Inverse and Saturation Theorems for Radial Basis Function Interpolation

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Abstract. While direct theorems for interpolation with radial basis functions are intensively investigated, little is known about inverse theorems so far. This paper deals with both inverse and saturation theorems. As an inverse theorem we especially show that a function that can be approximated sufficiently fast must belong to the native space of the basis function in use. In case of thin plate spline interpolation we also give certain saturation theorems.

1. Introduction

Direct and inverse theorems play an important role in classical approximation theory. Examples can be found in [2, 8]. The main idea can be described as follows. Suppose the elements of a linear space $(V, \|\cdot\|)$ should be approximated by elements of finite dimension subspaces $V_h \subseteq V$, where h serves as a discretisation index. Denote the approximation process by $S_h : V \to V_h$. Then the *direct theorems* conclude error estimates from additional information on the elements to be approximated: If f is an element of a subspace $\mathcal{G} \subseteq V$ then the error can be bounded by

$$\|f - S_h f\| \le c_f h^{\mu}. \tag{1.1}$$

On the other hand the *inverse theorems* try to conclude information on f from the way f can be approximated: If $f \in V$ satisfies (1.1) then f must belong to a certain subspace $\mathcal{G} \subseteq V$. The situation is optimal if the subspaces and the approximation orders coincide in both the direct and the inverse theorems.

Finally, *saturation theorems* give upper bounds on the possible approximation order: If $f \in \mathcal{G}$ can be approximated by

$$\|f - S_h f\| \le c_f h^{\nu},$$

where ν is a certain number larger than μ, then f must belong to a trivial subspace $\mathcal{N} \subseteq V$.

It is the aim of this paper to give both inverse and saturation theorems in the context of radial basis function interpolation. In case of direct and inverse theorems we shall take the native space $\mathcal{G}_{\Omega,\Phi}$, which we introduce in the third

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section, as the space $\mathcal{G}$ of smoother functions. We will look for the approximation order we can achieve from this fact and for the order we need, to show that a function belongs to the native space.

In case of saturation theorems we restrict ourselves to thin plate spline interpolation and show that functions that can be approximated with a high order are necessarily polyharmonic functions.

2. Radial basis function approximation

The theory of interpolation by radial basis functions has become popular in the last years to reconstruct multivariate functions from scattered data. The starting point of the reconstruction process is the choice of a conditionally positive definite function $\Phi : \mathbb{R}^d \to \mathbb{R}$.

Definition 2.1. A continuous and even function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be conditionally positive definite of order $m \in \mathbb{N}_0$, iff for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X = \{x_1, \ldots, x_N\} \subseteq \mathbb{R}^d$, and all $\alpha \in \mathbb{R}^N \setminus \{0\}$ satisfying

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0 \qquad \text{for all } p \in \mathcal{P}_m^d$$

the quadratic form

$$\sum_{j,k=1}^{N} \alpha_j \alpha_k \Phi(x_j - x_k)$$

is positive. Here, $\mathcal{P}_M^d$ denotes the set of all d-variate polynomials with a total degree less then m. We will denote the set of all conditionally positive definite functions of order m by $\texttt{cpd}(m)$.

Having a $\Phi \in \texttt{cpd}(m)$ the interpolant $s_{f,X}$ to a function f in $X = \{x_1, \ldots, x_N\}$ is given by

$$s_{f,X}(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - x_j) + \sum_{j=1}^{Q} \beta_j p_j(x) \tag{2.2}$$

where $p_1, \ldots, p_Q$ form a basis of $\mathcal{P}_m^d$. To cope with the additional degrees of freedom, the interpolation conditions

$$s_{f,X}(x_j) = f(x_j), \qquad 1 \le j \le N, \tag{2.3}$$

are completed by the further conditions

$$\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0, \qquad 1 \le k \le Q. \tag{2.4}$$

We summarize some standard statements on the interpolation process in

Lemma 2.2. *The interpolating function $s_{f,X}$ is well defined and unique, if X contains a $\mathcal{P}_m^d$-unisolvent subset. In this case the operator that maps f to $s_{f,X}$ is linear and reproduces polynomials up to degree m.*

In this paper we only want to deal with conditionally positive definite functions of order m that possess a generalized Fourier transform that coincides with a continuous function $\widehat{\Phi}$ on $\mathbb{R}^d \setminus \{0\}$. We are mainly interested in cases, where this Fourier transform decays only algebraically, i. e. there exist constants $0 < c_1 \leq c_2$ with

$$c_1 \|\omega\|_2^{-d-s_\infty} \leq \widehat{\Phi}(\omega) \leq c_2 \|\omega\|_2^{-d-s_\infty} \tag{2.5}$$

for $\|\omega\| \to \infty$. The upper bound is important for the direct theorems, while the lower bound is necessary for the inverse theorems. This decay condition is, for instance, covered by the thin plate splines and the compactly supported radial basis functions of minimal degree (cf. [13]). But we shall state the inverse theorems also in case of exponentially decaying Fourier transforms which covers Gaussian and (inverse) multiquadrics.

