $[0,1]$ vanishing at both ends. By standard transformations, any heat equation problem with constant boundary values can be brought into this form. More precisely, the u_{xx} part of the heat equation vanishes on affine-linear spatial functions. Thus one can change each problem with constant boundary values by subtracting an affine-linear function into one with zero boundary conditions, and finally transform back by adding the affine-linear function.

For simplicity, we discretize $[0,1]$ by

$$0 = x_0 < x_1 < \ldots < x_{n+1} = 1 \tag{4.1}$$

using equidistant points $x_j = jh,\ 0 \le j \le n+1$ with distance $h = 1/(n+1)$, but any other discretization (4.1) with *fill distance*

$$h := \sup_{x \in [0,1]} \min_{1 \le j \le n} |x - x_j|$$

will do.

Let K be a smooth positive definite symmetric kernel on $\mathbb{R}$ that vanishes in $x_0 = 0$ and $x_{n+1} = 1$, and let $u_1, \ldots, u_n$ be the Lagrange basis for interpolation using translates of K in the points $x_1, \ldots, x_n$. Note that $u_1, \ldots, u_n$ will automatically satisfy the zero boundary conditions at x_0 and x_{n+1}, because the kernel vanishes there.

When starting from a kernel K without zero boundary conditions, one can form the *power kernel* [8] for kernel–based interpolation in 0 and 1 as the new kernel K_2 with

$$\begin{aligned} K_1(x,y) &:= K(x,y) - \frac{K(x,0)K(y,0)}{K(0,0)}, \\ K_2(x,y) &:= K_1(x,y) - \frac{K_1(x,1)K_1(y,1)}{K_1(1,1)}, \end{aligned}$$

which will then vanish at 0 and 1 while being still positive definite.

Another possibility is to use kernels of the form

$$K(x,y) = \sum_{k=1}^{\infty} \mu_k \sin(\pi k x) \sin(\pi k y) \tag{4.2}$$

with suitably decaying positive coefficients μ_k. We shall use this construction [9] in our examples.

We use interpolatory trial functions

$$s(x,t) = \sum_{j=1}^{n} s(x_j,t) u_j(x)$$

vanishing at $x = 0$ nd $x = 1$ and interpolate the starting function g by requiring

$$s(x_k, 0) = g(x_k),\ 1 \le k \le n.$$

The Method of Lines uses functions

$$y(x,t) = \sum_{j=1}^{n} y_j(t) u_j(x)$$

with $y(x_j,t) = y_j(t)$ and poses the linear ODE system

$$y_k'(t) = \sum_{j=1}^{n} y_j(t) u_j''(x_k),\ 1 \le k \le n$$

with starting values

$$y_k(0) = s(x_k,0) = g(x_k),\ 1 \le k \le n.$$

Introducing the matrix-vector notation for values at the points $x_1,\ldots,x_n$ as in the previous section, we get the linear first–order system

$$y'(t) = U'' y(t) \tag{4.3}$$

with the solution

$$y(t) = \exp\left(U''t\right) y(0).$$

The solution satisfies

$$y_t(x_k,t) = y_{xx}(x_k,t),\ 1 \le k \le n$$

by construction, since

$$\begin{aligned} y_t(x_k,t) &= \sum_{j=1}^{n} y_j'(t) u_j(x_k) \\ &= y_k'(t) \\ &= \sum_{j=1}^{n} y_j(t) u_j''(x_k),\ 1 \le k \le n \\ &= y_{xx}(x_k,t),\ 1 \le k \le n. \end{aligned}$$

5. Error Analysis. We introduce the interpolant $v(x,t)$ to the true solution $u(x,t)$ at the points $x_1,\ldots,x_n$ for all t. This is

$$v(x,t) = \sum_{j=1}^{n} u(x_j,t) u_j(x)$$

and we use it to insert the true solution into the ODE system for the Method of Lines. Then

$$\begin{aligned} u_t(x_k,t) &= u_{xx}(x_k,t) \\ &= \sum_{j=1}^{n} u(x_j,t) u_j''(x_k) + u_{xx}(x_k,t) - \sum_{j=1}^{n} u(x_j,t) u_j''(x_k) \\ &= \sum_{j=1}^{n} u(x_j,t) u_j''(x_k) + u_{xx}(x_k,t) - v_{xx}(x_k,t). \end{aligned}$$

Introducing vectors for values at the x_k again, we get

$$u'(t) = U'' u(t) + (u_{xx} - v_{xx})(t)$$

and

$$(u-y)'(t) = U''(u-y)(t) + (u_{xx} - v_{xx})(t).$$

Since the discrete starting values $(u-y)(0)$ are zero, the standard formula for inhomogeneous linear first–order systems yields

$$(u-y)(t) = \int_0^t \exp(U''(t-s))(u_{xx} - v_{xx})(s)ds. \tag{5.1}$$

This is an exact formula for the error at the discrete points.

