The Multiwave Range Test

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Applied Mathematics
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Outline

1. Introduction
2. Multiwave Range Test
3. Summary and Outlook

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1. Introduction
   - Acoustic Scattering
   - Overview of the Range Test
   - Multi-Modality
   - Adaptive Sensing
2. Multiwave Range Test
3. Summary and Outlook

Acoustic Scattering Problem

- Consider a bounded domain D of class C^2.
- Let $u^i \in D$ be the incident field that solves the *Helmholtz equation* in $\mathbb{R}^m$

$$\Delta u^i + \kappa^2 u^i = 0$$

  with *wave number* $\kappa > 0$.
- scattered Field u^s solves the Helmholtz equation and satisfies the *Sommerfeld radiation condition* such that the total field $u = u^i + u^s$ satisfies a boundary condition (Dirichlet, Neumann, Impedance).

Acoustic Scattering Problem

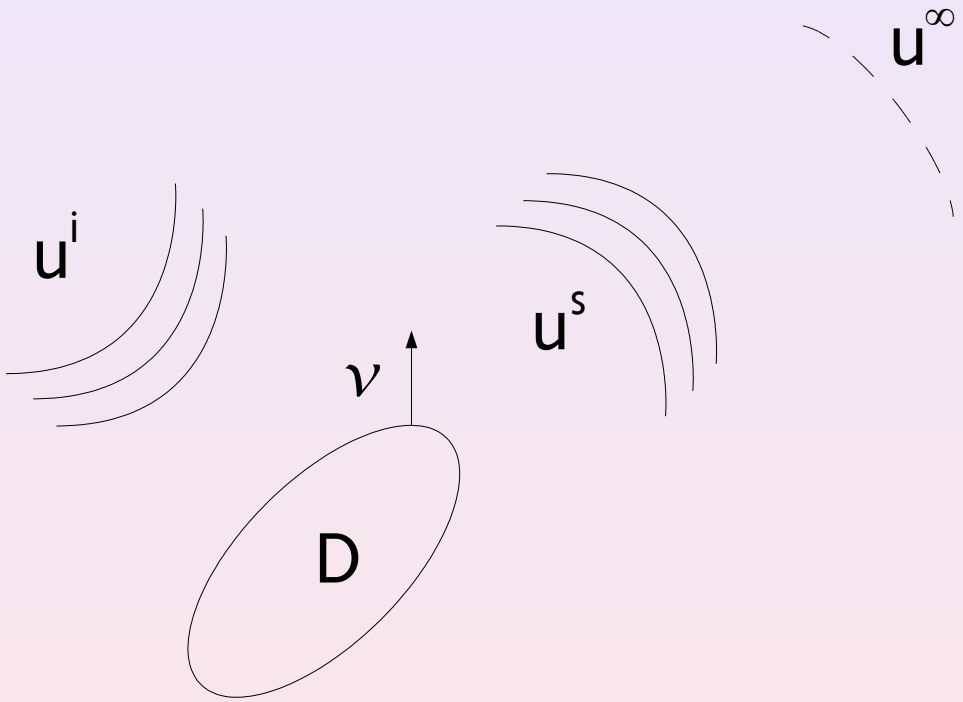

Figure: Model representation

Acoustic Scattering Problem

Direct Problem

Given the shape and physical properties of the domain D and the incident field u^i, find the scattered field u^s.

Inverse Problem (shape reconstruction)

Given the farfield pattern u^∞ and the incident field u^i find the shape of the domain D.

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Overview of the *Range Test*

- The *Multiwave Range Test* is an extension to the *Range Test* [Kusiak/Potthast/Sylvester].
  It handles multiple incident waves in an inverse acoustic scattering framework to find the shape of an scatterer.
- The *Range Test* tests the analytic continuability of a field into the exterior of some test domain.
- The method is a generalized sampling method which samples with test domains and finds the unknown shape as the intersection of those test domains.
- It does not need the boundary condition of the scatterer.

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Multi-Modality and Probe Methods

- The *Range Test* can used for an arbitrary PDE.
- It is possible to reconstruct the shape of an unknown object independently in different modalities and then merge the reconstructed images.
- In comparison, optimization shemes can also naturally combine different modalities.
  Drawbacks: they are usually computationally much more expensive. They often need knowledge of the boundary condition of the object.

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Adaptive Sensing

Completeness

If the far field pattern $u^\infty(\cdot, d)$ for scattering by all incident plane waves with direction $d \in \mathbb{S}$ of a fixed frequency is known, then the response for scattering of arbitrary incident patterns can be decomposed from the knowledge of this far field pattern.

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Adaptive Sensing

- SSM uses all far field data to reconstruct the scattered field of a pointsource. The field at the source point goes to infinity if the source reaches the boundary from which we find the boundary.
- We can construct particular singular incident probes and scan the unknown area.
- The *No Response Test* locates the support of a scatterer from the far field pattern of one wave. It uses a set of sampling domains on which incident fields are small. If the response (the far field pattern) is small, the unknown object is inside the test domain, and outside otherwise.
- We can construct incident fields which are small on some area and employ a "domain-sampling".