3. Direct theorems

There are several papers dealing with direct theorems, but only few have tried to establish inverse theorems. We will briefly repeat direct theorems as far as we need them for our further analysis.

To state error estimates two preparing steps have to be done. On the one hand the function space has to be introduced for which the error bounds shall apply. On the other hand a measure of the data density has to be given. We start with the function space by introducing the native space.

Let $\Omega \subseteq \mathbb{R}^d$ be given. Let us denote by

$$(\mathcal{P}_m^d)_\Omega^\perp = \{\lambda_{\alpha,X} = \sum_{j=1}^M \alpha_j \delta_{x_j} : M \in \mathbb{N}, \alpha_j \in \mathbb{R}, x_j \in \Omega, \lambda_{\alpha,X}|\mathcal{P}_m^d \equiv 0\}$$

the set of all point evaluation functionals of finite support in Ω vanishing on $\mathcal{P}_m^d$. Every conditionally positive definite function Φ of order m allows us to equip $(\mathcal{P}_m^d)_\Omega^\perp$ with an inner product

$$(\lambda, \mu)_\Phi = \lambda^x \mu^y \Phi(x-y)$$

where λ^x means the action of λ with respect to the variable x.

Then we can follow [6, 7] to introduce the function space

$$\mathcal{G}_{\Omega,\Phi} = \{f \in C(\Omega) : |\lambda(f)| \leq c_f \|\lambda\|_\Phi \text{ for all } \lambda \in (\mathcal{P}_m^d)_\Omega^\perp\}.$$

We denote the smallest constant c_f in the definition of $G_{\Omega,\Phi}$ by $\|f\|_\Phi$, i. e.

$$\|f\|_\Phi := \max_{\lambda \in (\mathcal{P}_m^d)_\Omega^\perp \setminus \{0\}} \frac{|\lambda(f)|}{\|\lambda\|_\Phi}.$$

Then $\|\cdot\|_\Phi$ is a semi-norm on $\mathcal{G}_{\Omega,\Phi}$ with null space $\mathcal{P}_m^d$. Thus

$$\mathcal{F}_{\Omega,\Phi} := \mathcal{G}_{\Omega,\Phi}/\mathcal{P}_m^d$$

is a normed linear space which turns out to be complete. There are several other possiblities to introduce the native space (cf. [10, 11, 14]) but the chosen approach serves our purposes best.

Not only the space $\mathcal{P}_m^d$ is a subspace of $\mathcal{G}_{\Omega,\Phi}$, but also all interpolating functions (2.2) are contained.

Lemma 3.1. *The map*

$$\begin{aligned} F : (\mathcal{P}_m^d)_\Omega^\perp &\to F((\mathcal{P}_m^d)_\Omega^\perp) \subseteq G_{\Omega,\Phi} \\ \lambda_{\alpha,X} &\mapsto \lambda_{\alpha,X}^y \Phi(\cdot - y) \end{aligned}$$

is well defined and bijective. Furthermore, we have the relations

$$\|\lambda_{\alpha,X}\|_\Phi = \|F(\lambda_{\alpha,X})\|_\Phi$$

and

$$\lambda_{\alpha,X}(F(\lambda_{\beta,Y})) = (\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi = \lambda_{\beta,Y}(F(\lambda_{\alpha,X})).$$

The proof is straightforward and will be omitted.

The first step in bounding the interpolation error is to define the power function as the norm of the pointwise error functional

$$P_{X,\Phi}(x) = \sup_{f\in\mathcal{G}_{\Omega,\Phi}\setminus\mathcal{P}_m^d} \frac{|f(x) - s_{f,X}(x)|}{\|f\|_\Phi}.$$

which leads immediately to

$$|f(x) - s_{f,X}(x)| \le P_{X,\Phi}(x)\|f\|_\Phi.$$

Then the power function has to be bounded in terms of the fill distance, defined by

$$h_{X,\Omega} \equiv h_X := \sup_{x\in\Omega} \min_{x_j\in X} \|x - x_j\|_2$$

which was done in [14], for instance.