We shall use a smooth positive definite translation–invariant kernel K on $\mathbb{R}$ of the form (4.2) with coefficients satisfying

$$0 < \mu_k \leq Ck^{-2m} \text{ for all } k \geq 1 \tag{5.2}$$

for some fixed $m > 1/2$. It will be reproducing in a "native" Hilbert space of at least continuous functions which is contained in the Sobolev space $W_2^m[0,1]$ of functions with 1–periodic extensions into $\mathbb{R}$. All functions of this space vanish on 0 and 1. Then we can get

$$\|u(\cdot,t) - v(\cdot,t)\|_{L_\infty[0,1]} \leq Ch^{m-1/2}\|u(\cdot,t)\|_K \tag{5.3}$$

due to [9, Cor. 3.6, p. 78] since we can also assume by standard results on the heat equation that the solution is smooth enough to lie in the spatial native space of the kernel for all times. Similarly,

$$\|u_{xx}(\cdot,t) - v_{xx}(\cdot,t)\|_{L_2[0,1]} \leq Ch^{m-2}\|u(\cdot,t)\|_K \tag{5.4}$$

if we use *sampling inequalities* [18],[11, Thm. 1]. The corresponding result for L_∞ errors of derivatives in 1D is

$$\|u_{xx}(\cdot,t) - v_{xx}(\cdot,t)\|_{L_\infty[0,1]} \leq Ch^{m-2-1/2}\|u(\cdot,t)\|_K. \tag{5.5}$$

All of this follows from standard literature on kernel–based methods, see e.g. [17] for the background of the cited papers.

6. Bounding the Exponential. The remaining problem now is to bound the matrix exponential

$$\exp(U''t)c = \sum_{n=0}^{\infty} \frac{t^n}{n!}(U'')^n c$$

somehow, e.g. via

$$\|\exp(U''t)c\|_2 \leq \sum_{n=0}^{\infty} \frac{t^n}{n!}\rho(U'')^n\|c\|_2$$

where ρ is the spectral radius. A special way to deal with the matrix exponential in a better way than above is to use that the matrix U'' will be negative definite. In fact, if A is the standard kernel matrix for the given points, and A'' is the same, but with second derivatives of a kernel of the form (4.2), we can use that $-A''$ will be positive definite, because the kernel expansion coefficients μ_k are going over to

$k^2\mu_k$ and thus stay positive. But U'' is $A^{-1}A''$, thus negative definite. Therefore the matrix exponential decays for increasing time, and can be bounded by a constant when looking at the integral (5.1). By Cauchy–Schwarz applied to (5.1), we have

$$\begin{aligned}\|u(t)-y(t)\|_2 &\leq \int_0^t \|\exp(U''(x_j-s))(u_{xx}-v_{xx})(s)\|_2\, ds \\ &\leq C\sqrt{t}\left(\int_0^t \|u_{xx}(s)-v_{xx}(s)\|_2^2 ds\right)^{1/2}\end{aligned}$$

and with (5.4) this yields

$$|u(x_j,t)-y(x_j,t)| \leq C\sqrt{t}h^{m-2}\left(\int_0^t |u(\cdot,s)\|_K^2 ds\right)^{1/2}$$

the final error bound on data then is

$$|u(x_j,t)-y(x_j,t)| \leq \sqrt{t}Ch^{m-2}\left(\int_0^t \|u(\cdot,s)\|_K^2 ds\right)^{1/2}. \tag{6.1}$$

THEOREM 6.1. *If the Method of Lines is carried out using a kernel of order m in the sense of (5.2), the error on the discretization points and up to all fixed times will be given by (6.1).*□

Since the true solution is C^∞ and vanishes for increasing t due to its standard series representation based on separation of variables, we know that the second factor in (6.1) is uniformly bounded.