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   - Idea, Definitions and Algorithm
   - Numerical Implementation
   - Numerical Examples
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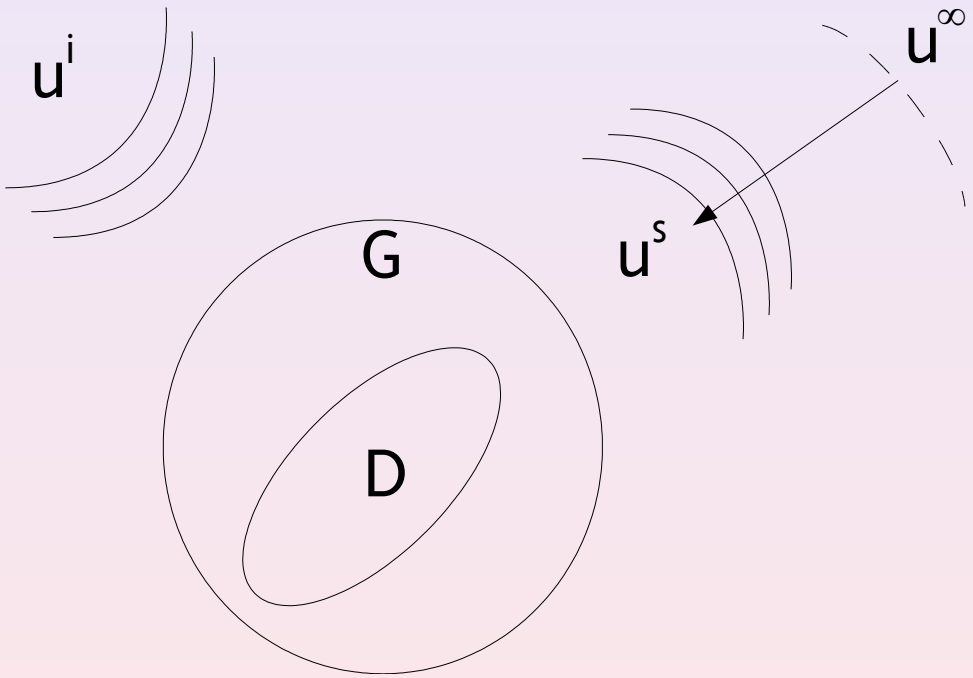
Idea, Range Test



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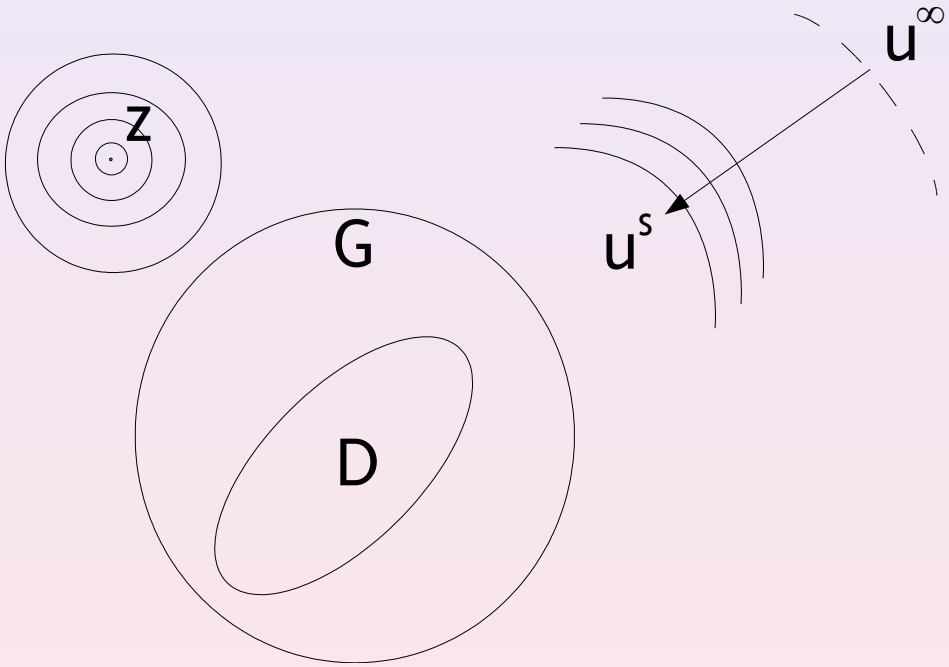

Multiwave Range Test(1)

- Test domain G of class C^2 with the property that the interior homogenous Dirichlet problem has only the trivial solution. Construct a set of test domains $G = G_\tau$ with the index $\tau \in T$.
- Single-layer potential on ∂G

$$(S_{\partial G}\varphi)(x) = \int_{\partial G} \Phi(x,y)\varphi(y)ds(y), \quad x \in \mathbb{R}^m \backslash G$$

- Far field pattern of single-layer potential

$$(S^\infty_{\partial G}\varphi)(\hat{x}) = \gamma_m \int_{\partial G} e^{i\kappa\hat{x}\cdot y}\varphi(y)ds(y), \quad \hat{x} \in \mathbb{S}^{m-1}$$

- Let $\Lambda \subset \mathbb{S}$ be the set of directions of given incident plane waves. Let the far field pattern $u^\infty(\cdot, d)$ be given $\forall d \in \Lambda$.

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Multiwave Range Test(2)

- Assume the extensiability of u^s into $\mathbb{R}^m \backslash G$ and calculate the density with Tikhonov Regularization

$$\varphi_d = (\alpha I + S_{\partial G}^{\infty,*} S_{\partial G}^{\infty})^{-1} S_{\partial G}^{\infty,*} u^{\infty}(\cdot, d) \ \ \forall d \in \Lambda$$

- The mixed reciprocity relation (see [Potthast,2001] chapter 2) leads to

$$\Phi^{\infty}(-d, z) = \gamma_m u^s(z, d) = \gamma_m (S_{\partial G} \varphi_d)(z), \ \ \forall d \in \Lambda, \forall z \in \mathbb{R}^m \backslash G$$

where $\Phi^{\infty}(\cdot, z)$ is the far field pattern of a point source in z.

- Using $S_{\partial G}^{\infty}$ in the same way as above

$$\psi_z = (\alpha I + S_{\partial G}^{\infty,*} S_{\partial G}^{\infty})^{-1} S_{\partial G}^{\infty,*} \Phi^{\infty}(\cdot, z), \ \ \forall z \in \mathbb{R}^m \backslash G$$

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- With this characterization we know that if all densities ψ_z are uniformly bounded for $z \in \mathbb{R}^m \backslash \overline{G}$ we know that $\overline{D} \subset G$.
- Checking of the boundedness is done via the norm

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Numerical Implementation

- construct test domains G_x from a reference domain G_0 by

$$G_x := G_0 + x$$

- calculations for different test domains G_x is done in a efficient way:

$$(S^{\infty}_{\partial G_x}\varphi)(\hat{x}) = \int_{\partial G_x} e^{-i\kappa\hat{x}\cdot y}\varphi(y)ds(y) = e^{-i\kappa\hat{x}\cdot x}(S^{\infty}_{\partial G_0}\tilde{\varphi})(\hat{x})$$

with $\tilde{\varphi}(y) := \varphi(y + x)$

- choose evaluation points $z \in \mathbb{R}\backslash G_x$ around the boundary and consider a fixed constellation of z and G_x. This can be done because we know