Theorem 3.2. *Let $\Phi \in$ $\mathtt{cpd}(m)$ satisfy (2.5). Let Ω be a bounded and open domain satisfying an interior cone condition. Then there exist constants h_0, C, such that for all sets of centers X with $h_X \le h_0$ and all $x \in \Omega$ the power function can be bounded by*

$$P_{X,\Phi}(x) \le C h_X^{s_\infty/2} \tag{3.6}$$

yielding the error bound

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \le C h^{s_\infty/2}\|f\|_\Phi \tag{3.7}$$

for $f \in \mathcal{G}_{\Omega,\Phi}$.

Actually, in [14] the theorem is stated in a more localized version, but the proof holds true in this situation. There are several other papers giving error bounds of this form, some of them are [1, 3, 5, 12].

Next, we need a stability result on the interpolation process. Therefore, we define the separation distance

$$q_X := \frac{1}{2}\min_{j\neq k}\|x_j - x_k\|_2$$

and cite from [9]

Theorem 3.3. *Let $\Phi \in \mathtt{cpd}(m)$ satisfy the decay condition (2.5). For $X = \{x_1,\dots,x_N\} \subseteq \Omega$ denote by $A_{X,\Phi}$ the matrix*

$$A_{X,\Phi} = (\Phi(x_j - x_k))_{1\le j,k\le N}$$

and by γ_X the smallest non vanishing eigenvalue of $A_{X,\Phi}$. Then the following holds true:

1) *(Stability) For all $\alpha \in \mathbb{R}^N$ satisfying (2.4) we have*

$$\alpha^T A_{X,\Phi}\alpha \ge \gamma_X\|\alpha\|_2 \ge c q_X^{s_\infty}$$

and therefore

$$\|(A_{X,\Phi}|V_X)^{-1}\|_{2,2} \le c_\Phi q_X^{-s_\infty}$$

with $V_X := \{\alpha : \lambda_{\alpha,X} \in (\mathcal{P}_m^d)_\Omega^\perp\}$ and a constant c_Φ depending only on Φ.

2) *(Uncertainty Relation) For all $x \in \Omega\setminus X$ we have*

$$P^2_{X,\Phi}(x) \ge \gamma_{X\cup\{x\}}. \tag{3.8}$$

4. Inverse theorems concerning Φ

Principally, there are two possibilities to state inverse theorems for interpolation by radial basis functions. The first one is based only on the basis function Φ and draws conclusions on the basis function from the fact that the power function can be bounded like (3.6). This will be done in this section. The second is to draw conclusions on f from estimates like (3.7) which will be subject of the sixth section.

Theorem 4.1. *Let $\Phi \in \mathtt{cpd}(m)$ satisfy (2.5). Let Ω be bounded and open, satisfying an interior cone condition. If there exist constants $\beta, c > 0$ such that the power function $P_{X,\Phi}$ can bounded by*

$$\|P_{X,\Phi}\|_{L_\infty(\Omega)} \le c h_X^{\beta/2} \tag{4.9}$$

for all sets $X \subseteq \Omega$ with sufficiently small h_X, then

$$s_\infty \ge \beta$$

must be satisfied.

Proof. On account of the conditions on Ω there exists a $\delta > 0$ and quasi-uniform sets $X = \{x_1, \dots, x_N\} \subseteq \Omega$ with respect to this $\delta > 0$. Here, we call a set of pairwise distinct centers $X = \{x_1, \dots, x_N\} \subseteq \Omega$ *quasi-uniform* with respect to $\delta > 0$, iff

1) $X \setminus \{x_j\}$ is $\mathcal{P}_m^d$-regular for $1 \le j \le N$,
2) $q_X \ge \delta h_X$.

Then we have (cf. [9]) $h_{X\setminus\{x_j\}} \le 2h_X$ for h_X sufficiently small. Therefore we can use (2.5), (4.9) and the Uncertainty Relation (3.8) to derive

$$c2^\beta h_X^\beta \ge c h_{X\setminus\{x_j\}}^\beta \ge P_{X\setminus\{x_j\}}^2(x_j) \ge \gamma_X \ge c_\Phi q_X^{s_\infty} \ge c_\Phi \delta^{s_\infty} h_X^{s_\infty}.$$

Choosing a sequence of such X with $h_X \to 0$, this leads to $\beta \le s_\infty$.

Theorem 4.1 shows that the decay (4.9) of the power function determines the decay of the generalized Fourier transform of the basis function and therefore the smoothness of the basis function itself. It also shows that there is no possibility to improve error estimates of the form (3.7) based on upper bounds of the power function.