The error outside the data can be bounded by a–posteriori analysis, using the Lagrange basis again. Writing

$$y(x,t)=\sum_{j=1}^n u_j(x)y(x_j,t)$$

and making use of the fact [3] that the Lagrange basis functions are uniformly bounded if the distribution of spatial data points is not too irregular, we have

$$\begin{aligned}|v(x,t)-y(x,t)| &= \left|\sum_{j=1}^n u_j(x)(v(x_j,t)-y(x_j,t))\right| \\ &= \left|\sum_{j=1}^n u_j(x)(u(x_j,t)-y(x_j,t))\right| \\ &\leq Cn\sqrt{t}h^{m-2} \\ &\leq C\sqrt{t}h^{m-3}\end{aligned} \tag{6.2}$$

for the global error between the interpolant v to the true solution and the solution y via the Method of Lines. The error between the true solution u and its interpolant v has the better bound (5.3). This implies

THEOREM 6.2. *If the Method of Lines is carried out using a kernel of order m in the sense of (5.2), the error on the discretization points and up to all fixed times will be of order $\mathcal{O}(\sqrt{t}h^{m-3})$.*□

Note that there is no CFL condition here, since there is no time step. At this point, it is assumed that the ODE system induced by the Method of Lines is solved

exactly, and propagation of roundoff is ignored. Any ODE solver has to cope with the linear system (4.3) somehow, and since we shall see in the next section that U'' has negative eigenvalues of absolute value $\mathcal{O}(h^{-2})$, the ODE system will be hard to integrate with good quality if h is small. All instability issues are shifted back to the ODE system when using the Method of Lines, but there is a CFL condition behind the scene, as we shall see in the next section.

7. Time Stepping Techniques. Using the above spatial discretization, and using the notation $w(x,t)$ for our approximate solution, a variation of a forward Euler time–stepping method would then be

$$\frac{w(x_k,t+\Delta t)-w(x_k,t)}{\Delta t} = w_{xx}(x_k,t) = \sum_{j=1}^{n} w(x_j,t)u_j''(x_k).$$

It still has a time discretization, but the space discretization is hidden in the exact differentiation of the spatial interpolant. In our vector notation, it is

$$E(w(\cdot,t+\Delta t)) = (I+U''\Delta t)E(w(\cdot,t)).$$

Its stability can thus be analyzed via linear algebra, and the algorithm is the same as a forward Euler step for the linear ODE system

$$w'(t) = U''w(t)$$

we encountered before. We shall have to analyze the spectrum of U'' for asserting stability, and this will follow below.

By standard approximation results like (5.5) for kernel–based methods, we get

THEOREM 7.1. *For kernels with orders $m > 5/2$, the forward Euler method (and others discretizing the spatial second derivative in the same way) will be consistent of order $m - 5/2$.*

Note that for stable methods the consistency order will be the convergence order.

To check stability and to get a CFL condition, we need

THEOREM 7.2. *The spectral radius of U'' satisfies*

$$\rho(U'') \le Ch^{-2}$$

if spatial discretization is done with $m > 5/2$.

Proof: If λ is an eigenvalue of U'' with eigenvector c, then

$$\begin{aligned} U''c &= \lambda c, \\ \sum_{j=1}^{n} u_j''(x_k)c_j &= \lambda c_k,\ 1 \le k \le n, \end{aligned}$$

and the function

$$s_\lambda(x) := \sum_{j=1}^{n} c_j u_j(x)$$

satisfies

$$s_\lambda''(x_k) = \lambda s_\lambda(x_k),\ 1 \le k \le n$$

and vanishes on both $x_0 = 0$ and $x_{n+1} = 1$. We invoke the "sampling" inequality

$$\|s''\|_{\infty,[0,1]} \le C\left(h^{m-5/2}\|s\|_m + h^{-2}\|s\|_{\infty,X}\right) \text{ for all } s \in W_2^m[0,1]$$

from [18] for $m > 5/2$ and normalize s to satisfy $\|s\|_m = 1$. Then

$$\begin{array}{rcl} |\lambda||s_\lambda(x_k)| & \le & \|s''\|_{\infty,[0,1]} \\ & \le & Ch^{-2}\|s\|_{\infty,X}. \end{array}$$

Picking k with $|s_\lambda(x_k)| = \|s\|_{\infty,X}$ yields the assertion.□

Thus, for Euler time–stepping in the ODE system (4.3), a spectral radius of order h^{-2} means that there must be a CFL condition of the form

$$\Delta t \le C(\Delta x)^2$$

as is to be expected.