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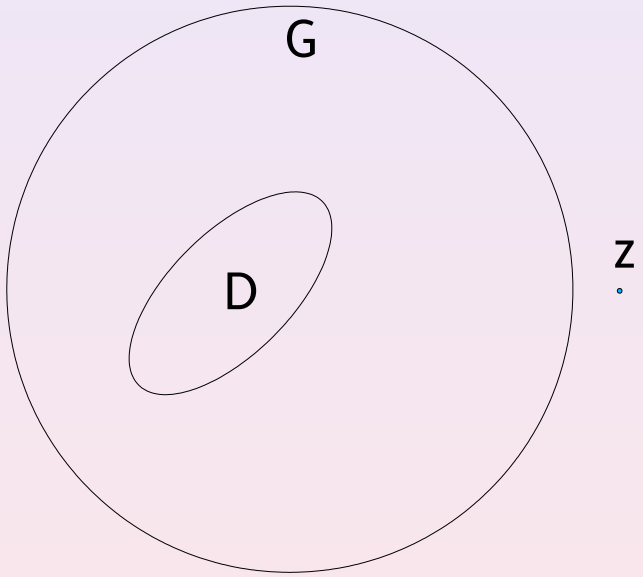

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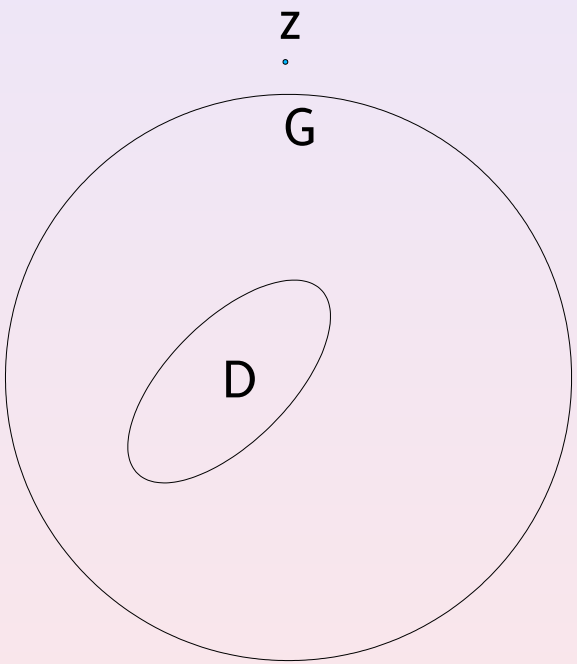

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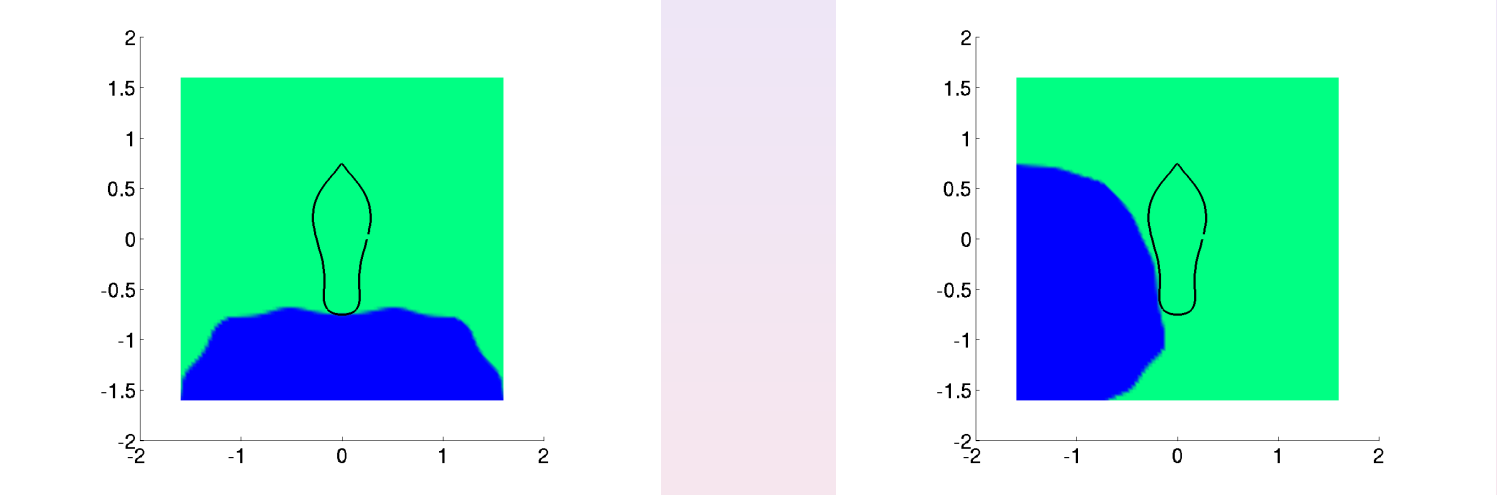

Figure: Reconstruction steps

- Repeat calculation for all constellations.
- Take the unification of the minima over all images and make a cutoff.