5. Characterisation of the native space

Our next result characterises the functions f from the native space $\mathcal{G}_{\Omega,\Phi}$ by uniform boundedness of their interpolating functions with respect to the semi-norm of the native space.

Theorem 5.1. *Denote by $s_{f,X}$ the interpolant (2.2) to a function $f \in C(\Omega)$ on X using a basis function $\Phi \in \mathtt{cpd}(m)$. Then f belongs to the native space $\mathcal{G}_{\Omega,\Phi}$ if and only if there exists a constant c_f such that $\|s_{f,X}\|_\Phi \le c_f$ for all $X \subseteq \Omega$.*

Proof. Assume $f \in \mathcal{G}_{\Omega,\Phi}$. Then $s_{f,X}$ is the best approximation to f from $\operatorname{span}\{\Phi(\cdot - x) : x \in X\} + \mathcal{P}_m^d$. with respect to the $\|\cdot\|_\Phi$-semi-norm. Thus we have

$$\|f - s_{f,X}\|_\Phi^2 + \|s_{f,X}\|_\Phi^2 = \|f\|_\Phi^2,$$

which gives the bound $\|s_{f,X}\|_\Phi \le \|f\|_\Phi$ at once.

Now, let us assume $\|s_{f,X}\|_\Phi \le c_f$ for all $X \subseteq \Omega$. For an arbitrary

$$\lambda_{\alpha,X} := \sum_{j=1}^N \alpha_j \delta_{x_j} \in (\mathcal{P}_m^d)_\Omega^\perp$$

we choose $s_{f,X}$ to be the interpolant on X to f satisfying the interpolation conditions (2.3) and (2.4). Then $s_{f,X}$ belongs to $\mathcal{G}_{\Omega,\Phi}$ and we have

$$\lambda_{\alpha,X}(f - s_{f,X}) = 0.$$

Thus we can estimate

$$\begin{aligned} |\lambda_{\alpha,X}(f)| &\le |\lambda_{\alpha,X}(f - s_{f,X})| + |\lambda_{\alpha,X}(s_{f,X})| \\ &\le \|\lambda_{\alpha,X}\|_\Phi \, \|s_{f,X}\|_\Phi \\ &\le c_f \|\lambda_{\alpha,X}\|_\Phi \end{aligned}$$

As this holds for all $\lambda_{\alpha,X}$ we have $f \in \mathcal{G}_{\Omega,\Phi}$.

6. Inverse theorems concerning f

Now we draw conclusions about a function from L_∞-convergence orders of its interpolants. To be more precise, we show that a function $f \in C(\Omega)$ which can be approximated sufficently fast by radial basis function interpolants in the L_∞-norm must belong to the native space of the basis function.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded and open domain satisfying an interior cone condition. The basis function $\Phi \in$ `cpd`(m) should satisfy the decay condition (2.5). Suppose further that for some $f \in C(\Omega)$ there exist constants $\mu > 0$ and $c_f > 0$ such that $\|f - s_{f,X}\|_{L_\infty(\Omega)} \le c_f h_X^\mu$ for all $X \subseteq \Omega$ with h_X sufficiently small. If $2\mu > s_\infty + d$, then f must belong to the native space $\mathcal{G}_{\Omega,\Phi}$.*

Proof. All sets of centers X that may appear in this proof shall be quasi-uniform with respect to a fixed $\delta > 0$.

Every interpolant $s_{f,X}$ defines a linear functional from $(\mathcal{P}_m^d)_\Omega^\perp$ which we shall denote by $\lambda_{\alpha,X}$. From Theorem 3.3 and Lemma 3.1 we have

$$\begin{aligned} \|s_{f,X}\|_\Phi^2 &= \|\lambda_{\alpha,X}\|_\Phi^2 \\ &= \alpha^T A_{X,\Phi} \alpha \\ &= \alpha^T A_{X,\Phi} (A_{X,\Phi}|V_X)^{-1} (A_{X,\Phi}|V_X) \alpha \\ &\le \|(A_{X,\Phi}|V_X)^{-1}\|_{2,2} \, \|A_X \alpha\|_{L_2(X)}^2 \\ &= \|(A_{X,\Phi}|V_X)^{-1}\|_{2,2} \, \|s_{f,X} - p_X\|_{L_2(X)}^2 \end{aligned} \tag{6.10}$$

Here, V_X denotes again the space $\{\beta \in \mathbb{R}^{|X|} : \lambda_{\beta,X} \in (\mathcal{P}_m^d)_\Omega^\perp\}$, and p_X denotes the polynomial in the definition of $s_{f,X}$.