8. Example. Figure 8.1 shows the approximate solution and absolute error for the following parabolic equation

$$\begin{array}{rll} u_t(x,t) = & u_{xx}(x,t), & x \in (0,1), 0 \le t \le 0.2, \\ u(x,0) = & 2\min(x, 1-x), & x \in (0,1), \\ u(0,t) = & 0, \quad u(1,t) = 0, & 0 \le t \le 0.2, \end{array}$$

with the exact solution

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{(2k-1)^2\pi^2} \sin((2k-1)\pi x) \exp -(2k-1)^2\pi^2 t\,.$$

Using the Method of Lines for 150 equidistant spatial points, using the kernel in (4.2).

9. General Parabolic Equations. We consider the problem

$$\begin{array}{rcll} u_t(x,t) & = & \nabla \cdot (a(x,t)\nabla u(x,t)) + f(x,t) + c(x,t)u(x,t) & x \in \Omega \subset \mathbb{R}^d, 0 \le t \le T, \\ u(x,0) & = & g(x), & x \in \Omega, \\ u(y,t) & = & u_B(y,t), & y \in \Gamma := \partial\Omega,\ 0 \le t \le T. \end{array}$$

A meshless discretization in terms of values at nodes can be carried out using $u(x_j,t)$ for $x_j \in \Omega \setminus \Gamma,\ 1 \le j \le N$. Known values are

$$u(y_k,t) = u_B(y_k,t),\ y_K \in \Gamma,\ 1 \le k \le K.$$

We assemble all points into

$$Z = \{x_1, \ldots, x_N, y_1, \ldots, y_k\} = \{z_1, \ldots, z_{N+K}\}$$

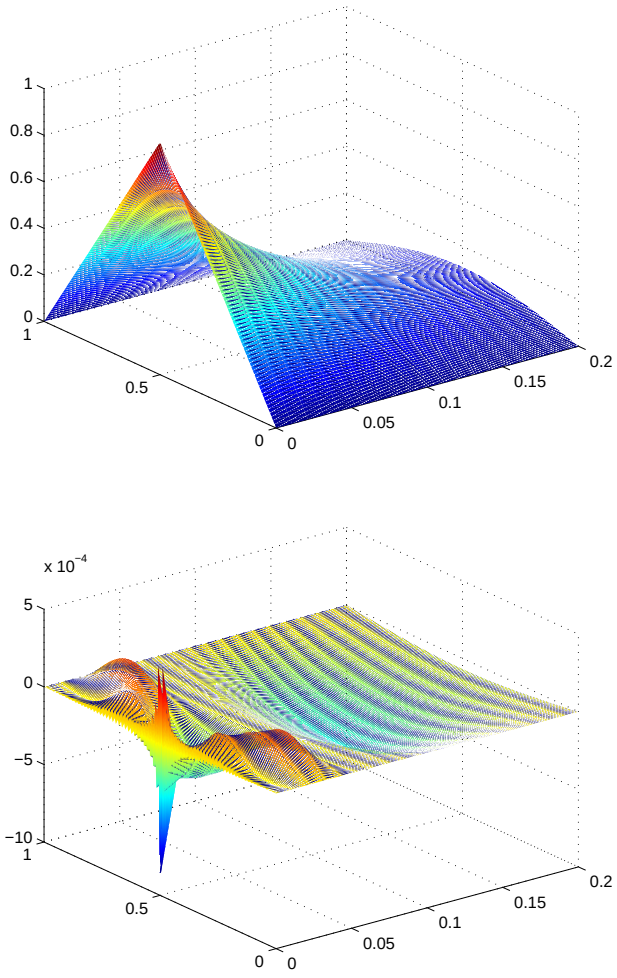

FIG. 8.1. *Approximation and absolute error*

and represent meshless trial functions in Lagrange form by the Lagrange basis $\{v_j\}_{j=1}^{N+K}$ as

$$\begin{aligned}
v(x) &= \sum_{j=1}^{N+K} v(z_j)v_j(x) \\
v_j(z_i) &= \delta_{ji},\ 1 \le j,i \le N+K \\
v(x,t) &= \sum_{j=1}^{N+K} v(z_j,t)v_j(x) \\
\nabla v(x,t) &= \sum_{j=1}^{N+K} v(z_j,t)\nabla v_j(x) \\
a(x,t)\nabla v(x,t) &= \sum_{j=1}^{N+K} v(z_j,t)a(x,t)\nabla v_j(x) \\
\nabla\cdot(a(x,t)\nabla v(x,t)) &= \sum_{j=1}^{N+K} v(z_j,t)\underbrace{\nabla\cdot(a(x,t)\nabla v_j(x))}_{=:w_j(x,t)} \\
&= \sum_{j=1}^{N+K} v(z_j,t)w_j(x,t).
\end{aligned}$$