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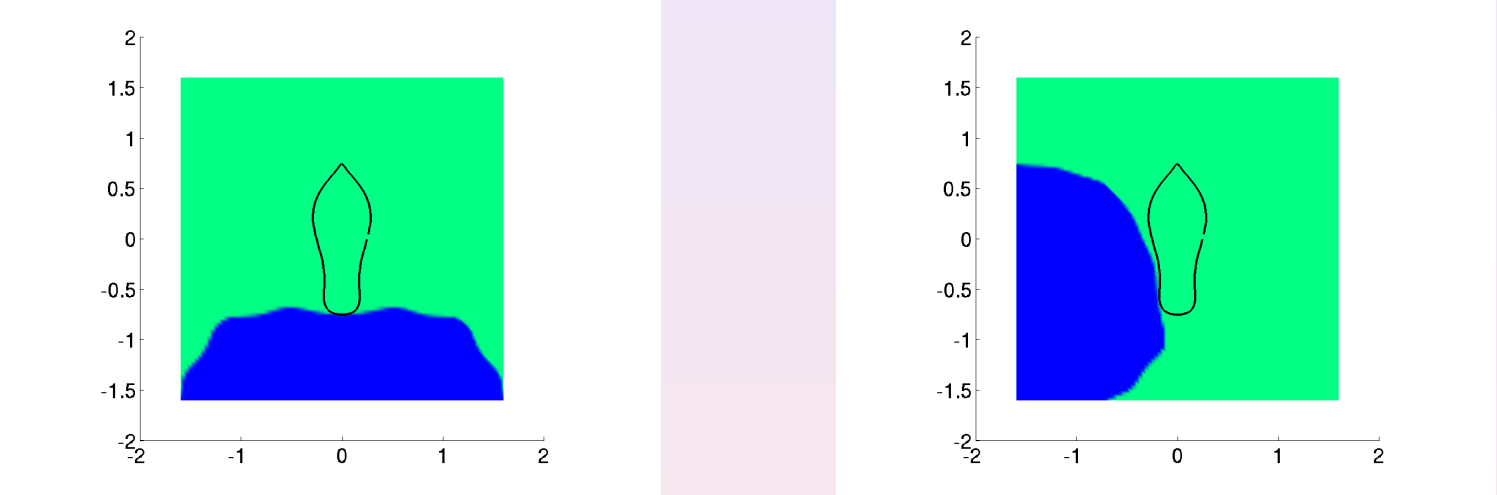

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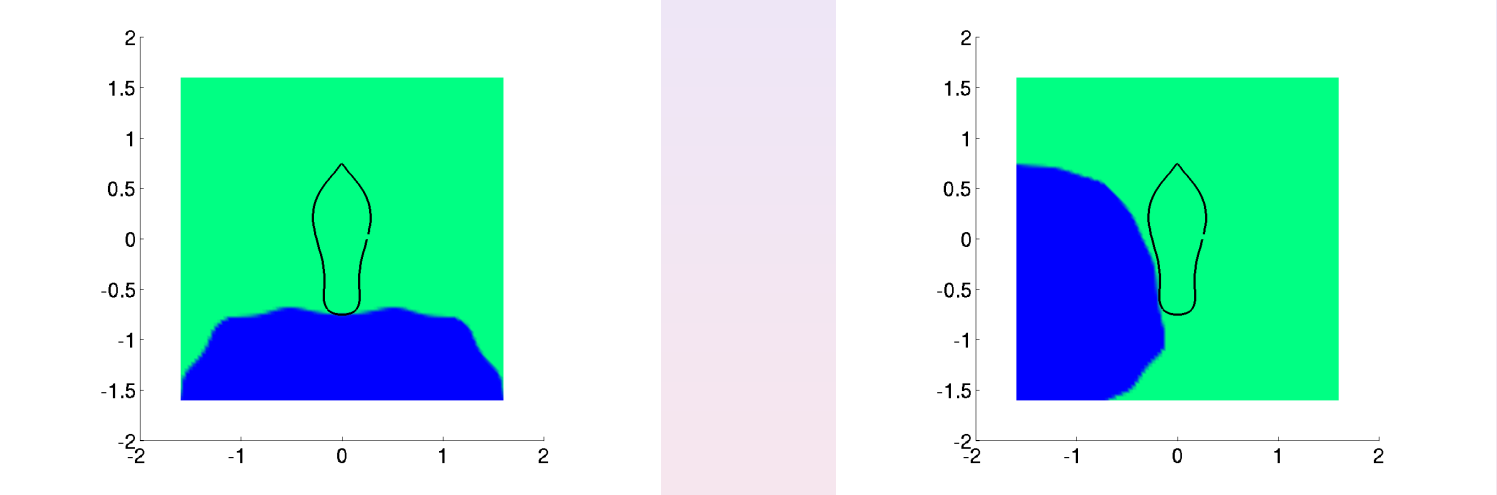

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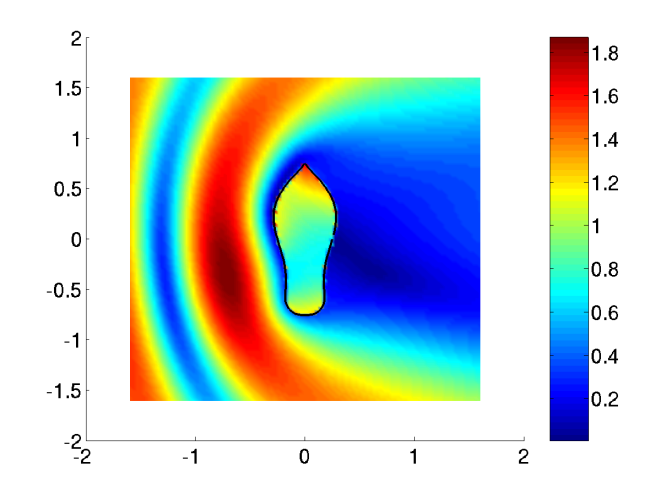

Figure: Total field resulting from an incident plane wave from the left.