If we have two sets of centers $X \subseteq Y$ and compare the two interpolating functions $s_{f,Y}$ and $s_{f,X}$, we can interpret $s_{f,X}$ as the interpolant to $s_{f,Y}$ and use the polynomial reproduction property of Lemma 2.2 to get

$$s_{f,X} - s_{f,Y} = \sum_{j=1}^{|X|} \alpha_j^X \Phi(\cdot - x_j) - \sum_{j=1}^{|Y|} \alpha_j^Y \Phi(\cdot - y_j).$$

On the other hand the difference can be interpreted as the interpolating function on Y to $s_{f,Y} - s_{f,X}$. This leads us to

$$\begin{aligned}\|s_{f,Y} - s_{f,X}\|_\Phi^2 &\le \|(A_{Y,\Phi}|V_Y)^{-1}\|_{2,2}\ \|s_{f,Y} - s_{f,X}\|^2_{L_2(Y)}\\ &\le \frac{1}{\gamma_Y}\sum_{y\in Y}|f(y) - s_{f,X}(y)|^2\\ &\le c_\Phi^{-1} q_Y^{-s_\infty}|Y|c_f^2 h_X^{2\mu}.\end{aligned}$$

In what follows c will denote a generic constant. Now we consider a special family of quasi-uniform sets of centers. We assume X_n to satisfy $|X_n| \le c2^{nd}$ and

$$c_1 2^{-n} \le q_{X_n} \le h_{X_n} \le c_2 2^{-n}.$$

Such a choice is always possible because of the assumption made on Ω. If we take $X = X_k \subseteq Y = X_n$ with $n \ge k$ we get

$$\begin{aligned}\|s_{f,X_n} - s_{f,X_k}\|_\Phi^2 &\le c2^{s_\infty n + dn - 2\mu k}\\ &= c2^{(d+s_\infty)n - 2\mu k}\\ &= c2^{2\mu(n-k) - 2\sigma n},\end{aligned}$$

where $\sigma > 0$ is defined by $d + s_\infty + 2\sigma = 2\mu$. Thus we can estimate the Φ-norm of two succeeding interpolants by

$$\|s_{f,X_{k+1}} - s_{f,X_k}\|_\Phi \le c2^{-k\sigma}.$$

A telescoping sum argument leads to

$$\begin{aligned}\|s_{f,X_K}\|_\Phi &\le \sum_{k=0}^{K}\|s_{f,X_{k+1}} - s_{f,X_k}\|_\Phi + \|s_{f,X_0}\|_\Phi\\ &\le c\sum_{j=0}^{\infty} 2^{-\sigma k} + \|s_{f,X_0}\|_\Phi.\\ &\le \frac{c}{1 - 2^{-\sigma}} + \|s_{f,X_0}\|_\Phi.\end{aligned}$$

Thus, the sequence $\|s_{f,X_k}\|_\Phi$ is bounded. But for $n \ge k$ the interpolant s_{f,X_k} is also the interpolant to s_{f,X_n} and therefore a best approximant to s_{f,X_n} from $S(X_k) := \mathrm{span}\{\Phi(\cdot - x) : x \in X_k\} + \mathcal{P}_m^d$. This leads to

$$\|s_{f,X_n} - s_{f,X_k}\|_\Phi^2 + \|s_{f,X_k}\|_\Phi^2 = \|s_{f,X_n}\|_\Phi^2 \tag{6.11}$$

which shows that the sequence $\|s_{f,X_k}\|_\Phi$ is also increasing and therefore convergent. Furthermore, (6.11) implies that s_{f,X_n} is a Cauchy sequence in $\mathcal{G}_{\Omega,\Phi}$ with a limit $\tilde{s}$, which is uniquely determined up to a polynomial of degree less than m.

Finally, we have to show that f coincides on Ω with $\tilde{s}$ modulo $\mathcal{P}_m^d$. Let us choose a fixed $\mathcal{P}_m^d$-unisolvent set $\Xi = \{\xi_1, \ldots, \xi_Q\} \subseteq \Omega$ and denote with $u_j(x) \in \mathcal{P}_m^d$, $1 \le j \le Q$ a Lagrange basis with respect to Ξ. Then the functional

$$\delta^{(x)} := \delta_x - \sum_{j=1}^{Q} u_j(x)\delta_{\xi_j}$$

lies in $(\mathcal{P}_m^d)_\Omega^\perp$ for every $x \in \Omega$. Thus we have

$$\begin{aligned}
|\delta^{(x)}(f - \tilde{s})| &\le |\delta^{(x)}(f - s_{f,X_n})| + |\delta^{(x)}(s_{f,X_n} - \tilde{s})| \\
&\le \|\delta^{(x)}\|_\Phi \|\tilde{s} - s_{f,X_n}\|_\Phi + |(f - s_{f,X_n})(x) + \sum_{j=1}^{Q} u_j(x)(f - s_{f,X_n})(\xi_j)| \\
&\le \|\delta^{(x)}\|_\Phi \ \|\tilde{s} - s_{f,X_n}\|_\Phi + ch_{X_n}^\mu .
\end{aligned}$$