We now state the PDE on the trial functions:

$$\begin{aligned} v_t(x,t) &= \nabla\cdot(a(x,t)\nabla v(x,t)) \\ &\quad +c(x,t)v(x,t)+f(x,t) \\ \sum_{j=1}^{N+K} v_t(z_j,t)v_j(x) &= \sum_{j=1}^{N+K} v(z_j,t)w_j(x,t) \\ &\quad +c(x,t)\sum_{j=1}^{N+K} v(z_j,t)v_j(x)+f(x,t) \end{aligned}$$

and collocate on points $z_i,\ 1\le i\le N+K$ to get

$$\begin{aligned} \sum_{j=1}^{N+K} v_t(z_j,t)v_j(z_i) &= \sum_{j=1}^{N+K} v(z_j,t)w_j(z_i,t) \\ &\quad +c(z_i,t)\sum_{j=1}^{N+K} v(z_j,t)v_j(z_i)+f(z_i,t) \\ v_t(z_i,t) &= \sum_{j=1}^{N+K} v(z_j,t)w_j(z_i,t) \\ &\quad +c(z_i,t)v(z_i,t)+f(z_i,t). \end{aligned}$$

Due to the known boundary values $v(z_j,t)=u_B(z_j,t)$ with $j=N+1,\cdots,N+K$, in terms of vectors $\mathbf{v}(t):=(v(z_1,t),\ldots,v(z_N))^T$, $\mathbf{v}_B(t):=(v(z_{N+1},t),\ldots,v(z_{N+K}))^T$, and $\mathbf{f}(t):=(f(z_1,t),\ldots,f(z_{N+K}))^T$, we get the system

$$\mathbf{v}'(t)=\mathbf{A}(t)\mathbf{v}(t)+\mathbf{B}(t)\mathbf{v}_B(t)+\mathbf{f}(t)$$

with the matrix $\mathbf{A}(t)$ having the entries

$$w_j(z_i,t)+c(z_i,t)\delta_{ij},\ 1\le i,j\le N,$$

and the matrix $\mathbf{B}(t)$ having the entries

$$w_j(z_i,t),\ N\le i,j\le N+K.$$

This can be solved via the method of Lines, the initial values provided by interpolation of g.

A further generalization to nonlinear problems is

$$u_t(x,t)=F(t,x,u)$$

which similarly leads to

$$v_t(z_i,t)=F\left(t,z_i,\sum_{j=1}^{N+K} v(z_j,t)v_j(x)\right)$$

or

$$\mathbf{v}'(t)=\mathbf{G}(t,\mathbf{v}(t))$$

with suitable nonlinear mappings F and $\mathbf{G}$.

10. Example for a General Parabolic Equation. We consider the one dimensional parabolic equation

$$\begin{aligned} u_t(x,t) &= xu_{xx}(x,t) + (\pi^2 x + 1)e^t \sin \pi x, \quad x \in (0,1), 0 \le t \le 0.8, \\ u(x,0) &= \sin \pi x, \qquad\qquad\qquad\qquad\quad x \in (0,1), \\ u(0,t) &= 0, \quad u(1,t) = 0, \qquad\qquad\qquad\quad 0 \le t \le 0.8, \end{aligned}$$

with the exact solution

$$u(x,t) = e^t \sin \pi x \, .$$

Utilizing the Method of Lines for 100 equidistant spatial points, using the kernel in (4.2), we plot the absolute error between the exact solution and approximation in Figure 10.1.

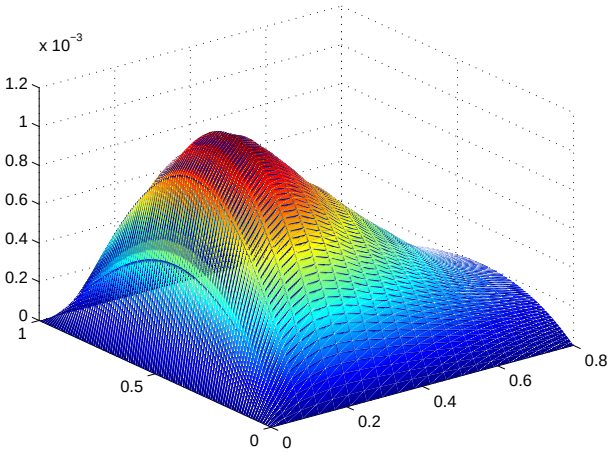

FIG. 10.1. *Absolute error.*

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