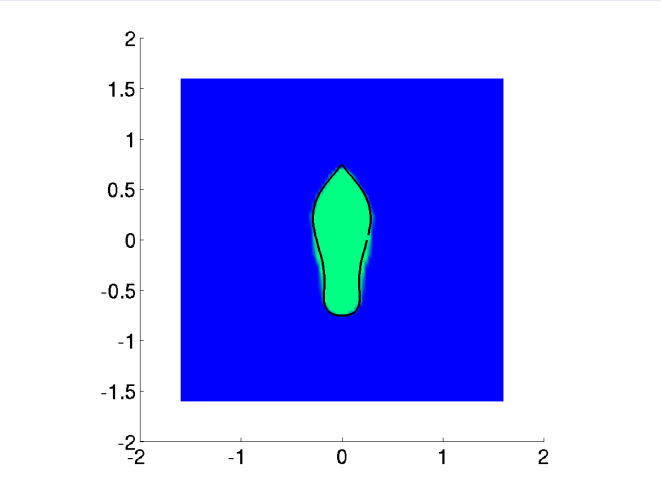

Figure: Reconstruction of a torpedo with 20 rotations

Parameters

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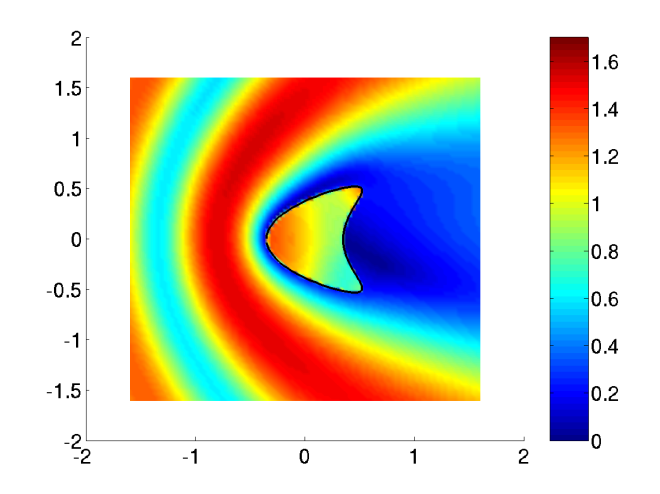

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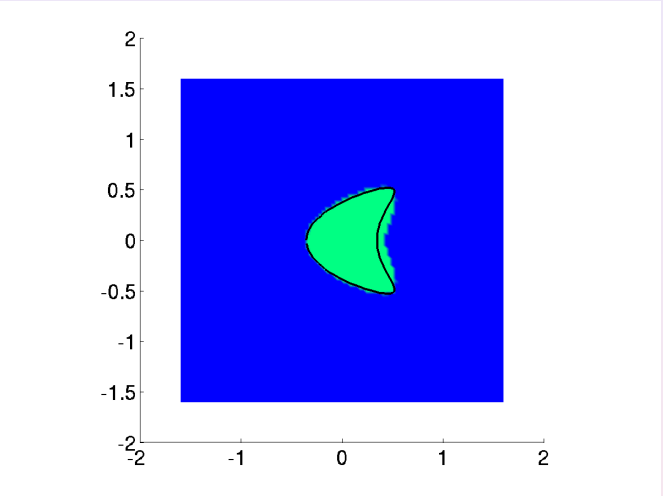

Figure: Reconstruction of a kite with 20 rotations

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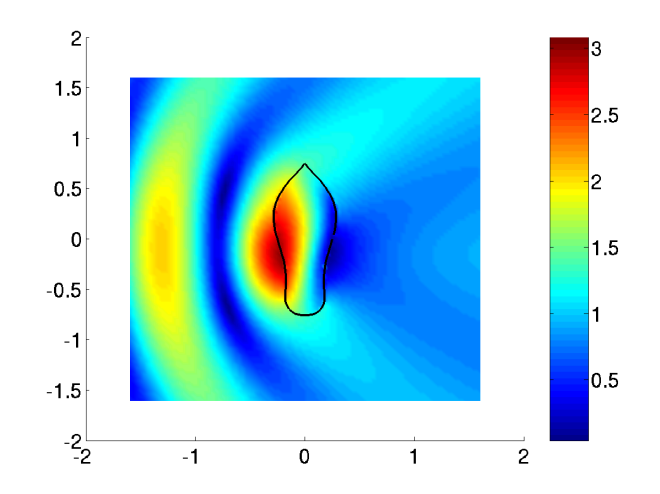
Numerical Example (Neumann)



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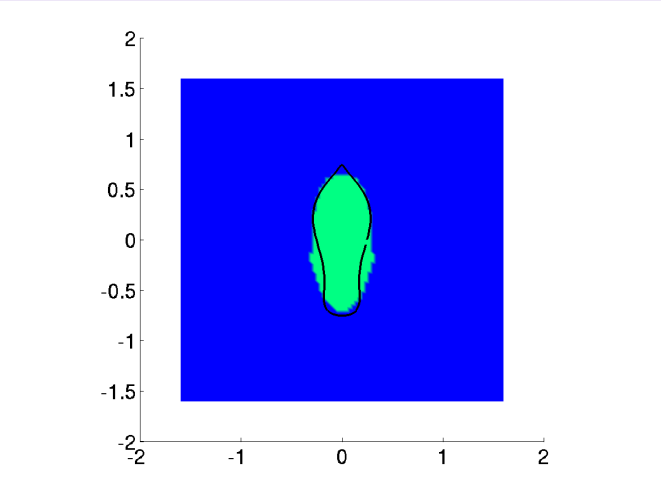

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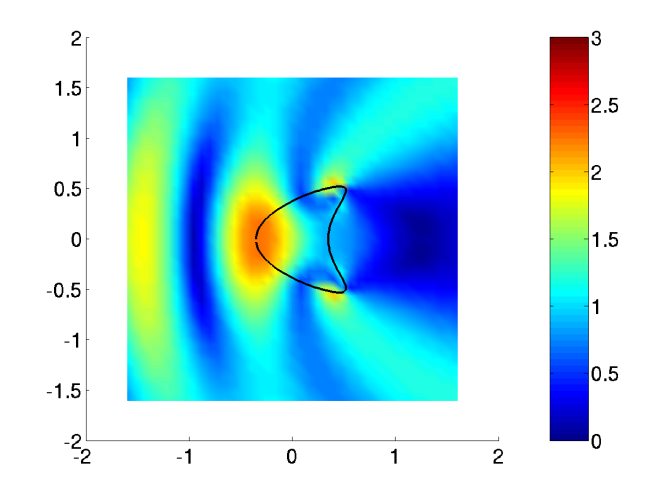