Thus we can derive

$$\delta^{(x)}(f) = \delta^{(x)}(\tilde{s})$$

for all $x \in \Omega$ or in other words

$$f(x) = \tilde{s}(x) + \sum_{j=1}^{N} u_j(x)(\tilde{s} - f)(\xi_j).$$

This implies

$$\lambda_{\alpha,X}(f) = \lambda_{\alpha,X}(\tilde{s}) \le c_{\tilde{s}} \ \|\lambda_{\alpha,X}\|_\Phi$$

for general $\lambda_{\alpha,X} \in (\mathcal{P}_m^d)_\Omega^\perp$, which completes the proof.

Note that there is a gap of $d/2$ between the necessary and sufficient approximation order for functions in the native space $\mathcal{G}_{\Omega,\Phi}$. A closer look shows that the direct theorem 3.2 implies for $f \in \mathcal{G}_{\Omega,\Phi}$:

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \le C_\Phi h^{s_\infty/2} \|f - s_{f,X}\|_\Phi. \tag{6.12}$$

The $h^{s_\infty/2}$ term comes from the estimate on the power function and is optimal in the sense of Theorem 4.1. On the other hand

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \le C_f h^{s_\infty/2} \ h^{d/2+\varepsilon}$$

is so far necessary for showing $f \in \mathcal{G}_{\Omega,\Phi}$ via Theorem 6.1. Thus the gap could be closed either by showing

$$\|f - s_{f,X}\|_\Phi \le C_f h^{d/2+\varepsilon},$$

or by improving our inverse theorem.

Before we come to inverse theorems for Gaussian and multiquadrics, let us remark that in case of an unconditionally positive definite function Φ and a quasi-uniform set X equation (6.10) can be rewritten as

$$\|s_{f,X}\|_\Phi \le c h_X^{-(s_\infty+d)/2} \|s_{f,X}\|_{L_\infty(\Omega)},$$

which can be seen as a kind of *Bernstein inequality.*

Now, let us assume for the rest of the section that the Fourier transform satisfies

$$\widehat{\Phi}(\omega) \ge c e^{-\tilde{c}_1 \|\omega\|_2^\alpha}. \tag{6.13}$$

This leads to estimates of the form

$$\|(A_{X,\Phi}|V_X)^{-1}\|_{2,2} \le c e^{c_1 h_X^{-\alpha}} \tag{6.14}$$

where we used the subspace V_X again and where c always denotes a generic constant. In case of multiquadrics and Gaussians the constants α, $\tilde{c}_1$ and c_1 can be found in [9].

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded and open domain satisfying an interior cone condition. The basis function $\Phi \in \mathtt{cpd}(m)$ should satisfy the decay condition (6.13). Suppose further that for some $f \in C(\Omega)$ there exist constants $c_2 > c_1$ and $c_f > 0$ such that $\|f - s_{f,X}\|_{L_\infty(\Omega)} \le c_f e^{-c_2 h_X^\alpha}$ for all $X \subseteq \Omega$ with h_X sufficiently small. Then f must belong to the native space $\mathcal{G}_{\Omega,\Phi}$.*

Proof. The proof of Theorem 6.1 applies completely if we show that the native space norm of the difference of two interpolants can be bounded in such a way that the telescoping sum argument still works. But this is the case: for $X \subseteq Y$ we can derive

$$\begin{aligned}\|s_{f,X} - s_{f,Y}\|_\Phi &\le c e^{c_1 h_X^{-\alpha}} \sum_{y\in Y} |f(y) - s_{f,X}(y)|^2 \\ &\le c e^{-c_3 h_x^{-\alpha}} |Y|\end{aligned}$$

with $c_3 := c_2 - c_1 > 0$. Taking the same sequence of sets of centers X_n as in Theorem 6.1 we see that the cardinality of Y is only polynomial in h_Y, which means that we can bound two succeeding interpolants by

$$\|s_{f,X_{k+1}} - s_{f,X_k}\|_\Phi \le c e^{-\tilde{c}_3 2^{(n\alpha)/2}}.$$

This ensures the convergence of the telescoping sum.