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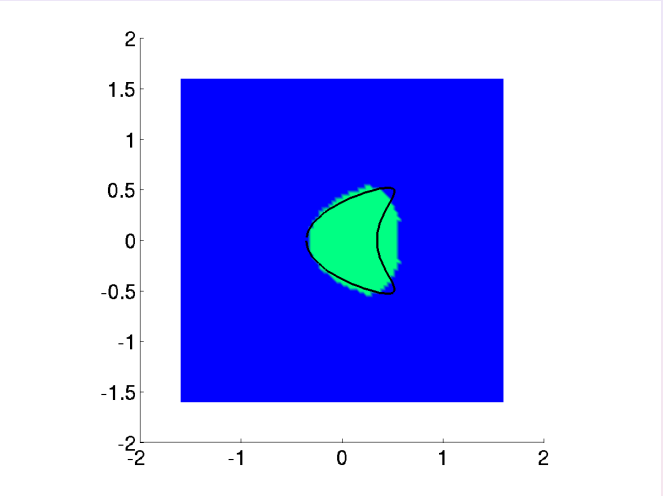

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further Tests

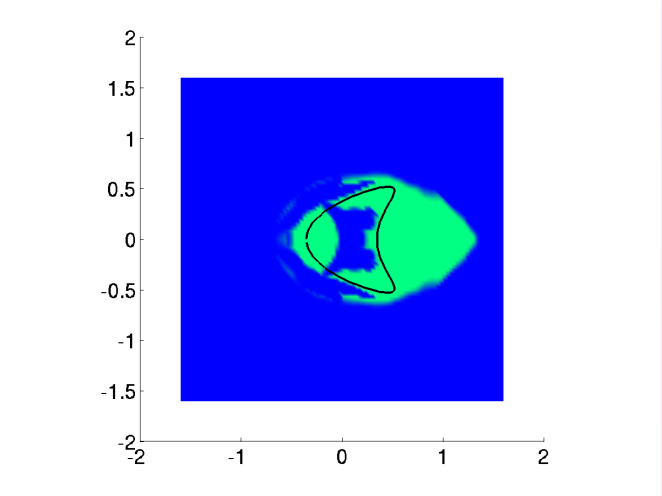

Figure: Multiwave Range Test. 2 incident plane waves.

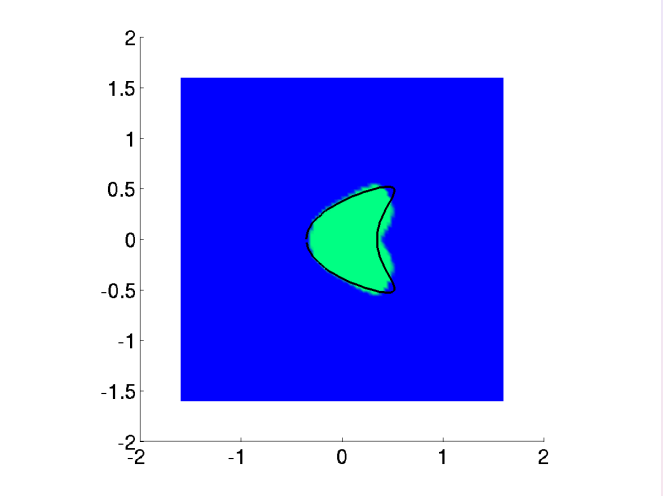

Figure: Multiwave Range Test. 3 incident plane waves.

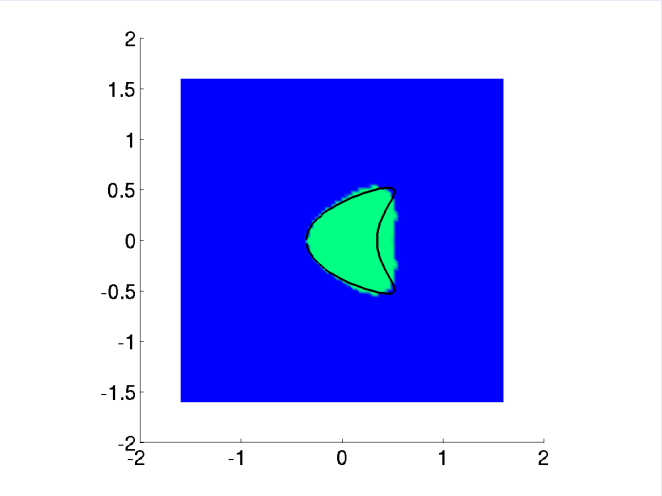

Figure: Multiwave Range Test. 4 incident plane waves.

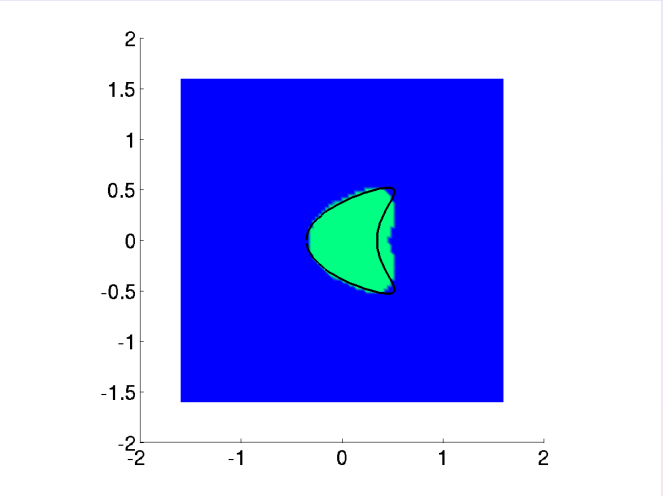

Figure: Multiwave Range Test. 5 incident plane waves.

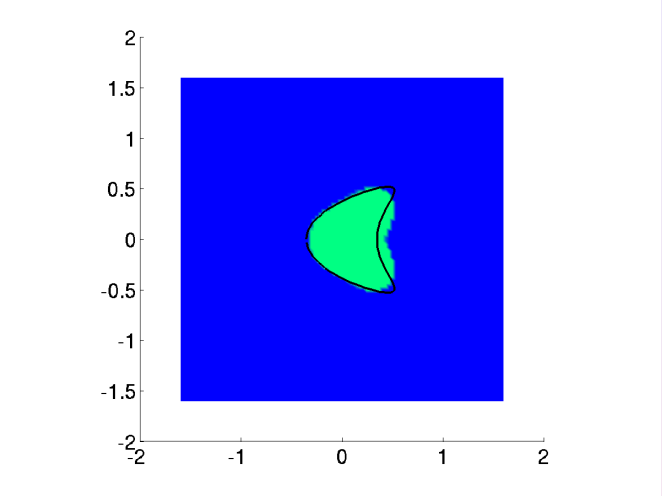

Figure: Multiwave Range Test. 10 incident plane waves.