7. Saturation for thin plate spline interpolation

In this section we concentrate on interpolation by thin plate or polyharmonic splines. To be more precise we consider the functions $\Phi_{d,\ell} = \phi_{d,\ell}(\|\cdot\|_2)$ with

$$\phi_{d,\ell}(r) = \begin{cases} c_{d,\ell}\, r^{2\ell-d} & \text{for odd } d \\ e_{d,\ell}\, r^{2\ell-d}\log r & \text{for even } d \end{cases} \tag{7.15}$$

with $d \le 2\ell$ where the constants

$$c_{d,\ell} = \frac{(-1)^{\ell}\Gamma\left(\frac{d}{2}-\ell\right)}{2^{2\ell}\pi^{d/2}\Gamma(\ell)},$$
$$e_{d,\ell} = \frac{(-1)^{(d-2)/2}}{2^{2\ell-1}\pi^{d/2}(\ell-1)!\left(\ell-\frac{d}{2}\right)!}.$$

are determined by the fact that these functions should be the fundamental solutions of the iterated Laplacian (see Lemma 7.2).

The functions $\Phi_{d,\ell}$ are conditionally positive definite of order m with $m = \ell - \lfloor d/2 \rfloor + 1$ on $\mathbb{R}^d$ and possess a generalized Fourier transform $\widehat{\Phi}_{d,\ell}$ which is $\|\cdot\|_2^{-2\ell}$ up to a constant factor. Thus interpolants come from the space

$$S(X) = \operatorname{span}\{\Phi_{d,\ell}(\cdot - x) : x \in X\} + \mathcal{P}_m^d$$

and Theorem 3.2 leads to the error bound

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \le c h_X^{\ell-\frac{d}{2}} \|f\|_{\Phi_{d,\ell}}. \tag{7.16}$$

For a restricted set of functions f, an improvement in [10] yields

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} \le c_f h_X^{2\ell-d}. \tag{7.17}$$

In [1] the following improved error estimate is given:

Theorem 7.1. *Suppose Ω is a cube in $\mathbb{R}^d$ and the set of centers X_h are given by the grid points $h\mathbb{Z}^d \cap \Omega$. If $f \in \operatorname{Lip}(2\ell+1, \Omega)$, then the error can be bounded by*

$$\|f - s_{f,X_h}\|_{L_\infty(K)} \le c_f h^{2\ell} \tag{7.18}$$

for every compact subset K of the interior of Ω as $h \to 0$.

See [1] for the exact definition of the space $\operatorname{Lip}(2\ell+1, \Omega)$. Note that this estimate is based on three additional assumptions:

- The function f is supposed to be smoother than $f \in \mathcal{G}_{\Phi,\Omega}$. This is a natural assumption.
- The domain has to be a cube and the centers have to form a grid. This is a consequence of the proof given in [1]. A generalization to arbitrary centers would be useful.

– The estimates are restricted to compact subsets of the interior of the cube. This means that boundary effects are neglected.

Nonetheless, the result gives a hint on the possible local convergence order and we shall show that this order is also the saturation order. But before we can do that, we need two auxiliary results:

Lemma 7.2. *For every test function $\gamma \in C_0^\infty(\mathbb{R}^d)$ and every $y \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} \Phi_{d,\ell}(x-y)\Delta^\ell\gamma(x)dx = \gamma(y).$$

A proof can be found in [4].

Lemma 7.3. *Suppose $X = \{x_1, \dots, x_N\} \subseteq \Omega$ is given. Suppose further that $\gamma \in C_0^\infty(\Omega)$ satisfies $X \cap$ supp $\gamma = \emptyset$. Then for every $s \in S(X)$:*

$$(\Delta^\ell s, \gamma)_{L_2(\Omega)} = 0.$$

Proof. Choose an arbitrary $s(x) = \sum_{j=1}^N \alpha_j \Phi_{d,\ell}(x-x_j) + p(x) \in S(X)$. As $\Delta^\ell \mathcal{P}_m^d \equiv 0$ we can use Lemma 7.2 to obtain

$$\begin{aligned}(\Delta^\ell s, \gamma)_{L_2(\Omega)} &= \sum_{j=1}^N \alpha_j \int_{\mathbb{R}^d} \Delta^\ell\gamma(x)\Phi_{d,\ell}(x-x_j)dx \\ &= \sum_{j=1}^N \alpha_j \gamma(x_j) \\ &= 0.\end{aligned}$$

Now we can give our saturation result.