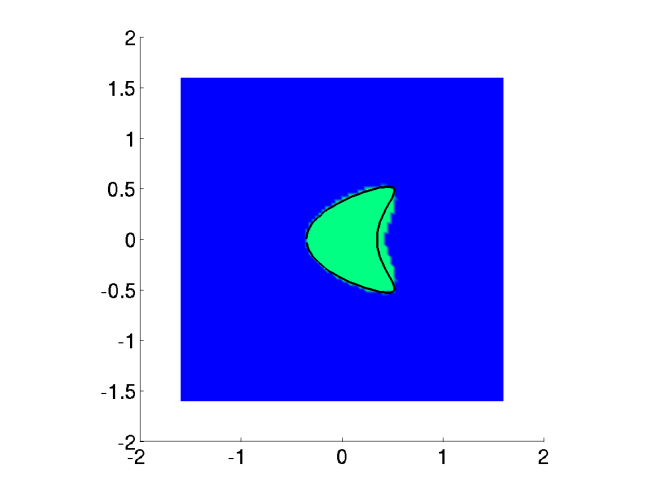

Figure: Multiwave Range Test. 40 incident plane waves.

Outline

1. Introduction
2. Multiwave Range Test
3. Summary and Outlook

Summary

Summary

- SSM and *No Response Test* can be used in adaptive sensing.
- The *Multiwave Range Test* can be used in multi-modality frameworks.
- *Multiwave Range Test* is an extension to the (one wave) *Range Test* to handle multiple incident waves in a special way.
- it uses all (or many) waves to reconstruct domains without using boundary conditions.
- It can fully reconstruct the shape of the scatterer and numerics show it can reconstruct non-convex domains.

Outlook

Outlook

- Numerics in multi-modality and adaptive Sensing.
- 3D
- Use non-sphere test domains. E.g. Sphere with slit.
- Implementation of intersection of test domains.
- (Direct) comparisons to other multiwave algorithms.

Bibliography

R. Potthast.
Point sources and multipoles in inverse scattering theory, volume 427 of *Chapman & Hall/CRC Research Notes in Mathematics*.
Chapman & Hall/CRC, Boca Raton, FL, 2001.

Outline

4 Appendix

Numerical Example (Dirichlet)

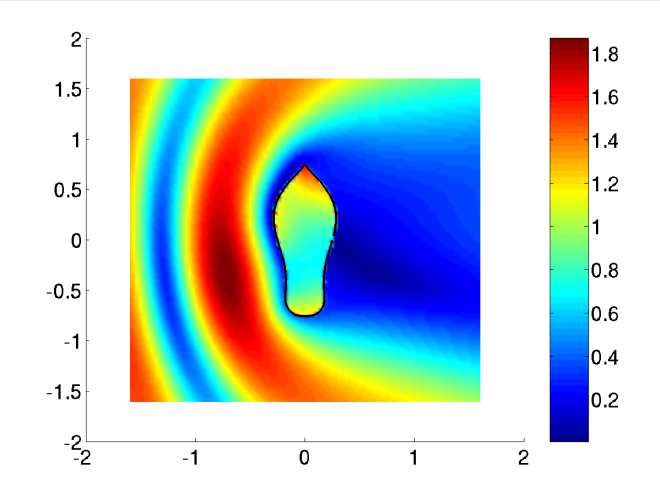

Figure: Total field resulting from an incident plane wave from the left.

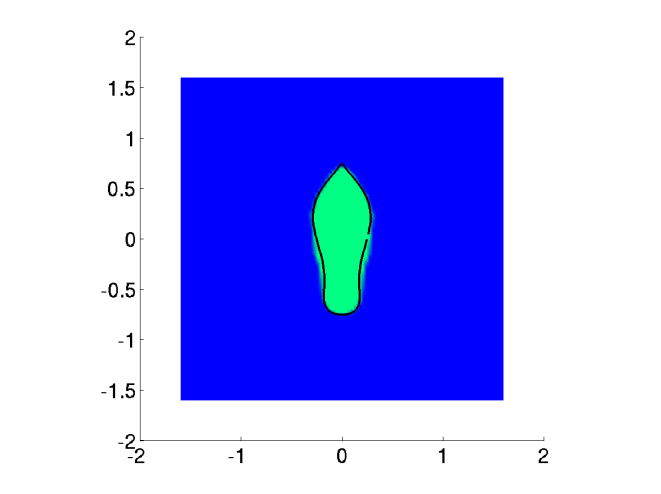

Figure: Reconstruction of a torpedo with 20 rotations

ninnen=100, ngrid=100, ndir=40, nfar=80,κ=3

Return

Numerical Example (Dirichlet)

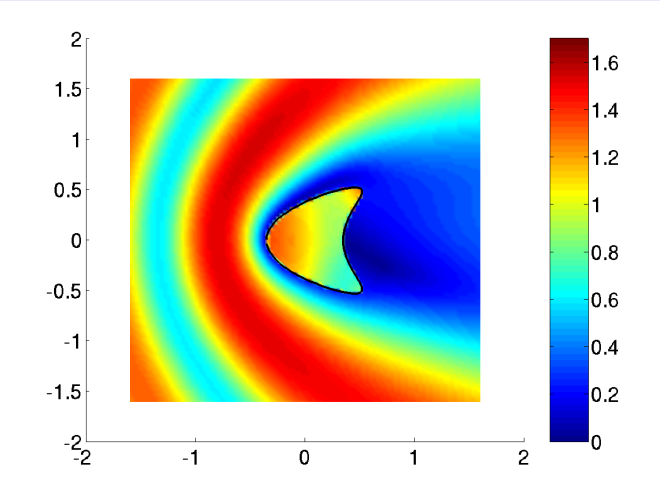

Figure: Total field resulting from an incident plane wave from the left.

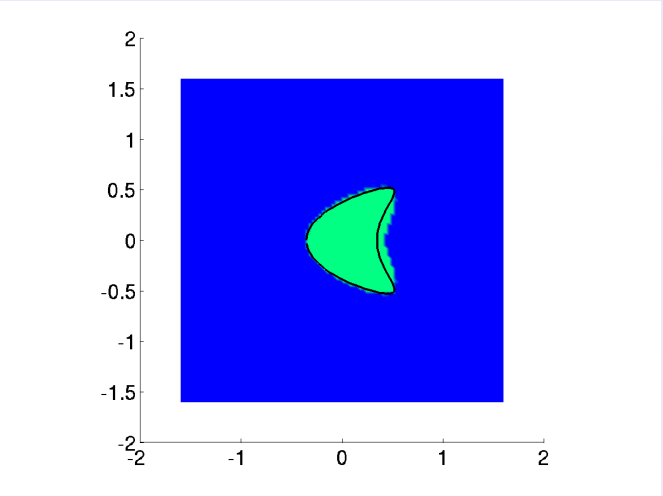

Figure: Reconstruction of a kite with 20 rotations

ninnen=100, ngrid=100, ndir=40, nfar=80,κ=3

Return

Numerical Example (Neumann)

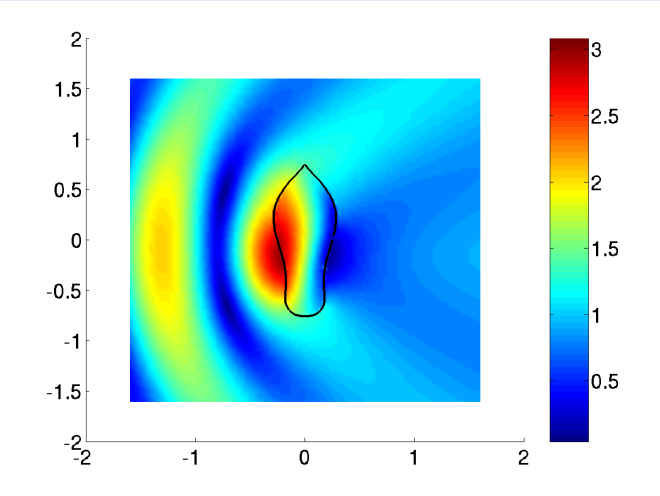

Figure: Total field resulting from an incident plane wave from the left.

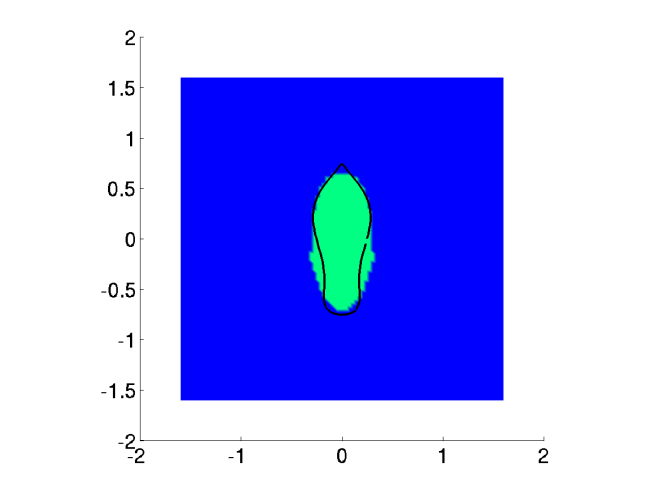

Figure: Reconstruction of a torpedo with 20 rotations

ninnen=100, ngrid=100, ndir=40, nfar=80,κ=3

Return

Numerical Example (Neumann)

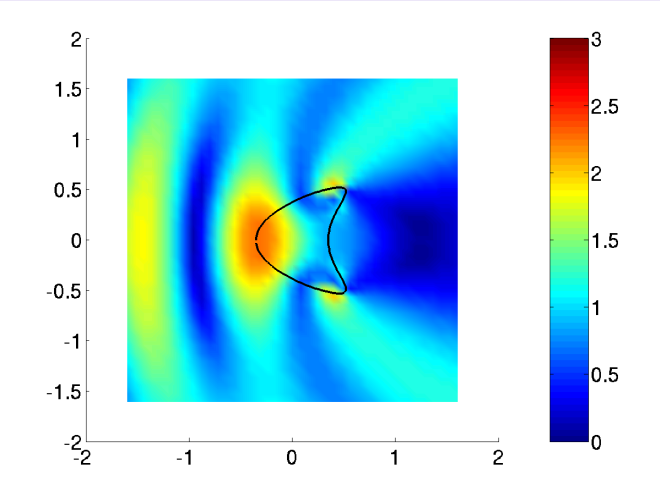

Figure: Total field resulting from an incident plane wave from the left.

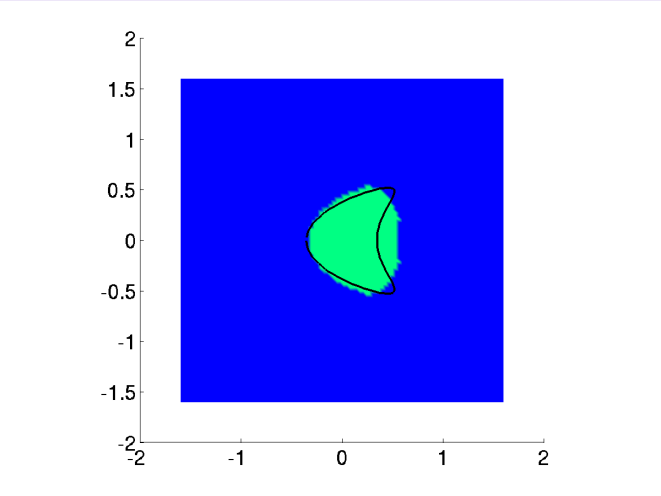

Figure: Reconstruction of a kite with 20 rotations

ninnen=100, ngrid=100, ndir=40, nfar=80,κ=3

Return