Theorem 7.4. *Let $\phi_{d,\ell}$ be any of the thin plate splines defined in (7.15). Suppose $\Omega \subseteq \mathbb{R}^d$ to be open and bounded, satisfying an interior cone condition. Suppose that for $f \in C^{2\ell}(\Omega)$ the interpolating functions $s_{f,X}$ on X satisfy*

$$\|f - s_{f,X}\|_{L_\infty(K)} = o(h_X^{2\ell}) \qquad \text{as } h_X \to 0$$

for every compact subset K of Ω. Then f satisfies

$$\Delta^\ell f = 0 \qquad \text{on } \Omega.$$

Proof. Fix $x_0 \in \Omega$. Choose $X \subseteq \Omega$ to be quasi-uniform with respect to a fixed $\delta > 0$, such that $\min_{x\in X}\|x - x_0\|_2 = c_0 h_X$ with c_0 independent of h_X. Choose a test function $\gamma_0 \in C_0^\infty(\mathbb{R}^d)$ with supp$\gamma_0 = B_1(0) = \{x \in \mathbb{R}^d : \|x\|_2 \le 1\}$ and $\int \gamma_0(x)dx = 1$. Set $\tilde{h} = c_0 h_X/2$ and $\gamma_h := \tilde{h}^{-d}\gamma_0((\cdot - x_0)/\tilde{h})$. Then the support of γ_h is given by $B_{\tilde{h}}(x_0)$ and satisfies $B_{\tilde{h}}(x_0) \cap X = \emptyset$. Thus we can use Lemma 7.3 to get

$$\begin{aligned}(\Delta^\ell f, \gamma_h)_{L_2(\Omega)} &= (\Delta^\ell (f-s), \gamma_h)_{L_2(\Omega)} \\ &= (f-s, \Delta^\ell \gamma_h)_{L_2(\Omega)} \\ &\le \|f-s\|_{L_2(B_{\tilde{h}}(x_0))} \, \|\Delta^\ell \gamma_h\|_{L_2(B_{\tilde{h}}(x_0))} \\ &\le c\, \tilde{h}^{\frac{d}{2}} \|f-s\|_{L_\infty(B_{\tilde{h}}(x_0))} \, \|\Delta^\ell \gamma_h\|_{L_2(B_{\tilde{h}}(x_0))}\end{aligned}$$

with $s = s_{f,X}$. And because of

$$\begin{aligned}\|\Delta^\ell \gamma_h\|^2_{L_2(B_{\tilde{h}}(x_0))} &= \tilde{h}^{-2d} \int_{\mathbb{R}^d} |\Delta^\ell \gamma_h(x)|^2 dx \\ &= \tilde{h}^{-d-4\ell} \int_{\mathbb{R}^d} |\gamma_0(x)|^2 dx \\ &=: \tilde{h}^{-d-4\ell} c_0^2\end{aligned}$$

we can conclude

$$(\Delta^\ell f, \gamma_h)_{L_2(\Omega)} \le c\tilde{h}^{-2\ell} \|f-s\|_{L_\infty(B_{\tilde{h}}(x_0))}.$$

On account of the assumptions this leads to

$$\lim_{h\to 0+} (\Delta^\ell f, \gamma_h)_{L_2(\Omega)} = 0.$$

On the other hand we have

$$\lim_{h\to 0+} (\Delta^\ell f, \gamma_h)_{L_2(\Omega)} = \lim_{h\to 0+} \int_{\mathbb{R}^d} (\Delta^\ell f)(x_0 + hx)\gamma_0(x) dx = \Delta^\ell f(x_0)$$

which proves $\Delta^\ell f(x_0) = 0$.

Note that our proof also applies to the situation of classical splines. As in the latter case, functions in the saturation class are already determined by their values on the boundary of the domain:

Corollary 7.5. *Suppose in addition to the assumptions of the last theorem that Ω has a C^ℓ boundary $\partial\Omega$. Then f is already determined by the values of $\partial^j f/\partial^j \nu$, $0 \le j \le \ell - 1$, on the boundary $\partial\Omega$. Here ν denotes the outer unit normal vector.*

This sheds some light on the influence of boundary conditions on the possibilities to improve the approximation order $\ell - d/2$ of (7.16) towards 2ℓ.

Finally, we want to draw the reader's attention to the $d/2$-gap arising not only in the discussion around (6.12), but also in (7.17) when compared to (7.18). If (7.16) could be improved by $h^{d/2}$, then (7.17) would by [10] improve to $h^{2\ell}$ and coincide with (7.18). We consider closing the $d/2$-gap to be a challenging research task.

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